The Einstein Equation is Fully Compatible with Purely Newtonian Gravity, but Einstein's Coordinate Condition Self-Consistently Enforces Lorentz Covariance

Steven Kenneth Kauffmann*

Abstract In 1915 Einstein adopted a new coordinate condition for the Einstein equation, namely that the metric tensor's determinant keeps the value -1 it has in the Minkowskian case. In his landmark November 18, 1915 paper, Einstein showed that applying his new coordinate condition to the approximate calculation of the metric of a static point mass (the sun) produces agreement with the previously unaccounted-for part of Mercury's perihelion shift, and also doubles the deflection of light by the sun's gravity from his previous calculation which didn't use his new coordinate condition; a 1919 solar-eclipse expedition verified his new result. In January, 1916 Schwarzschild published the exact version of Einstein's new static point-mass metric; as expected, it slightly lengthens circular-orbit periods. In May 1916 Droste published a much simpler exact metric that fails to satisfy the Einstein equation at all empty-space points and doesn't lengthen circular-orbit periods. In 1922 Friedmann replaced Einstein's coordinate condition with setting the metric's time-time component to unity; this eliminates gravitational time dilation and sends c to infinity, causing the Einstein equation to yield purely Newtonian gravity. We revisit the Oppenheimer-Snyder model using Einstein's coordinate condition instead; the considerably different results reflect gravitational time dilation.

1. Detailed review of Einstein's relativistic gravity and Friedmann's Newtonian regression

The *central entity* of Einstein's gravity theory is the dimensionless Riemann space-time metric tensor $g_{\mu\nu}(x)$, whose *physical role* is that of a multicomponent *gravitational potential* which, *via a stationary-action principle,* determines a test-particle's *gravitational trajectory in that gravitational potential.* The *action involving the metric tensor $g_{\mu\nu}(x)$ that is relevant to the determination of a test particle's gravitational trajectory* is,

$$-mc \int ds = -mc \int \sqrt{g_{\mu\nu}(x)\, dx^\mu\, dx^\nu}, \tag{1.1a}$$

where m is the test particle's rest mass. The path of stationary action *is also clearly the path of stationary length $\int ds$,* namely *the geodesic of the Riemannian geometry.* We take note of the fact that when $g_{\mu\nu}(x)$ becomes the metric of special relativity $\eta_{\mu\nu}$, *so that gravity is absent,* the action $-mc\int ds$ reduces to,

$$-mc \int \sqrt{\eta_{\mu\nu}\, dx^\mu\, dx^\nu} = -mc \int \sqrt{(c\,dt)^2 - |d\mathbf{x}|^2} = \int \Big(-mc^2\sqrt{1 - |\dot{\mathbf{x}}/c|^2}\Big) dt, \tag{1.1b}$$

where we recognize that $-mc^2\sqrt{1-|\dot{\mathbf{x}}/c|^2}$ *is the Lagrangian of the relativistic free particle.* When $|\dot{\mathbf{x}}| \ll c$, it reduces to $-mc^2 + \frac{1}{2}m|\dot{\mathbf{x}}|^2$, where $\frac{1}{2}m|\dot{\mathbf{x}}|^2$ *of course is the Lagrangian of the nonrelativistic free particle.*

Riemannian geometry originally was concerned with interesting extensions of the concept of *space* (e.g., curved surfaces in various dimensions), whereas *gravity's arena* of course *is four-dimensional space-time.* In *particular,* when $g_{\mu\nu}(x)$ becomes special relativity's $\eta_{\mu\nu}$ and gravity *is absent,*

$$ds = \sqrt{\eta_{\mu\nu}\, dx^\mu\, dx^\nu} = \sqrt{(c\,dt)^2 - |d\mathbf{x}|^2} = c\sqrt{1-|\dot{\mathbf{x}}/c|^2}\, dt = c\, d\tau, \tag{1.2a}$$

where $d\tau = \big(\sqrt{1-|\dot{\mathbf{x}}/c|^2}\, dt\big)$ *is special relativity's Lorentz-invariant differential time.* When gravity *is present,* we *extend* the Eq. (1.2a) relation $ds = c\, d\tau$ *to accommodate the gravitational metric tensor $g_{\mu\nu}(x)$,*

$$ds = \sqrt{g_{\mu\nu}(x)\, dx^\mu\, dx^\nu} = c\, d\tau, \quad \text{so,} \quad c = ds/d\tau = \sqrt{g_{\mu\nu}(x)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau)}, \tag{1.2b}$$

and we in addition *assume transformation properties of the metric tensor $g_{\mu\nu}(x)$ which are consistent with the invariance of $d\tau$ under general transformations of the space-time coordinates x^μ.* We postpone presentation of the details of such transformations until after we obtain a test particle's gravitational equation of motion.

The equation of motion for a test particle's gravitational trajectory which connects two fixed space-time points (and likewise the equation for the geodesic of the Riemannian geometry which connects those two points) is obtained by setting to zero the first-order variation of the length $\int ds$ of that trajectory with respect to an infinitesimal change δx^λ of the trajectory x^λ between those two fixed points. Thus,

$$0 = \delta \int ds = \delta \int \sqrt{g_{\mu\nu}(x)\, dx^\mu\, dx^\nu} = \delta \int \big(\sqrt{g_{\mu\nu}(x)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau)}\big)\, d\tau =$$

$$\int \Big(\sqrt{g_{\mu\nu}(x^\lambda + \delta x^\lambda)\,(d(x^\mu + \delta x^\mu)/d\tau)\,(d(x^\nu + \delta x^\nu/d\tau)} - \sqrt{g_{\mu\nu}(x)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau)}\Big) d\tau. \tag{1.3a}$$

Through first order in the infinitesimal change δx^λ of the trajectory x^λ between the two points we have that,

*Retired, APS Senior Member, SKKauffmann@gmail.com.

$$g_{\mu\nu}(x^\lambda + \delta x^\lambda)\,(d(x^\mu + \delta x^\mu)/d\tau)\,(d(x^\nu + \delta x^\nu)/d\tau) \simeq g_{\mu\nu}(x)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau) +$$
$$(\partial g_{\mu\nu}/\partial x^\lambda)\,(\delta x^\lambda)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau) + g_{\mu\nu}(x)\,(d(\delta x^\mu)/d\tau)\,(dx^\nu/d\tau) + g_{\mu\nu}(x)\,(dx^\mu/d\tau)\,(d(\delta x^\nu)/d\tau). \quad (1.3b)$$

In light of the result obtained in Eq. (1.3b), we see that the integrand of the last integral of Eq. (1.3a) is of the form $(\sqrt{\alpha+\epsilon} - \sqrt{\alpha})$, where ϵ is infinitesimal. Evaluation of $(\sqrt{\alpha+\epsilon} - \sqrt{\alpha})$ to first order in ϵ yields,

$$(\sqrt{\alpha+\epsilon} - \sqrt{\alpha}) = \sqrt{\alpha}(\sqrt{1 + (\epsilon/\alpha)} - 1) \simeq \sqrt{\alpha}(1 + \tfrac{1}{2}(\epsilon/\alpha) - 1) = \tfrac{1}{2}\epsilon/\sqrt{\alpha}. \quad (1.3c)$$

For the integrand of the last integral of Eq. (1.3a) we have, in the language of Eq. (1.3c), given Eq. (1.3b),

$$\alpha = g_{\mu\nu}(x)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau) \text{ and } \epsilon = (\partial g_{\mu\nu}/\partial x^\lambda)\,(\delta x^\lambda)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau) +$$
$$g_{\mu\nu}(x)\,(d(\delta x^\mu)/d\tau)\,(dx^\nu/d\tau) + g_{\mu\nu}(x)\,(dx^\mu/d\tau)\,(d(\delta x^\nu)/d\tau). \quad (1.3d)$$

From Eq. (1.2b), $\sqrt{g_{\mu\nu}(x)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau)} = c$, and from Eq. (1.3c), $(\sqrt{\alpha+\epsilon} - \sqrt{\alpha}) = \frac{1}{2}\epsilon/\sqrt{\alpha}$. We next combine these two facts with Eqs. (1.3d) and (1.3b) to present Eq. (1.3a) as,

$$0 = \delta\int ds = \tfrac{1}{2}(1/c)\int\Big((\partial g_{\mu\nu}/\partial x^\lambda)\,(\delta x^\lambda)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau)$$
$$+ g_{\lambda\nu}(x)\,(d(\delta x^\lambda)/d\tau)\,(dx^\nu/d\tau) + g_{\mu\lambda}(x)\,(dx^\mu/d\tau)\,(d(\delta x^\lambda)/d\tau)\Big)d\tau. \quad (1.3e)$$

Since the two endpoints of the trajectory *are fixed,* the *infinitesimal change* δx^λ in the trajectory *vanishes at those two endpoints,* which makes the needed integrations by parts in Eq. (1.3e) straightforward. Bearing in mind that $g_{\mu\nu}(x) = g_{\nu\mu}(x)$, these integrations by parts cause Eq. (1.3e) to read,

$$0 = \delta\int ds = \tfrac{1}{2}(1/c)\int\Big((\partial g_{\mu\nu}/\partial x^\lambda)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau)$$
$$-2g_{\lambda\mu}(x)\,(d^2x^\mu/d\tau^2) - 2(dg_{\lambda\mu}(x)/d\tau)\,(dx^\mu/d\tau)\Big)\delta x^\lambda d\tau. \quad (1.3f)$$

Since $(dg_{\lambda\mu}(x)/d\tau) = (\partial g_{\lambda\mu}/\partial x^\nu)\,(dx^\nu/d\tau)$ Eq. (1.3f) becomes,

$$0 = \delta\int ds = (1/c)\int\Big(\big(\tfrac{1}{2}(\partial g_{\mu\nu}/\partial x^\lambda) - (\partial g_{\lambda\mu}/\partial x^\nu)\big)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau) - g_{\lambda\mu}(x)\,(d^2x^\mu/d\tau^2)\Big)\delta x^\lambda d\tau. \quad (1.3g)$$

Since, except from the two fixed endpoints of the trajectory, the trajectory's infinitesimal change δx^λ is essentially arbitrary, we can conclude from Eq. (1.3g) that,

$$g_{\lambda\mu}(x)\,(d^2x^\mu/d\tau^2) + \tfrac{1}{2}\big((\partial g_{\lambda\mu}/\partial x^\nu) + (\partial g_{\lambda\nu}/\partial x^\mu) - (\partial g_{\mu\nu}/\partial x^\lambda)\big)\,(dx^\mu/d\tau)\,(dx^\nu/d\tau) = 0. \quad (1.3h)$$

If the metric tensor $g_{\lambda\mu}(x)$ *happens to have a matrix inverse* $g^{\kappa\lambda}(x)$ *at every space-time point* x *such that* $g^{\kappa\lambda}(x)\,g_{\lambda\mu}(x) = \delta^\kappa_\mu$, then Eq. (1.3h) can be expressed in the form,

$$d^2x^\kappa/d\tau^2 + \Gamma^\kappa_{\mu\nu}\,(dx^\mu/d\tau)\,(dx^\nu/d\tau) = 0, \quad (1.3i)$$

where,

$$\Gamma^\kappa_{\mu\nu} \overset{\text{def}}{=} \tfrac{1}{2}g^{\kappa\lambda}(x)\big(\big(\partial g_{\lambda\mu}/\partial x^\nu\big) + \big(\partial g_{\lambda\nu}/\partial x^\mu\big) - \big(\partial g_{\mu\nu}/\partial x^\lambda\big)\big), \quad (1.3j)$$

is called *the affine connection.* A *critical difference* between the Eq. (1.3h) test-particle gravitational equation of motion versus the test-particle gravitational equation of motion of Eqs. (1.3i) and (1.3j) is that *only the latter adheres to the special-relativistic extension of Newton's Second Law for that test particle,* which is,

$$m\,d^2x^\kappa/d\tau^2 = F^\kappa. \quad (1.3k)$$

In the case that the metric tensor's matrix inverse $g^{\kappa\lambda}(x)$ *exists everywhere in space-time,* so that the test-particle gravitational equation of motion is that given by Eqs. (1.3i) and (1.3j), we have that,

$$F^\kappa = -m\,\Gamma^\kappa_{\mu\nu}\,(dx^\mu/d\tau)\,(dx^\nu/d\tau), \quad (1.3l)$$

but if the metric tensor's matrix inverse $g^{\kappa\lambda}(x)$ *fails to exist everywhere in space-time,* so that *only* Eq. (1.3h) holds for the test-particle's motion, the Eq. (1.3k) precept of special relativity *will be violated.*

Einstein's 1915 coordinate condition $\det(g_{\mu\nu}(x)) = -1$, which he used to calculate the previously unaccounted-for part of Mercury's perihelion shift, obviously *guarantees* the existence of the metric tensor's matrix inverse $g^{\kappa\lambda}(x)$, *but it has since been almost completely forgotten; it isn't mentioned anywhere* in Steven Weinberg's 657-page 1972 tome Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. Weinberg favors the harmonic coordinate equations $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$, which *prima facie assume the existence of the metric tensor's inverse* $g^{\mu\nu}(x)$ *without guaranteeing it,* whereas Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *clearly does guarantee the existence of the metric's inverse* $g^{\mu\nu}(x)$.

The issue of the existence of the metric tensor's inverse $g^{\kappa\lambda}(x)$ *goes far beyond* whether the test-particle gravitational equation of motion adheres to the special-relativistic extension of Newton's Second Law. The systematic construction of entities *which keep their form under general transformations of space-time coordinates relies on the affine connection* $\Gamma^{\kappa}_{\mu\nu} \stackrel{\text{def}}{=} \frac{1}{2}g^{\kappa\lambda}(x)\big((\partial g_{\lambda\mu}/\partial x^{\nu}) + (\partial g_{\lambda\nu}/\partial x^{\mu}) - (\partial g_{\mu\nu}/\partial x^{\lambda})\big)$ *of* Eq. (1.3j) *as a fundamental building block,* and it is of course *transparent that the existence of the affine connection is completely dependent on the existence of the metric tensor's inverse* $g^{\kappa\lambda}(x)$.

Before we turn to the details of general transformations of space-time coordinates, we point out the conditions under which the test-particle gravitational equation of motion of Eqs. (1.3i) and (1.3j) is consistent with Newtonian gravity, and, after that, we establish that two clocks at rest at two different points of a gravitational field will usually tick at two different rates, which is called gravitational time dilation.

We now show that when the metric tensor $g_{\mu\nu}(x)$ is static (independent of time), has components which differ by much less than unity from those of the special-relativistic metric tensor $\eta_{\mu\nu}$, and the test particle's speed is much less than c, then the test particle's behavior is consistent with Newtonian gravity.

We note that $(dx^{\mu}/d\tau) = (dt/d\tau)(dx^{\mu}/dt) = (dt/d\tau)(c, \dot{\mathbf{x}}) = c\,(dt/d\tau)(1, (\dot{\mathbf{x}}/c))$, which when $|\dot{\mathbf{x}}| \ll c$ is almost equal to $c\,(dt/d\tau)(1, \mathbf{0}) = c\,(dt/d\tau)\delta^{\mu}_{0}$. This approximate result for $(dx^{\mu}/d\tau)$ implies that when $|\dot{\mathbf{x}}| \ll c$, the Eq. (1.3i) dynamical equation for a test particle in a gravitational field, namely,

$$d^2x^{\kappa}/d\tau^2 + \Gamma^{\kappa}_{\mu\nu}(x)\,(dx^{\mu}/d\tau)\,(dx^{\nu}/d\tau) = 0, \tag{1.4a}$$

is well approximated by,

$$d^2x^{\kappa}/d\tau^2 + c^2(dt/d\tau)^2\,\Gamma^{\kappa}_{00}(x) = 0. \tag{1.4b}$$

We next insert into Γ^{κ}_{00}, as it is given by Eq. (1.3j), the metric tensor $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$, *where* $h_{\mu\nu}(x)$ *is assumed to be static, and contributions to* Γ^{κ}_{00} *which are second-order or higher in* $h_{\mu\nu}$ *are discarded,*

$$\Gamma^{\kappa}_{00}(x) = \tfrac{1}{2}g^{\kappa\lambda}(x)\big((\partial h_{\lambda 0}/\partial x^0) + (\partial h_{\lambda 0}/\partial x^0) - (\partial h_{00}/\partial x^{\lambda})\big) \approx -\tfrac{1}{2}\eta^{\kappa\lambda}(\partial h_{00}/\partial x^{\lambda}), \tag{1.4c}$$

which, upon insertion into Eq. (1.4b), yields,

$$d^2x^{\kappa}/d\tau^2 = \tfrac{1}{2}\,c^2\,(dt/d\tau)^2\eta^{\kappa\lambda}(\partial h_{00}/\partial x^{\lambda}). \tag{1.4d}$$

Since $h_{00}(x)$ *is static,* we call it $h_{00}(\mathbf{x})$, and since $(\partial h_{00}/\partial x^0) = 0$, the $\kappa = 0$ component of Eq. (1.4d) is,

$$d^2(ct)/d\tau^2 = 0, \tag{1.4e}$$

which implies that,

$$dt/d\tau \textit{ is a constant.} \tag{1.4f}$$

The $\kappa = 1$, 2 and 3 components of Eq. (1.4d) yield the *three-vector* equation,

$$(d^2\mathbf{x}/d\tau^2)/(dt/d\tau)^2 = -\nabla_{\mathbf{x}}(\tfrac{1}{2}\,c^2\,h_{00}(\mathbf{x})), \tag{1.4g}$$

which, because $d^2\mathbf{x}/d\tau^2 = (d^2\mathbf{x}/dt^2)(dt/d\tau)^2 + (d\mathbf{x}/dt)(d^2t/d\tau^2)$ and $dt/d\tau$ *is a constant,* implies that,

$$d^2\mathbf{x}/dt^2 = -\nabla_{\mathbf{x}}(\tfrac{1}{2}\,c^2\,h_{00}(\mathbf{x})). \tag{1.4h}$$

The corresponding *Newtonian* gravitational acceleration equation of course is,

$$d^2\mathbf{x}/dt^2 = -\nabla_{\mathbf{x}}\,\phi(\mathbf{x}), \tag{1.4i}$$

where *a typical example* of such a Newtonian-gravity potential $\phi(\mathbf{x})$ is $-GM/|\mathbf{x}|$, which is produced by a static point mass M at $\mathbf{x} = \mathbf{0}$. Comparison of Eq. (1.4h) with Eq. (1.4i) shows that,

$$h_{00}(\mathbf{x}) = 2\,\phi(\mathbf{x})/c^2. \tag{1.4j}$$

Thus, *in the Newtonian limit, where* $|h_{00}(\mathbf{x})| \ll 1$, Eq. (1.4j) *implies that,*

$$g_{00}(x) \approx g_{00}(\mathbf{x}) = \eta_{00} + h_{00}(\mathbf{x}) = 1 + 2\,\phi(\mathbf{x})/c^2, \text{ where } |\phi(\mathbf{x})| \ll \tfrac{1}{2}c^2. \tag{1.4k}$$

We next explore gravitational time dilation. The *ratio* of *the invariant differential time* $d\tau = ds/c = \big(\sqrt{g_{\mu\nu}(x)\, dx^\mu\, dx^\nu}\big)/c$ *to the differential time* (dx^0/c) *recorded by a clock embedded in a test particle that is moving arbitrarily in the gravitational field described by* $g_{\mu\nu}(x)$ *is,*

$$(d\tau)/(dx^0/c) = (c\, d\tau)/(dx^0) = \sqrt{g_{\mu\nu}(x)\,(dx^\mu/dx^0)\,(dx^\nu/dx^0)}\,. \tag{1.5a}$$

When the test particle with its embedded clock *is at rest with respect to the observer,*

$$(dx^\mu/dx^0) = \delta^\mu_0,\ \text{ so Eq. (1.5a) reduces to } (c\, d\tau)/(dx^0) = \sqrt{g_{00}(x)}\,. \tag{1.5b}$$

We now use Eq. (1.5b) to work out *the ratio* dx^0_1/dx^0_2 *of the two differential times* dx^0_1/c *and* dx^0_2/c *recorded by two clocks at rest with respect to the observer present at two different space-time points* x_1 *and* x_2*,*

$$(c\, d\tau)/(dx^0_1) = \sqrt{g_{00}(x_1)}\ \textit{and}\ (c\, d\tau)/(dx^0_2) = \sqrt{g_{00}(x_2)}\ \textit{together yield,}$$

$$dx^0_1/dx^0_2 = \big((c\, d\tau)/(dx^0_2)\big)/\big((c\, d\tau)/(dx^0_1)\big) = \sqrt{g_{00}(x_2)/g_{00}(x_1)}\,. \tag{1.5c}$$

Eq. (1.5c) implies that,

$$\big[(\text{the tick rate of the clock at } x_2)/(\text{the tick rate of the clock at } x_1)\big] = \sqrt{g_{00}(x_2)/g_{00}(x_1)}\,. \tag{1.5d}$$

In the Newtonian limit, $g_{00}(x)$ is static and very close to unity, i.e., $g_{00}(x) = g_{00}(\mathbf{x}) \approx 1 + 2\phi(\mathbf{x})/c^2$ where $|\phi(\mathbf{x})| \ll \frac{1}{2}c^2$ according to Eq. (1.4k). Therefore, in the Newtonian limit Eq. (1.5d) yields that,

$$\big[(\text{the tick rate of the clock at } \mathbf{x}_2)/(\text{the tick rate of the clock at } \mathbf{x}_1)\big] \approx \sqrt{(1 + 2\phi(\mathbf{x}_2)/c^2)/(1 + 2\phi(\mathbf{x}_1)/c^2)}$$

$$\approx \sqrt{1 - \big(2\big(\phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)\big)/c^2\big)} \approx \Big[1 - \big(\big(\phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)\big)/c^2\big)\Big], \tag{1.5e}$$

which has been verified using super-accurate atomic clocks by, for example, placing one clock 33 cm above another on a wall for a day or so. Eq. (1.5e) gives that the tick rate of the clock below divided by that of the clock above equals $[1 - (g(\Delta h)/c^2)]$, where $g = 9.8$ m/s^2, $\Delta h = 0.33$ m and $c = 3 \times 10^8$ m/s, so the tick rate of the clock below is slower by the factor $(1 - 3.6 \times 10^{-17})$ than that of the clock above, which implies that the clock below loses about 3 picoseconds per 24-hour day relative to the clock above. In this experiment the positions of the two atomic clocks are subsequently swapped as a check for systematic errors.

Since GPS satellites *are vastly more than 33 cm above the earth's surface, there is a far greater gravity-caused difference* between an atomic clock's tick rate on the earth's surface and its tick rate in a GPS satellite. There is, however, *also* a special-relativistic *speed effect* on the tick rate of an atomic clock in a fast-moving GPS satellite when that clock is viewed from the earth; GPS satellites *aren't geostationary.*

Atomic clocks and satellites *didn't exist in 1922,* the year that A. Friedmann decided *to try fixing* $g_{00}(x)$ *to unity everywhere in space-time,* and was thrilled by how that made it feasible *to analytically solve the Einstein equation in some cases.* In due course it was realized that the analytic solutions of the Einstein equation in those cases *exactly correspond to purely Newtonian gravity.*

It is *obvious* from Eq. (1.5d) that fixing $g_{00}(x)$ everywhere to unity *eliminates gravitational time dilation,* which in the present era of atomic clocks and satellites *casts an extremely unfavorable light on the practice of fixing* $g_{00}(x)$ *everywhere to unity* that A. Friedmann so "successfully" initiated in 1922. The *elimination of gravitational time dilation* that is the result of fixing $g_{00}(x)$ to unity everywhere supports the hypothesis that fixing $g_{00}(x)$ to unity everywhere *reduces relativistic gravity to its Newtonian counterpart.* The fact that *all known analytic solutions of the Einstein equation which result from fixing* $g_{00}(x)$ *to unity everywhere correspond exactly to purely Newtonian gravity further supports this hypothesis.* Another clue to the effect of fixing $g_{00}(x)$ to unity everywhere is the Eq. (1.4k) Newtonian-limit result that $g_{00}(x) \approx 1 + 2\,\phi(\mathbf{x})/c^2$, *which is only consistent with* $g_{00}(x) = 1$ *everywhere when* $c \to \infty$*, reducing relativistic gravity to its Newtonian counterpart.* We *furthermore* find in Steven Weinberg's 1972 tome in Section 4.1 on pages 92-93 the following statement that is motivated by a 1928 mathematical-journal article by K. O. Friedrichs, "In particular, general covariance does not imply Lorentz invariance—there are generally covariant theories of gravitation that allow the construction of inertial frames at any point in a gravitational field, but that satisfy Galilean relativity rather than special relativity in these frames." In this regard *we note that fixing* $g_{00}(x)$ *to unity everywhere isn't compatible with Lorentz covariance of* $g_{\mu\nu}(x)$*, but is compatible with Galilean covariance of* $g_{\mu\nu}(x)$. Weinberg's tome *fails to point out that fixing* $g_{00}(x)$ *to unity everywhere eliminates gravitational time dilation, despite its* page-80 Eq. (3.5.3) *being the same as* Eq. (1.5d) *above.*

It *is also usually asserted* that an energy-momentum source which is spatially maximally symmetric (spherically symmetric and homogeneous) *compels its associated metric to be of the Robertson-Walker form,*

$$(ds)^2 = (c\,dt)^2 - (R(t))^2\big[\big(1 - kr^2\big)^{-1}(dr)^2 + r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big)\big], \tag{1.6a}$$

which prima facie *is extremely implausible.* Since any metric of this Robertson-Walker form has $g_{00}(x)$ equal to unity everywhere, *it describes purely Newtonian gravity.* How can *purely Newtonian gravity* conceivably *be compelled* by the spherical symmetry and homogeneity of its energy-momentum source?

In fact, *at the beginning of* Subsection (C) of Section 13.5 on page 403 of his tome, Steven Weinberg *only obtains from the spatially maximal symmetry of the source the more general metric form,*

$$(ds)^2 = (T(t))^2(c\,dt)^2 - (S(t))^2\big[\big(1 - kr^2\big)^{-1}(dr)^2 + r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big)\big], \tag{1.6b}$$

which doesn't necessarily describe purely Newtonian gravity. But this Eq. (1.6b) metric form shown in Weinberg's tome *doesn't reflect the vast infinity of ways that its general coordinate transformations can differ from itself. One mustn't forget,* as Weinberg unaccountably did, *that every general coordinate transformation of a metric solution of an Einstein equation is also a metric solution of that equation.* Therefore the *only* assertion which can be made concerning the metric form given by Eq. (1.6b) with regard to metric solutions of Einstein equations whose energy-momentum source is spatially maximally symmetric is that *there exist metric solutions of Einstein equations whose energy-momentum source is spatially maximally symmetric which have the metric form given by* Eq. (1.6b), *but it absolutely cannot be asserted that all metric solutions of Einstein equations whose energy-momentum source is spatially maximally symmetric have the metric form given by* Eq. (1.6b). In fact, we next show that *there exists a general coordinate transformation of the metric form given by* Eq. (1.6b) *which satisfies Einstein's coordinate condition* $\det(g_{\mu\nu}(x)) = -1$.

To impose $\det(g_{\mu\nu}(x)) = -1$ *on a general coordinate transformation of the* Eq. (1.6b) *metric form,* we put $T(t)$ to $(S(t))^{-3}$ *and transform its radius variable from* r *to* ρ, which transforms that metric form to,

$$(ds)^2 = (S(t))^{-6}(c\,dt)^2 - (S(t))^2\big[\big(1 - k(r(\rho))^2\big)^{-1}(dr(\rho)/d\rho)^2(d\rho)^2 + (r(\rho)/\rho)^2\rho^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big)\big]. \tag{1.6c}$$

Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ will be satisfied by the Eq. (1.6c) metric form if the factor $\big(1 - k(r(\rho))^2\big)^{-1}(dr(\rho)/d\rho)^2$ in its second term is made equal to $(r(\rho)/\rho)^{-4}$, which reduces Eq. (1.6c) to,

$$(ds)^2 = (S(t))^{-6}(c\,dt)^2 - (S(t))^2\big[(r(\rho)/\rho)^{-4}(d\rho)^2 + (r(\rho)/\rho)^2\rho^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big)\big], \tag{1.6d}$$

which clearly satisfies Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$.

To solve the differential equation $\big(1 - k(r(\rho))^2\big)^{-1}(dr(\rho)/d\rho)^2 = (r(\rho)/\rho)^{-4}$, we note that equating the square roots of its two sides produces the differential equation $\big(1 - kr^2\big)^{-\frac{1}{2}}(dr/d\rho) = (\rho^2/r^2)$. This is equivalent to the integrable differential equality $\big(1 - kr^2\big)^{-\frac{1}{2}}\,r^2\,dr = \rho^2\,d\rho$, which, together with the specification $r(\rho = 0) = 0$, yields $\int_0^{r(\rho)}\big(1 - k(r')^2\big)^{-\frac{1}{2}}(r')^2\,dr' = \rho^3/3$.

We now *define the following* k*-indexed functions of* r: $V_k(r) \stackrel{\text{def}}{=} \int_0^r\big(1 - k(r')^2\big)^{-\frac{1}{2}}(r')^2\,dr'$. Since, as we see at the end of the foregoing paragraph, $V_k(r(\rho)) = \rho^3/3$, it follows that $r(\rho) = (V_k)^{-1}\big(\rho^3/3\big)$, where $(V_k)^{-1}$ *denotes the inverse function of the* k*-indexed function* V_k which is defined in the foregoing sentence. With *the solution* $r(\rho) = (V_k)^{-1}\big(\rho^3/3\big)$ *of the differential equation* $\big(1 - k(r(\rho))^2\big)^{-1}(dr(\rho)/d\rho)^2 = (r(\rho)/\rho)^{-4}$ *thus in hand,* we present the Eq. (1.6d) metric form as,

$$(ds)^2 = (S(t))^{-6}(c\,dt)^2 - (S(t))^2\big[(U_k(\rho))^{-4}(d\rho)^2 + (U_k(\rho))^2\rho^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big)\big], \tag{1.6e}$$

where $U_k(\rho) = (V_k)^{-1}\big(\rho^3/3\big)/\rho$ and $V_k(r) = \int_0^r\big(1 - k(r')^2\big)^{-\frac{1}{2}}(r')^2\,dr'$. The Eq. (1.6e) metric form *satisfies the Einstein coordinate condition* $\det(g_{\mu\nu}(x)) = -1$*, and for the appropriate function* $S(t)$ *it also satisfies the Einstein equation for a spatially maximally-symmetric source because it is a general coordinate transformation of the* Eq. (1.6b) *metric form which satisfies the Einstein equation for a spatially maximally-symmetric source.*

Inattention to the fact *that every general coordinate transformation of a metric solution of an Einstein equation is also a metric solution of that equation* has resulted in physically-inappropriate clinging *to the purely Newtonian* Eq. (1.6a) *Robertson-Walker metric form* that, in the throes of a Big Bang (or of gravitational collapse), sends subsets of its maximally-symmetric zero-pressure perfect-fluid source *to arbitrarily high speeds in gross violation of relativity's speed limit c. Relativistic upgrade* of $g_{00}(x) = 1$ purely Newtonian Friedmann/Lemaitre/Tolman/Robertson-Walker/Oppenheimer-Snyder metrics *is nearly a century overdue.*

It is important to realize that, *in the most general case,* Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *adds four further equations* to the six independent equations which comprise the Einstein equation. These four further equations *are a bit similar to the four harmonic coordinate equations* $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$ *favored by Weinberg, which have been briefly discussed in the next paragraph after* Eq. (1.3l). However the four equations *which immediately follow from* $\det(g_{\mu\nu}(x)) = -1$ are of course,

$$\partial \det(g_{\mu\nu}(x))/\partial x^{\lambda} = 0. \tag{1.7a}$$

But for any matrix M(x) *which shares the property of* $g_{\mu\nu}(x)$ *that* $\det(M(x)) < 0$ *everywhere,*

$$\partial \det(M(x))/\partial x^{\lambda} = \det(M(x))\,\mathrm{Tr}\big(M^{-1}(x)\big(\partial M(x)/\partial x^{\lambda}\big)\big). \tag{1.7b}$$

To obtain the Eq. (1.7b) *result,* we first *take note of the fact that,*

$$\partial \ln[-\det(M(x))]/\partial x^{\lambda} = \big(\partial(-\det(M(x)))/\partial x^{\lambda}\big)/(-\det(M(x))),\text{ so,}$$

$$\partial(\det(M(x)))/\partial x^{\lambda} = \det(M(x))\big(\partial \ln[-\det(M(x))]/\partial x^{\lambda}\big). \tag{1.7c}$$

With the Eq. (1.7c) *result in hand, we alternatively work out* $\partial \ln[-\det(M(x))]/\partial x^{\lambda}$ *as an* $\epsilon \to 0$ *limit,*

$$\partial \ln[-\det(M(x))]/\partial x^{\lambda} \overset{\epsilon\to 0}{=} (1/\epsilon)\big\{\ln\big[-\det\big(M(x) + \epsilon\big(\partial M(x)/\partial x^{\lambda}\big)\big)\big] - \ln[-\det(M(x))]\big\} =$$

$$(1/\epsilon)\ln\big[\det\big(M(x) + \epsilon\big(\partial M(x)/\partial x^{\lambda}\big)\big)\big/\det(M(x))\big] = (1/\epsilon)\ln\big[\det\big(I + \epsilon\, M^{-1}(x)\big(\partial M(x)/\partial x^{\lambda}\big)\big)\big] \overset{\epsilon\to 0}{=}$$

$$(1/\epsilon)\ln\big[1 + \epsilon\,\mathrm{Tr}\big(M^{-1}(x)\big(\partial M(x)/\partial x^{\lambda}\big)\big)\big] \overset{\epsilon\to 0}{=} \mathrm{Tr}\big(M^{-1}(x)\big(\partial M(x)/\partial x^{\lambda}\big)\big), \tag{1.7d}$$

which inserted into Eq. (1.7c) yields Eq. (1.7b). Putting $M(x)$ to $g_{\mu\nu}(x)$ in Eq. (1.7b) produces,

$$\partial \det(g_{\mu\nu}(x))/\partial x^{\lambda} = \det(g_{\mu\nu}(x))\big(g^{\kappa\alpha}(x)\big(\partial g_{\alpha\kappa}/\partial x^{\lambda}\big)\big). \tag{1.7e}$$

Eq. (1.3j) tells us that, $\Gamma^{\kappa}_{\mu\lambda} = \frac{1}{2}g^{\kappa\alpha}(x)\big(\big(\partial g_{\alpha\mu}/\partial x^{\lambda}\big) + \big(\partial g_{\alpha\lambda}/\partial x^{\mu}\big) - \big(\partial g_{\mu\lambda}/\partial x^{\alpha}\big)\big)$, which implies that,

$$\Gamma^{\kappa}_{\kappa\lambda} = \tfrac{1}{2}g^{\kappa\alpha}(x)\big(\big(\partial g_{\alpha\kappa}/\partial x^{\lambda}\big) + \big(\partial g_{\alpha\lambda}/\partial x^{\kappa}\big) - \big(\partial g_{\kappa\lambda}/\partial x^{\alpha}\big)\big) = \tfrac{1}{2}\big(g^{\kappa\alpha}(x)\big(\partial g_{\alpha\kappa}/\partial x^{\lambda}\big)\big), \tag{1.7f}$$

since $\frac{1}{2}g^{\kappa\alpha}(x)((\partial g_{\alpha\lambda}/\partial x^{\kappa}) - (\partial g_{\kappa\lambda}/\partial x^{\alpha})) = 0$ because $((\partial g_{\alpha\lambda}/\partial x^{\kappa}) - (\partial g_{\kappa\lambda}/\partial x^{\alpha}))$ is antisymmetric and $\frac{1}{2}g^{\kappa\alpha}(x)$ is symmetric under interchange of α and κ. Eqs. (1.7a), (1.7e) and (1.7f) together imply that,

$$0 = \partial \det(g_{\mu\nu}(x))/\partial x^{\lambda} = \det(g_{\mu\nu}(x))\big(g^{\kappa\alpha}(x)\big(\partial g_{\alpha\kappa}/\partial x^{\lambda}\big)\big) = 2\det(g_{\mu\nu}(x))\,\Gamma^{\kappa}_{\kappa\lambda}, \tag{1.7g}$$

so Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *implies the four equations,*

$$\Gamma^{\kappa}_{\kappa\lambda} = 0, \tag{1.7h}$$

which, as mentioned above Eq. (1.7a), *are a bit similar to the four harmonic coordinate equations* $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$.

We also noted above Eq. (1.7a) that, *in the most general case,* a bona fide coordinate condition *must add four further equations* to the six independent equations which comprise the Einstein equation *in order to determine the ten independent components of the metric tensor.* Both the four harmonic equations $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$ and the four equations $\Gamma^{\kappa}_{\kappa\lambda} = 0$ of Eq. (1.7h) that follow from Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ fulfill that requirement, and both are compatible with Lorentz covariance of the metric tensor $g_{\mu\nu}(x)$. However, as was noted in the next paragraph after Eq. (1.3l), Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *guarantees the existence of the matrix inverse* $g^{\mu\nu}(x)$ *of the metric tensor* $g_{\mu\nu}(x)$*, which isn't the case for the four harmonic equations* $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$*, notwithstanding that their very definition requires the existence of the metric's matrix inverse.* In Eqs. (1.3h) through (1.3l) we also saw that the existence of the metric's matrix inverse $g^{\mu\nu}(x)$ *is crucial to the existence of the affine connection* $\Gamma^{\kappa}_{\mu\nu}$, an entity which makes the geodesic gravitational equation of motion for a test particle *consistent with the relativistic extension of Newton's Second Law.* Moreover, we will shortly see that the affine connection $\Gamma^{\kappa}_{\mu\nu}$ *is a fundamental building block for the construction of equations which keep their form under general coordinate transformations.* In a nutshell, there would seem to be *no simple alternative to Einstein's coordinate condition* $\det(g_{\mu\nu}(x)) = -1$ *that fulfills two essential requirements of gravity theory:* 1) *four further Lorentz-covariant equations in the most general case* and 2) *guaranteed existence of the metric tensor's matrix inverse* $g^{\mu\nu}(x)$. Certainly *the simplest way to fulfill these two requirements is to assert that* $\det(g_{\mu\nu}(x)) = k$, where k is a nonzero constant; the fact that $k = -1$ is then determined *by the crucial special case* $g_{\mu\nu}(x) = \eta_{\mu\nu}$.

We *once again* note here that the condition $g_{00}(x) = 1$ is *incompatible* with Lorentz covariance of the metric tensor $g_{\mu\nu}(x)$, but *is* compatible with Galilean covariance of that tensor, and *that it totally eliminates gravitational time dilation.* Indeed the condition $g_{00}(x) = 1$ effectively sends c to infinity *and effectively reduces the Einstein equation to its Newtonian counterpart.* In the wake of A. Friedmann's opening of the $g_{00}(x) = 1$ Pandora's box in 1922, imposition of $g_{00}(x) = 1$ on metrics was pursued or strongly advocated over almost the next two decades by G. Lemaitre, R. C. Tolman, H. P. Robertson, A. G. Walker, J. R. Oppenheimer and H. Snyder without comprehension of the fundamental fact that imposing $g_{00}(x) = 1$ on the metric forces purely Newtonian gravity on the problem being treated—wholly *technical* matters such as *the Oppenheimer-Snyder use of the uniform density instead of the radius* of their zero-pressure, uniform-density spherical perfect-fluid energy-momentum source *as the time-dependent variable,* helped *to obscure the purely Newtonian nature of their results.* In the absence of a pointless desire to pursue purely Newtonian approximate gravitational calculations *in the unnecessarily-involved context of a multicomponent metric tensor,* setting $g_{00}(x)$ to unity *is to be shunned;* instead, setting $\det(g_{\mu\nu}(x))$ to -1, as Einstein did in November, 1915 to obtain the previously unexplained part of Mercury's perihelion shift, *is to be regarded as absolutely mandatory in relativistic gravity theory* because 1) it is *the simplest possible way to guarantee the existence of the metric tensor's matrix inverse* $g^{\mu\nu}(x)$ *and affine connection* $\Gamma^{\kappa}_{\mu\lambda}$, 2) *it is compatible with Lorentz covariance of the metric tensor* $g_{\mu\nu}$ and 3) in the most general case it provides the four further equations $\Gamma^{\kappa}_{\kappa\lambda} = 0$ which complement the six independent equations that comprise the Einstein equation to determine the ten components of the metric tensor $g_{\mu\nu}$. In the case that the energy-momentum source is spatially maximally symmetric, the Eq. (1.6e) metric form, which both satisfies Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ and as well the Einstein equation for the metric tensor of such a source, applies, with the proviso that its function $S(t)$ is determined by the details of each such spatially maximally-symmetric source; for example, if its energy-momentum source is a sphere of uniformly-dense perfect fluid of nonzero pressure, $S(t)$ depends on that perfect fluid's equation of state which relates its pressure to its density. The Eq. (1.6a) $g_{00}(x) = 1$ Robertson-Walker metric form is to be shunned unless one pointlessly desires to carry out a purely Newtonian approximate gravitational calculations *in the unnecessarily-involved context of a multicomponent metric tensor.* When the energy-momentum source is a uniformly-dense sphere of perfect fluid *of zero pressure,* application of the Birkhoff theorem offers a far simpler route to the solution than does head-on application of the Eq. (1.6e) metric form, *but it is absolutely crucial that the static point-mass metric used to apply the Birkhoff theorem be precisely the one which actually satisfies Einstein's coordinate condition* $\det(g_{\mu\nu}(x)) = -1$ *everywhere except at the location of the static point mass itself, namely that the static point-mass metric be the almost unknown one Schwarzschild himself published on January 13, 1916.*

We now present basics of the construction of equations which keep their form under general coordinate transformations. The interested reader can find a multitude of extremely worthwhile facts and derivations which are omitted here in chapters 4 through 7 and chapter 12 of Steven Weinberg's 1972 657-page tome Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.

General coordinate transformations map the four space-time coordinates x^μ one-to-one and onto four other space-time coordinates $y^\alpha(x)$, the mapping $y^\alpha(x)$ being multiply times continuously differentiable. Consequences include the existence of the inverse mapping $x^\mu(y)$ with the same properties, and, from the chain rule of the calculus, the familiar basic partial-derivative matrix identities,

$$\frac{\partial y^\alpha}{\partial x^\lambda}\frac{\partial x^\lambda}{\partial y^\beta} = \frac{\partial y^\alpha}{\partial y^\beta} = \delta^\alpha_\beta \quad \text{and} \quad \frac{\partial x^\mu}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu. \tag{1.8a}$$

Also from the calculus chain rule, general coordinate transformations from dx^μ to dy^α have the form,

$$dy^\alpha = \frac{\partial y^\alpha}{\partial x^\mu}dx^\mu, \tag{1.8b}$$

a *contravariant* (upper index) *vector transformation,* as is *its inverse transformation from* dy^α *to* dx^ν,

$$dx^\nu = \frac{\partial x^\nu}{\partial y^\alpha}dy^\alpha. \tag{1.8c}$$

To obtain Eq. (1.8c) we contract $\frac{\partial x^\nu}{\partial y^\alpha}$ into both sides of Eq. (1.8b) and then apply Eq. (1.8a),

$$\frac{\partial x^\nu}{\partial y^\alpha}dy^\alpha = \frac{\partial x^\nu}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\mu}dx^\mu = \delta^\nu_\mu\, dx^\mu = dx^\nu. \tag{1.8d}$$

Below Eq. (1.2b) we noted that the general coordinate transformation properties of the metric tensor $g_{\mu\nu}(x)$ *must be consistent with the invariance of* $d\tau(x) = \left(\sqrt{g_{\mu\nu}(x)\, dx^\mu\, dx^\nu}/c\right)$. In light of Eq. (1.8c),

$$(c\,d\tau(x))^2 = g_{\mu\nu}(x)\,dx^\mu\,dx^\nu = g_{\mu\nu}(x)\,\frac{\partial x^\mu}{\partial y^\alpha}dy^\alpha\frac{\partial x^\nu}{\partial y^\beta}dy^\beta = \left\{\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}g_{\mu\nu}(x)\right\}dy^\alpha\,dy^\beta. \tag{1.8e}$$

Therefore, *provided that the general coordinate transformation of the metric tensor is taken to be,*

$$g_{\alpha\beta}(y) = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}g_{\mu\nu}(x), \tag{1.8f}$$

then, from Eqs. (1.8f) *and* (1.8e),

$$(c\,d\tau(x))^2 = g_{\mu\nu}(x)\,dx^\mu\,dx^\nu = g_{\alpha\beta}(y)\,dy^\alpha\,dy^\beta = (c\,d\tau(y))^2, \tag{1.8g}$$

so $d\tau(x) = d\tau(y)$, i.e., *$d\tau$ is invariant under general coordinate transformations.* Eq. (1.8f) shows that the metric tensor transforms *as a covariant* (lower index) *second-rank tensor,* which is "upside down" from the Eq. (1.8b) *contravariant* (upper index) *transformation $dy^\alpha = \frac{\partial y^\alpha}{\partial x^\mu}dx^\mu$ of the vector dx^μ.*

If the metric tensor $g_{\mu\nu}(x)$ *has a matrix inverse* (which it will if it satisfies Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$), then a coordinate transformation of this metric tensor's matrix inverse *is equal to the matrix inverse* of this metric tensor's *corresponding coordinate transformation.* As a consequence, the matrix inverse of a metric tensor transforms *as a contravariant* (upper index) *second-rank tensor,* which is "right side up" relative to the Eq. (1.8b) *contravariant* (upper index) *transformation of the vector dx^μ.*

In detail, if a metric tensor $g_{\mu\nu}(x)$ *has the matrix inverse* $g^{\nu\lambda}(x)$ (which it will if it satisfies Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$), then $g_{\mu\nu}(x)g^{\nu\lambda}(x) = \delta^\lambda_\mu$, a relation whose coordinate transformation must correspondingly be $g_{\alpha\beta}(y)g^{\beta\gamma}(y) = \delta^\gamma_\alpha$. We next insert $g_{\alpha\beta}(y)$, as it is given by Eq. (1.8f) above, into the equation $g_{\alpha\beta}(y)g^{\beta\gamma}(y) = \delta^\gamma_\alpha$ to obtain,

$$\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}g_{\mu\nu}(x)g^{\beta\gamma}(y) = \delta^\gamma_\alpha. \tag{1.8h}$$

It is now more convenient to write Eq. (1.8h) as,

$$\frac{\partial x^\mu}{\partial y^\alpha}\,g_{\mu\nu}(x)\,\frac{\partial x^\nu}{\partial y^\beta}\,g^{\beta\gamma}(y) = \delta^\gamma_\alpha.$$

We contract $\frac{\partial y^\alpha}{\partial x^\lambda}$ into both sides of the above equation and apply Eq. (1.8a), which produces,

$$g_{\lambda\nu}(x)\,\frac{\partial x^\nu}{\partial y^\beta}\,g^{\beta\gamma}(y) = \frac{\partial y^\gamma}{\partial x^\lambda}.$$

We next contract $g^{\mu\lambda}(x)$ into both sides of the above equation and apply $g^{\mu\lambda}(x)g_{\lambda\nu}(x) = \delta^\mu_\nu$, which produces,

$$\frac{\partial x^\mu}{\partial y^\beta}\,g^{\beta\gamma}(y) = g^{\mu\lambda}(x)\frac{\partial y^\gamma}{\partial x^\lambda}.$$

We finally contract $\frac{\partial y^\alpha}{\partial x^\mu}$ into both sides of the above equation and apply Eq. (1.8a), which produces,

$$g^{\alpha\gamma}(y) = \frac{\partial y^\alpha}{\partial x^\mu}g^{\mu\lambda}(x)\frac{\partial y^\gamma}{\partial x^\lambda},$$

which it is now more convenient to write as,

$$g^{\alpha\beta}(y) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}g^{\mu\nu}(x). \tag{1.8i}$$

From Eqs. (1.8f) and (1.8i) we clearly see that under general coordinate transformations the metric tensor $g_{\mu\nu}(x)$ transforms *as a covariant second-rank tensor,* whereas the matrix inverse $g^{\mu\nu}(x)$ of the metric tensor transforms *as a contravariant second-rank tensor.* The *contraction* of the metric tensor $g_{\mu\nu}(x)$ with *a contravariant upper index* of any tensor *lowers that index to one which transforms covariantly under general coordinate transformations.* Likewise, the *contraction* of the matrix inverse $g^{\mu\nu}(x)$ of the metric tensor with *a covariant lower index* of any tensor *raises that index to one which transforms contravariantly.*

We have so far considered upper and lower indexed entities whose *upper* indices transform *contravariantly* under general coordinate transformations in the manner of dx^μ in Eq. (1.8b) and of $g^{\mu\nu}(x)$ in Eq. (1.8i)), and whose *lower* indices transform *covariantly* under those transformations in the manner of $g_{\mu\nu}(x)$ in Eq. (1.8f). Such upper and lower indexed entities are termed *tensors under general coordinate transformations,* or are simply termed *tensors* for short. However, *partial derivatives of almost all tensors under general coordinate transformations fail to transform as tensors under general coordinate transformations.* The upper and lower indexed *affine connection* $\Gamma^\kappa_{\mu\nu}(x)$ of Eq. (1.3j), which involves *partial derivatives* of the metric tensor, *fails to transform as a tensor,* but it is possible *to merge partial differentiation with the affine connection* in such a way *that the result of applying the merged operation to a tensor is itself a tensor.*

We next work out the general coordinate transformation of the non-tensor affine connection $\Gamma^{\kappa}_{\mu\nu}(x)$ of Eq. (1.3j), which is *a lengthy exercise.* We *begin* by working out the general coordinate transformation of $\partial g_{\mu\nu}(x)/\partial x^{\lambda}$ via the Eq. (1.8f) general coordinate transformation of the covariant tensor $g_{\mu\nu}(x)$,

$$\frac{\partial g_{\alpha\beta}(y)}{\partial y^{\gamma}} = \frac{\partial}{\partial y^{\gamma}}\Big[\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}g_{\mu\nu}(x)\Big] = \frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\frac{\partial x^{\lambda}}{\partial y^{\gamma}}\frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}} + g_{\mu\nu}(x)\left[\frac{\partial^2 x^{\mu}}{\partial y^{\gamma}\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}} + \frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial^2 x^{\nu}}{\partial y^{\gamma}\partial y^{\beta}}\right], \tag{1.8j}$$

which illustrates *how partial differentiation can turn a covariant tensor into a non-tensor.* If *only the first term* of the final expression in Eq. (1.8j) were present, $\partial g_{\mu\nu}(x)/\partial x^{\lambda}$ would be a covariant third-rank tensor. We next use Eq. (1.8j) *to combine terms* of the general coordinate transformation of the entity $(\partial g_{\lambda\mu}(x)/\partial x^{\nu}) + (\partial g_{\lambda\nu}(x)/\partial x^{\mu}) - (\partial g_{\mu\nu}(x)/\partial x^{\lambda})$,

$$\begin{aligned}
\frac{\partial g_{\gamma\alpha}(y)}{\partial y^{\beta}} + \frac{\partial g_{\gamma\beta}(y)}{\partial y^{\alpha}} - \frac{\partial g_{\alpha\beta}(y)}{\partial y^{\gamma}} &= \frac{\partial x^{\mu}}{\partial y^{\gamma}}\frac{\partial x^{\nu}}{\partial y^{\alpha}}\frac{\partial x^{\lambda}}{\partial y^{\beta}}\frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}} + g_{\mu\nu}(x)\left[\frac{\partial^2 x^{\mu}}{\partial y^{\beta}\partial y^{\gamma}}\frac{\partial x^{\nu}}{\partial y^{\alpha}} + \frac{\partial x^{\mu}}{\partial y^{\gamma}}\frac{\partial^2 x^{\nu}}{\partial y^{\beta}\partial y^{\alpha}}\right]\\
&+\frac{\partial x^{\mu}}{\partial y^{\gamma}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\frac{\partial x^{\lambda}}{\partial y^{\alpha}}\frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}} + g_{\mu\nu}(x)\left[\frac{\partial^2 x^{\mu}}{\partial y^{\alpha}\partial y^{\gamma}}\frac{\partial x^{\nu}}{\partial y^{\beta}} + \frac{\partial x^{\mu}}{\partial y^{\gamma}}\frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}}\right]\\
&-\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\frac{\partial x^{\lambda}}{\partial y^{\gamma}}\frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}} - g_{\mu\nu}(x)\left[\frac{\partial^2 x^{\mu}}{\partial y^{\gamma}\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}} + \frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial^2 x^{\nu}}{\partial y^{\gamma}\partial y^{\beta}}\right]\\
&= \frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\frac{\partial x^{\lambda}}{\partial y^{\gamma}}\left(\frac{\partial g_{\lambda\mu}(x)}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}(x)}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}}\right) + 2\, g_{\mu\nu}(x)\frac{\partial x^{\mu}}{\partial y^{\gamma}}\frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}}\\
&= \frac{\partial x^{\lambda}}{\partial y^{\gamma}}\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\left(\frac{\partial g_{\lambda\mu}(x)}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}(x)}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}}\right) + 2\,\frac{\partial x^{\mu}}{\partial y^{\gamma}}g_{\mu\nu}(x)\frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}}.
\end{aligned} \tag{1.8k}$$

We are now in a position to obtain the general coordinate transformation of the Eq. (1.3j) affine connection $\Gamma^{\kappa}_{\mu\nu}(x) = \frac{1}{2}g^{\kappa\lambda}(x)\big((\partial g_{\lambda\mu}(x)/\partial x^{\nu}) + (\partial g_{\lambda\nu}(x)/\partial x^{\mu}) - (\partial g_{\mu\nu}(x)/\partial x^{\lambda})\big)$ by combining the general coordinate transformation result for $(\partial g_{\lambda\mu}(x)/\partial x^{\nu}) + (\partial g_{\lambda\nu}(x)/\partial x^{\mu}) - (\partial g_{\mu\nu}(x)/\partial x^{\lambda})$ given by Eq. (1.8k) with that for $g^{\mu\nu}(x)$ given by Eq. (1.8i), which for this particular purpose it is much more convenient to restate as,

$$g^{\sigma\gamma}(y) = \frac{\partial y^{\sigma}}{\partial x^{\kappa}}g^{\kappa\upsilon}(x)\frac{\partial y^{\gamma}}{\partial x^{\upsilon}}. \tag{1.8l}$$

Combining Eq. (1.8l) with the Eq. (1.8k) result yields for the general coordinate transformation of $\Gamma^{\kappa}_{\mu\nu}(x)$,

$$\begin{gathered}
\Gamma^{\sigma}_{\alpha\beta}(y) = \tfrac{1}{2}g^{\sigma\gamma}(y)\left(\frac{\partial g_{\gamma\alpha}(y)}{\partial y^{\beta}} + \frac{\partial g_{\gamma\beta}(y)}{\partial y^{\alpha}} - \frac{\partial g_{\alpha\beta}(y)}{\partial y^{\gamma}}\right) =\\
\frac{1}{2}\frac{\partial y^{\sigma}}{\partial x^{\kappa}}g^{\kappa\upsilon}(x)\frac{\partial y^{\gamma}}{\partial x^{\upsilon}}\frac{\partial x^{\lambda}}{\partial y^{\gamma}}\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\left(\frac{\partial g_{\lambda\mu}(x)}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}(x)}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}}\right) + \frac{\partial y^{\sigma}}{\partial x^{\kappa}}g^{\kappa\upsilon}(x)\frac{\partial y^{\gamma}}{\partial x^{\upsilon}}\frac{\partial x^{\mu}}{\partial y^{\gamma}}g_{\mu\nu}(x)\frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}} =\\
\frac{1}{2}\frac{\partial y^{\sigma}}{\partial x^{\kappa}}g^{\kappa\lambda}(x)\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\left(\frac{\partial g_{\lambda\mu}(x)}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}(x)}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}}\right) + \frac{\partial y^{\sigma}}{\partial x^{\kappa}}g^{\kappa\mu}(x)g_{\mu\nu}(x)\frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}} =\\
\frac{\partial y^{\sigma}}{\partial x^{\kappa}}\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\frac{1}{2}g^{\kappa\lambda}(x)\left(\frac{\partial g_{\lambda\mu}(x)}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}(x)}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}(x)}{\partial x^{\lambda}}\right) + \frac{\partial y^{\sigma}}{\partial x^{\nu}}\frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}} =\\
\frac{\partial y^{\sigma}}{\partial x^{\kappa}}\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\Gamma^{\kappa}_{\mu\nu}(x) + \frac{\partial y^{\sigma}}{\partial x^{\nu}}\frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}},
\end{gathered} \tag{1.8m}$$

where we used $\frac{\partial y^{\gamma}}{\partial x^{\upsilon}}\frac{\partial x^{\lambda}}{\partial y^{\gamma}} = \delta^{\lambda}_{\upsilon}$ as well as $\frac{\partial y^{\gamma}}{\partial x^{\upsilon}}\frac{\partial x^{\mu}}{\partial y^{\gamma}} = \delta^{\mu}_{\upsilon}$, which are aspects of Eq. (1.8a), and we also used $g^{\kappa\mu}(x)g_{\mu\nu}(x) = \delta^{\kappa}_{\nu}$. If *only the first term* of the final expression in Eq. (1.8m) were present, the affine connection $\Gamma^{\kappa}_{\mu\nu}(x)$ would be a "mixed" contravariant/covariant third-rank tensor, *but the presence of the second term* $\frac{\partial y^{\sigma}}{\partial x^{\nu}}\frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}}$ *makes it a non-tensor.* A *slightly different* (but equivalent) form of the general coordinate transformation of the affine connection turns out to be important. Taking the partial derivative with respect to y^{β} of the identity $\frac{\partial y^{\sigma}}{\partial x^{\nu}}\frac{\partial x^{\nu}}{\partial y^{\alpha}} = \delta^{\sigma}_{\alpha}$ produces *the further identity,*

$$\frac{\partial y^{\sigma}}{\partial x^{\nu}}\frac{\partial^2 x^{\nu}}{\partial y^{\alpha}\partial y^{\beta}} = -\frac{\partial x^{\nu}}{\partial y^{\alpha}}\frac{\partial x^{\lambda}}{\partial y^{\beta}}\frac{\partial^2 y^{\sigma}}{\partial x^{\nu}\partial x^{\lambda}}, \tag{1.8n}$$

so *a form equivalent to* Eq. (1.8m) of the general coordinate transformation of the affine connection is,

$$\Gamma^{\sigma}_{\alpha\beta}(y) = \frac{\partial y^{\sigma}}{\partial x^{\kappa}}\frac{\partial x^{\mu}}{\partial y^{\alpha}}\frac{\partial x^{\nu}}{\partial y^{\beta}}\Gamma^{\kappa}_{\mu\nu}(x) - \frac{\partial x^{\nu}}{\partial y^{\alpha}}\frac{\partial x^{\lambda}}{\partial y^{\beta}}\frac{\partial^2 y^{\sigma}}{\partial x^{\nu}\partial x^{\lambda}}. \tag{1.8o}$$

Although the partial derivative $\frac{\partial V_{\mu}(x)}{\partial x^{\nu}}$ of *a covariant vector $V_{\mu}(x)$ doesn't transform as a covariant second-rank tensor,* we now show that that partial derivative *minus the affine connection's particular contraction $\Gamma^{\lambda}_{\mu\nu}(x)V_{\lambda}(x)$ with that covariant vector $V_{\mu}(x)$ does* transform as a covariant second-rank tensor. The general coordinate transformation of the covariant vector $V_{\mu}(x)$ of course is $V_{\alpha}(y) = \frac{\partial x^{\mu}}{\partial y^{\alpha}}V_{\mu}(x)$, so,

$$\frac{\partial V_\alpha(y)}{\partial y^\beta} = \frac{\partial}{\partial y^\beta}\left[\frac{\partial x^\mu}{\partial y^\alpha}V_\mu(x)\right] = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial V_\mu(x)}{\partial x^\nu} + \frac{\partial^2 x^\mu}{\partial y^\alpha \partial y^\beta}V_\mu(x) = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial V_\mu(x)}{\partial x^\nu} + \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta}V_\nu(x). \tag{1.8p}$$

The double-lower-index partial derivative $\frac{\partial V_\mu(x)}{\partial x^\nu}$ *fails to transform as a covariant second-rank tensor due to the bad term* $\frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta}V_\nu(x)$ *in* Eq. (1.8p). However we now use the Eq. (1.8m) transformation of the affine connection $\Gamma^\lambda_{\mu\nu}(x)$ to calculate *the transformation of the $\mu\nu$ double-lower-index contraction* $\Gamma^\lambda_{\mu\nu}(x)V_\lambda(x)$,

$$\Gamma^\gamma_{\alpha\beta}(y)V_\gamma(y) = \left[\frac{\partial y^\gamma}{\partial x^\lambda}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\Gamma^\lambda_{\mu\nu}(x) + \frac{\partial y^\gamma}{\partial x^\nu}\frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta}\right]\left[\frac{\partial x^\kappa}{\partial y^\gamma}V_\kappa(x)\right] =$$

$$\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\Gamma^\lambda_{\mu\nu}(x)\left[\frac{\partial y^\gamma}{\partial x^\lambda}\frac{\partial x^\kappa}{\partial y^\gamma}V_\kappa(x)\right] + \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta}\left[\frac{\partial y^\gamma}{\partial x^\nu}\frac{\partial x^\kappa}{\partial y^\gamma}V_\kappa(x)\right] = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\Gamma^\lambda_{\mu\nu}(x)V_\lambda(x) + \frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta}V_\nu(x). \tag{1.8q}$$

Upon equating *the difference of the initial expressions* in Eqs. (1.8p) and (1.8q) to *the difference of their final expressions, the bad terms* $\frac{\partial^2 x^\nu}{\partial y^\alpha \partial y^\beta}V_\nu(x)$ *in each of their final expressions cancel,* producing the result,

$$\frac{\partial V_\alpha(y)}{\partial y^\beta} - \Gamma^\gamma_{\alpha\beta}(y)V_\gamma(y) = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial V_\mu(x)}{\partial x^\nu} - \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\Gamma^\lambda_{\mu\nu}(x)V_\lambda(x) = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\left[\frac{\partial V_\mu(x)}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu}(x)V_\lambda(x)\right], \tag{1.8r}$$

which shows that the entity $V_{\mu;\nu}(x) \stackrel{\text{def}}{=} \left[\frac{\partial V_\mu(x)}{\partial x^\nu} - \Gamma^\lambda_{\mu\nu}(x)V_\lambda(x)\right]$ *transforms as a covariant second-rank tensor even though the partial derivative* $\frac{\partial V_\mu(x)}{\partial x^\nu}$ *doesn't.* The entity $V_{\mu;\nu}(x)$ is called *the covariant derivative with respect to x^ν of the covariant vector $V_\mu(x)$*, and, like ordinary differentiation, covariant differentiation of $V_\mu(x)$ is a homogeneously linear operation on $V_\mu(x)$. When the metric tensor $g_{\mu\nu}(x)$ reduces to $\eta_{\mu\nu}$, the affine connection $\Gamma^\lambda_{\mu\nu}(x)$ reduces to zero, and covariant differentiation reduces to ordinary partial differentiation. The extension of covariant differentiation to covariant tensors of higher rank is achieved by,

$$T_{\mu_1\cdots\mu_n;\nu}(x) \stackrel{\text{def}}{=} \left[\frac{\partial T_{\mu_1\cdots\mu_n}(x)}{\partial x^\nu} - \sum_{k=1}^n \Gamma^\lambda_{\mu_k\nu}(x)T_{\mu_1\cdots\mu_{k-1}\,\lambda\,\mu_{k+1}\cdots\mu_n}(x)\right],$$

as is readily verified by calculations that are highly analogous to those of Eqs. (1.8p) through (1.8r).

The general coordinate transformation *of a contravariant vector* $V^\mu(x)$ is $V^\alpha(y) = \frac{\partial y^\alpha}{\partial x^\mu}V^\mu(x)$, and its partial derivative $\frac{\partial V^\alpha(y)}{\partial y^\beta}$ *has mixed index,* so Eq. (1.8p) *for the $\alpha\beta$ double-lower-index* $\frac{\partial V_\alpha(y)}{\partial y^\beta}$ *is replaced by,*

$$\frac{\partial V^\alpha(y)}{\partial y^\beta} = \frac{\partial}{\partial y^\beta}\left[\frac{\partial y^\alpha}{\partial x^\mu}V^\mu(x)\right] = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial V^\mu(x)}{\partial x^\nu} + \frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\mu}\frac{\partial x^\nu}{\partial y^\beta}V^\mu(x) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial V^\mu(x)}{\partial x^\nu} + \frac{\partial x^\nu}{\partial y^\beta}\frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\lambda}V^\lambda(x).$$

$\frac{\partial V^\mu(y)}{\partial y^\nu}$ *fails to transform as a mixed tensor due to the bad term* $\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\lambda}V^\lambda(x)$, which has *three factors* instead of the *two factors* of the Eq. (1.8p) bad term. The *contraction* $\Gamma^\mu_{\nu\lambda}(x)V^\lambda(x)$ *has mixed index,* and the *three factors* of the *above bad term* suggests *using* Eq. (1.8o) *for* $\Gamma^\sigma_{\alpha\beta}(y)$ *in place of* Eq. (1.8m),

$$\Gamma^\alpha_{\beta\gamma}(y)V^\gamma(y) = \left[\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial x^\lambda}{\partial y^\gamma}\Gamma^\mu_{\nu\lambda}(x) - \frac{\partial x^\nu}{\partial y^\beta}\frac{\partial x^\lambda}{\partial y^\gamma}\frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\lambda}\right]\left[\frac{\partial y^\gamma}{\partial x^\kappa}V^\kappa(x)\right] =$$

$$\frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^\beta}\Gamma^\mu_{\nu\lambda}(x)\left[\frac{\partial x^\lambda}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial x^\kappa}V^\kappa(x)\right] - \frac{\partial x^\nu}{\partial y^\beta}\frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\lambda}\left[\frac{\partial x^\lambda}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial x^\kappa}V^\kappa(x)\right] = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^\beta}\Gamma^\mu_{\nu\lambda}(x)V^\lambda(x) - \frac{\partial x^\nu}{\partial y^\beta}\frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\lambda}V^\lambda(x).$$

Upon equating *the sum of the initial expressions* of the two foregoing equality chains to *the sum of their final expressions, the bad terms* $\pm\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial^2 y^\alpha}{\partial x^\nu \partial x^\lambda}V^\lambda(x)$ *of their final expressions cancel,* producing the result,

$$\frac{\partial V^\alpha(y)}{\partial y^\beta} + \Gamma^\alpha_{\beta\gamma}(y)V^\gamma(y) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial V^\mu(x)}{\partial x^\nu} + \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^\beta}\Gamma^\mu_{\nu\lambda}(x)V^\lambda(x) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial y^\beta}\left[\frac{\partial V^\mu(x)}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda}(x)V^\lambda(x)\right],$$

which shows that the entity $V^\mu{}_{;\nu}(x) \stackrel{\text{def}}{=} \left[\frac{\partial V^\mu(x)}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda}(x)V^\lambda(x)\right]$ *transforms as a mixed second-rank tensor even though the partial derivative* $\frac{\partial V^\mu(x)}{\partial x^\nu}$ *doesn't.* The entity $V^\mu{}_{;\nu}(x)$ is *the covariant derivative with respect to x^ν of the contravariant vector $V^\mu(x)$*. The extension of covariant differentiation to contravariant tensors of higher rank is achieved by,

$$T^{\mu_1\cdots\mu_n}{}_{;\nu}(x) \stackrel{\text{def}}{=} \left[\frac{\partial T^{\mu_1\cdots\mu_n}(x)}{\partial x^\nu} + \sum_{k=1}^n \Gamma^{\mu_k}_{\nu\lambda}(x)T^{\mu_1\cdots\mu_{k-1}\,\lambda\,\mu_{k+1}\cdots\mu_n}(x)\right],$$

as is readily verified by calculations that are highly analogous to those of the three preceding equality chains. The extension of covariant differentiation to *mixed tensors* is correspondingly achieved, e.g.,

$$T^{\mu_1}{}_{\mu_2\mu_3}{}^{\mu_4}{}_{;\nu}(x) \stackrel{\text{def}}{=} \frac{\partial T^{\mu_1}{}_{\mu_2\mu_3}{}^{\mu_4}(x)}{\partial x^\nu}$$

$$+\Gamma^{\mu_1}_{\nu\lambda}(x)T^\lambda{}_{\mu_2\mu_3}{}^{\mu_4}(x) - \Gamma^\lambda_{\mu_2\nu}(x)T^{\mu_1}{}_{\lambda\mu_3}{}^{\mu_4}(x) - \Gamma^\lambda_{\mu_3\nu}(x)T^{\mu_1}{}_{\mu_2\lambda}{}^{\mu_4}(x) + \Gamma^{\mu_4}_{\nu\lambda}(x)T^{\mu_1}{}_{\mu_2\mu_3}{}^\lambda(x).$$

We next study the general coordinate transformation behavior of the geodesic equation for a test particle moving under the influence of the metric tensor (multicomponent gravitational potential) $g_{\mu\nu}(x)$. The Eq. (1.3h) form of the geodesic equation,

$$g_{\lambda\mu}(x)\frac{d^2x^\mu}{d\tau^2}+\tfrac{1}{2}\left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu}+\frac{\partial g_{\lambda\nu}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0,$$

is of course *equivalent to the functional-derivative equation,*

$$\delta\int ds\Big/\delta x^\lambda(\tau)=0,$$

from which it was obtained, where,

$$ds=\sqrt{g_{\mu\nu}(x)\,dx^\mu\,dx^\nu}=\left(\sqrt{g_{\mu\nu}(x(\tau))\,\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\right)d\tau,$$

is clearly *invariant* under general coordinate transformations. Since $\delta x^\lambda(\tau)$ transforms as a *contravariant vector* under general coordinate transformations, the above Eq. (1.3h) representation of the geodesic equation transforms as *a covariant vector equation* under general coordinate transformations. If the matrix inverse $g^{\kappa\lambda}(x)$ of the metric tensor exists everywhere (which is ensured by Einstein's coordinate condition $\det(g_{\mu\nu})=-1$), contracting this *contravariant second rank tensor* $g^{\kappa\lambda}(x)$ into the above Eq. (1.3h) covariant vector form of the geodesic equation *produces its* Eq. (1.3i) *standard contravariant vector form,*

$$\frac{d^2x^\kappa}{d\tau^2}+\Gamma^\kappa_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0,$$

where,

$$\Gamma^\kappa_{\mu\nu}(x)\stackrel{\text{def}}{=}\tfrac{1}{2}g^{\kappa\lambda}(x)\left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu}+\frac{\partial g_{\lambda\nu}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right),$$

is the affine connection. (This Eq. (1.3i) *standard form of the geodesic equation* is *compatible* with the relativistic version of Newton's Second Law.) The interesting point is that *the characteristics of the variational principle $\delta\int ds/\delta x^\lambda(\tau)=0$ from which the geodesic equation was obtained permit us to immediately grasp its general coordinate transformation properties.* However, we now *as well* give the tedious *customary proof* that the above Eq. (1.3i) *standard form of the geodesic equation is a contravariant vector equation,*

$$\frac{d^2y^\sigma}{d\tau^2}=\frac{d}{d\tau}\left[\frac{\partial y^\sigma}{\partial x^\kappa}\frac{dx^\kappa}{d\tau}\right]=\frac{\partial y^\sigma}{\partial x^\kappa}\frac{d^2x^\kappa}{d\tau^2}+\frac{\partial^2y^\sigma}{\partial x^\kappa\partial x^\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau}=\frac{\partial y^\sigma}{\partial x^\kappa}\frac{d^2x^\kappa}{d\tau^2}+\frac{\partial^2y^\sigma}{\partial x^\nu\partial x^\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau},$$

so $\frac{d^2x^\kappa}{d\tau^2}$ *isn't a contravariant vector because of the bad term* $\frac{\partial^2y^\sigma}{\partial x^\nu\partial x^\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}$. We next apply Eq. (1.8o),

$$\Gamma^\sigma_{\alpha\beta}(y)\frac{dy^\alpha}{d\tau}\frac{dy^\beta}{d\tau}=\left[\frac{\partial y^\sigma}{\partial x^\kappa}\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta}\Gamma^\kappa_{\mu\nu}(x)-\frac{\partial x^\nu}{\partial y^\alpha}\frac{\partial x^\lambda}{\partial y^\beta}\frac{\partial^2y^\sigma}{\partial x^\nu\partial x^\lambda}\right]\left[\frac{\partial y^\alpha}{\partial x^\varsigma}\frac{dx^\varsigma}{d\tau}\right]\left[\frac{\partial y^\beta}{\partial x^\upsilon}\frac{dx^\upsilon}{d\tau}\right]=$$

$$\frac{\partial y^\sigma}{\partial x^\kappa}\Gamma^\kappa_{\mu\nu}(x)\left[\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\varsigma}\frac{dx^\varsigma}{d\tau}\right]\left[\frac{\partial x^\nu}{\partial y^\beta}\frac{\partial y^\beta}{\partial x^\upsilon}\frac{dx^\upsilon}{d\tau}\right]-\frac{\partial^2y^\sigma}{\partial x^\nu\partial x^\lambda}\left[\frac{\partial x^\nu}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\varsigma}\frac{dx^\varsigma}{d\tau}\right]\left[\frac{\partial x^\lambda}{\partial y^\beta}\frac{\partial y^\beta}{\partial x^\upsilon}\frac{dx^\upsilon}{d\tau}\right]=$$

$$\frac{\partial y^\sigma}{\partial x^\kappa}\Gamma^\kappa_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}-\frac{\partial^2y^\sigma}{\partial x^\nu\partial x^\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}.$$

Upon equating *the sum of the initial expressions* of the two foregoing equality chains to *the sum of their final expressions, the bad terms* $\pm\frac{\partial^2y^\sigma}{\partial x^\nu\partial x^\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}$ *of their final expressions cancel,* producing the result,

$$\frac{d^2y^\sigma}{d\tau^2}+\Gamma^\sigma_{\alpha\beta}(y)\frac{dy^\alpha}{d\tau}\frac{dy^\beta}{d\tau}=\frac{\partial y^\sigma}{\partial x^\kappa}\frac{d^2x^\kappa}{d\tau^2}+\frac{\partial y^\sigma}{\partial x^\kappa}\Gamma^\kappa_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=\frac{\partial y^\sigma}{\partial x^\kappa}\left[\frac{d^2x^\kappa}{d\tau^2}+\Gamma^\kappa_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right],$$

so the above Eq. (1.3i) standard form of the geodesic equation *indeed transforms as a contravariant vector equation.* The Eq. (1.3i) geodesic equation *therefore keeps its form under general coordinate transformations.* That is true *as well* of the Einstein equation for the metric tensor $g_{\mu\nu}(x)$, *and the resulting arbitrariness in the four-vector argument x^λ of $g_{\mu\nu}(x)$, and therefore in $g_{\mu\nu}(x)$ itself, makes choosing the physically appropriate coordinate condition of crucial importance.* Just as the geodesic equation follows from the variation with respect to $\delta x^\lambda(\tau)$ of an action *which is invariant under general coordinate transformations,* the Einstein equation follows from the variation with respect to $\delta g_{\mu\nu}(x)$ of the Einstein-Hilbert action, *which is likewise invariant under general coordinate transformations* (see chapter 12 of Steven Weinberg's 657-page 1972 tome Gravitation and Cosmology . . .). The *invariant curvature scalar* is the *central purely gravitational part* of that invariant action. We next discuss *curvature's relation to successive covariant differentiations.*

If, on the earth's *curved surface,* one *starts* at the equator and *first* travels a short distance *directly toward the north pole, followed by* traveling the same short distance *directly east,* one ends up at *a slightly different point* than if one *first* travels that same distance *directly east, followed by* traveling that distance *directly toward the north pole.* Somewhat similarly, taking *two* covariant derivatives *in succession* of a tensor gives a result *that depends on the order in which the two covariant derivatives are taken.* The *difference* between *a second covariant derivative of a tensor* and *its reversed-order counterpart* highlights a combination of the affine connection and its first partial derivatives *that describes a metric tensor's intrinsic curvature,* and is called the metric tensor's *Riemann-Christoffel curvature tensor.* In detail,

$$V^\kappa{}_{;\mu;\nu} = \frac{\partial V^\kappa{}_{;\mu}}{\partial x^\nu} + \Gamma^\kappa_{\nu\sigma} V^\sigma{}_{;\mu} - \Gamma^\sigma_{\mu\nu} V^\kappa{}_{;\sigma} = \frac{\partial}{\partial x^\nu}\left[\frac{\partial V^\kappa}{\partial x^\mu} + \Gamma^\kappa_{\mu\lambda} V^\lambda\right] + \Gamma^\kappa_{\nu\sigma}\left[\frac{\partial V^\sigma}{\partial x^\mu} + \Gamma^\sigma_{\mu\lambda} V^\lambda\right] - \Gamma^\sigma_{\mu\nu} V^\kappa{}_{;\sigma} =$$

$$\left\{\frac{\partial^2 V^\kappa}{\partial x^\mu \partial x^\nu} + \left[\Gamma^\kappa_{\mu\sigma}\frac{\partial V^\sigma}{\partial x^\nu} + \Gamma^\kappa_{\nu\sigma}\frac{\partial V^\sigma}{\partial x^\mu}\right] - \Gamma^\sigma_{\mu\nu} V^\kappa{}_{;\sigma}\right\} + V^\lambda\left[\frac{\partial \Gamma^\kappa_{\lambda\mu}}{\partial x^\nu} + \Gamma^\sigma_{\lambda\mu}\Gamma^\kappa_{\nu\sigma}\right],$$

where we *renamed one summed-over pair of dummy indices from λ to σ,* and we *interchanged certain of the symmetric lower indices of affine connection symbols.* Since the entity enclosed in curly brackets *is symmetric under the interchange of its μ and ν indices,* we see that,

$$V^\kappa{}_{;\mu;\nu}(x) - V^\kappa{}_{;\nu;\mu}(x) = V^\lambda(x) R^\kappa{}_{\lambda\mu\nu}(x),$$

where,

$$R^\kappa{}_{\lambda\mu\nu}(x) \stackrel{\text{def}}{=} \frac{\partial \Gamma^\kappa_{\lambda\mu}(x)}{\partial x^\nu} - \frac{\partial \Gamma^\kappa_{\lambda\nu}(x)}{\partial x^\mu} + \Gamma^\sigma_{\lambda\mu}(x)\Gamma^\kappa_{\nu\sigma}(x) - \Gamma^\sigma_{\lambda\nu}(x)\Gamma^\kappa_{\mu\sigma}(x), \tag{1.8s}$$

is called the metric tensor's *Riemann-Christoffel curvature tensor.* Two key *contractions* are the *Ricci tensor,*

$$R_{\lambda\nu}(x) \stackrel{\text{def}}{=} R^\kappa{}_{\lambda\kappa\nu}(x),$$

and the invariant *curvature scalar,*

$$R(x) \stackrel{\text{def}}{=} g^{\lambda\nu}(x) R_{\lambda\nu}(x).$$

In Einstein's gravity theory, *the source of gravitational curvature is energy-momentum,* which supersedes the Newtonian concept that the source of gravitational forces and potentials is mass; a key motivating consideration is that mass isn't conserved, but energy-momentum is. The density and flux of a physical system's energy-momentum is given by *its energy-momentum tensor $T_{\mu\nu}(x)$, which transforms as a tensor* under general coordinate transformations. It is *symmetric in its two indices,*

$$T_{\mu\nu}(x) = T_{\nu\mu}(x),$$

and *its covariant divergence vanishes,*

$$T^\mu{}_{\nu;\mu}(x) = 0,$$

where, as usual, indices are raised using the contravariant matrix inverse $g^{\mu\nu}(x)$ of the metric tensor, and lowered using the covariant metric tensor $g_{\mu\nu}(x)$ itself; in particular, $T^\mu{}_\nu(x) = g^{\mu\lambda}(x) T_{\lambda\nu}(x)$. In Sections 12.2 and 12.3, on pages 360-363, Steven Weinberg's tome gives a general definition of a physical system's energy-momentum tensor in terms of its action integral, together with a proof of the above two properties of that tensor. To arrive at the definition of $T^{\mu\nu}(x)$, a physical system's Lorentz-invariant action integral is first converted to one that is invariant under general coordinate transformations by replacing occurrences of $\eta_{\mu\nu}$ by $g_{\mu\nu}(x)$, occurrences of partial derivatives by covariant derivatives, and the occurrence of d^4x by $\sqrt{-\det(g_{\mu\nu}(x))}\, d^4x$. Taking half of the functional derivative with respect to the metric tensor $g_{\mu\nu}(x)$ of that upgraded action integral then yields the energy-momentum tensor. In Section 12.4 on page 364 in Eq. (12.4.2), Weinberg then goes on to give the simple gravitational scalar-curvature based action integral,

$$I_G = -\frac{c^3}{16\pi G}\int R(x)\sqrt{-\det(g_{\mu\nu}(x))}\, d^4x,$$

which *when added* to the type of upgraded action integral for a physical system described above yields the Einstein equation upon first-order variation *of their sum* with respect to the metric tensor $g_{\mu\nu}(x)$.

We now present *another approach* to deriving the Einstein equation, which is set out in Section 7.1 on pages 151-154 of Steven Weinberg's tome. The idea is that in principle we can always make a general space-time coordinate transformation such that *the local effect of any gravitational field becomes extremely* *weak,* which is known as going into free fall in that local gravitational field. The earth is freely falling in its orbit around the sun, *so we have almost no sense of the the sun's gravitational field,* but *the curvature* of the sun's gravitational field *persists; no general space-time coordinate transformation can make the gravitational curvature tensor vanish.* The *curvature* of the sun's gravitational field indeed *exerts a seasonal effect on the earth's ocean tides.* Since the Einstein equation relates *only the curvature* of the gravitational field to *its energy-momentum source,* we are perfectly able *to derive the Einstein equation while simultaneously assuming that gravitational effects are extremely weak,* which is *exactly* what we will now do. That we *can do this* is a pointed reminder of how *useless* the Einstein equation, which *only determines the metric's "Einstein curvature"* for a given energy-momentum tensor, *is by itself for fully determining that metric,* and therefore how *crucial* it is that the Einstein equation's *accompanying coordinate condition be physically appropriate.*

To implement the assumption that the gravitational field produced by the energy-momentum tensor $T_{\mu\nu}(x)$ is extremely weak, we present its metric tensor $g_{\mu\nu}(x)$ as $\eta_{\mu\nu} + h_{\mu\nu}(x)$, where $|h_{\mu\nu}(x)| \ll 1$. Thus the Eq. (1.3i) affine connection $\Gamma^{\kappa}_{\mu\nu}(x)$ becomes,

$$\Gamma^{\kappa}_{\mu\nu}(x) = \tfrac{1}{2} g^{\kappa\lambda}(x)\left(\frac{\partial g_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\right) = \tfrac{1}{2} g^{\kappa\lambda}(x)\left(\frac{\partial h_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial h_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial h_{\mu\nu}}{\partial x^{\lambda}}\right)$$

$$\approx \tfrac{1}{2}\eta^{\kappa\lambda}\left(\frac{\partial h_{\lambda\mu}}{\partial x^{\nu}} + \frac{\partial h_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial h_{\mu\nu}}{\partial x^{\lambda}}\right) = \tfrac{1}{2}\left(\frac{\partial h^{\kappa}{}_{\mu}}{\partial x^{\nu}} + \frac{\partial h^{\kappa}{}_{\nu}}{\partial x^{\mu}} - \frac{\partial h_{\mu\nu}}{\partial x_{\kappa}}\right),$$

where in the final step we used $\eta^{\kappa\lambda} = \eta_{\kappa\lambda}$ to raise and lower indices. Therefore from Eq. (1.8s),

$$R^{\kappa}{}_{\mu\lambda\nu}(x) \approx \frac{\partial\Gamma^{\kappa}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial\Gamma^{\kappa}_{\mu\nu}}{\partial x^{\lambda}} \approx \frac{1}{2}\frac{\partial}{\partial x^{\nu}}\left(\frac{\partial h^{\kappa}{}_{\mu}}{\partial x^{\lambda}} + \frac{\partial h^{\kappa}{}_{\lambda}}{\partial x^{\mu}} - \frac{\partial h_{\mu\lambda}}{\partial x_{\kappa}}\right) - \frac{1}{2}\frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial h^{\kappa}{}_{\mu}}{\partial x^{\nu}} + \frac{\partial h^{\kappa}{}_{\nu}}{\partial x^{\mu}} - \frac{\partial h_{\mu\nu}}{\partial x_{\kappa}}\right) =$$

$$\frac{1}{2}\left[\frac{\partial^2 h^{\kappa}{}_{\lambda}}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial^2 h_{\mu\lambda}}{\partial x^{\nu}\partial x_{\kappa}} - \frac{\partial^2 h^{\kappa}{}_{\nu}}{\partial x^{\lambda}\partial x^{\mu}} + \frac{\partial^2 h_{\mu\nu}}{\partial x^{\lambda}\partial x_{\kappa}}\right],$$

the weak-field Riemann-Christoffel curvature tensor. Therefore the weak-field Ricci tensor is,

$$R_{\mu\nu}(x) = R^{\lambda}{}_{\mu\lambda\nu}(x) \approx \frac{1}{2}\left[\frac{\partial^2 h^{\lambda}{}_{\lambda}}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial^2 h_{\mu\lambda}}{\partial x^{\nu}\partial x_{\lambda}} - \frac{\partial^2 h^{\lambda}{}_{\nu}}{\partial x^{\lambda}\partial x^{\mu}} + \frac{\partial^2 h_{\mu\nu}}{\partial x^{\lambda}\partial x_{\lambda}}\right] = \frac{1}{2}\left[\frac{\partial^2 h^{\lambda}{}_{\lambda}}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial^2 h_{\mu\lambda}}{\partial x^{\nu}\partial x_{\lambda}} - \frac{\partial^2 h_{\nu\lambda}}{\partial x^{\mu}\partial x_{\lambda}} + \frac{\partial^2 h_{\mu\nu}}{\partial x^{\lambda}\partial x_{\lambda}}\right],$$

whose the third term is reexpressed to show that $R_{\mu\nu} = R_{\nu\mu}$. Therefore the weak-field *curvature scalar* is,

$$R(x) \approx \eta^{\mu\nu} R_{\mu\nu}(x) \approx \left[\frac{\partial^2 h^{\lambda}{}_{\lambda}}{\partial x^{\mu}\partial x_{\mu}} - \frac{\partial^2 h_{\mu\lambda}}{\partial x_{\mu}\partial x_{\lambda}}\right].$$

In the weak-field limit, the energy-momentum tensor's vanishing covariant divergence condition $T^{\mu}{}_{\nu;\mu}(x) = 0$ is of course replaced by the vanishing of its ordinary divergence, $\frac{\partial T^{\mu}{}_{\nu}(x)}{\partial x^{\mu}} = 0$. Therefore to create a self-consistent Einstein equation in the weak-field limit, we need a second-rank curvature-related tensor $E_{\mu\nu}(x)$ that is symmetric in its two indices, $E_{\mu\nu}(x) = E_{\nu\mu}(x)$, and whose divergence vanishes, $\frac{\partial E^{\mu}{}_{\nu}(x)}{\partial x^{\mu}} = 0$. In terms of such a curvature-related tensor $E_{\mu\nu}(x)$, the self-consistent Einstein equation would have the form $E_{\mu\nu}(x) = KT_{\mu\nu}(x)$, where the constant K is determined by Newtonian gravity when this equation's energy-momentum source $T_{\mu\nu}(x)$ is a weak static energy density. In looking for such a curvature-related $E_{\mu\nu}(x)$, we next calculate the weak-field Ricci tensor's divergence,

$$\frac{\partial R^{\mu}{}_{\nu}(x)}{\partial x^{\mu}} = \frac{1}{2}\left[\frac{\partial^3 h^{\lambda}{}_{\lambda}}{\partial x^{\nu}\partial x^{\mu}\partial x_{\mu}} - \frac{\partial^3 h^{\mu}{}_{\lambda}}{\partial x^{\nu}\partial x^{\mu}\partial x_{\lambda}} - \frac{\partial^3 h_{\nu\lambda}}{\partial x^{\mu}\partial x_{\mu}\partial x_{\lambda}} + \frac{\partial^3 h^{\mu}{}_{\nu}}{\partial x^{\mu}\partial x^{\lambda}\partial x_{\lambda}}\right] =$$

$$\frac{1}{2}\frac{\partial}{\partial x^{\nu}}\left[\frac{\partial^2 h^{\lambda}{}_{\lambda}}{\partial x^{\mu}\partial x_{\mu}} - \frac{\partial^2 h^{\mu}{}_{\lambda}}{\partial x^{\mu}\partial x_{\lambda}}\right] - \frac{1}{2}\left[\frac{\partial^3 h_{\nu\lambda}}{\partial x^{\mu}\partial x_{\mu}\partial x_{\lambda}} - \frac{\partial^3 h_{\mu\nu}}{\partial x_{\mu}\partial x^{\lambda}\partial x_{\lambda}}\right] = \frac{1}{2}\frac{\partial R(x)}{\partial x^{\nu}} - \frac{1}{2}\left[\frac{\partial^3 h_{\lambda\nu}}{\partial x_{\lambda}\partial x^{\mu}\partial x_{\mu}} - \frac{\partial^3 h_{\mu\nu}}{\partial x_{\mu}\partial x^{\lambda}\partial x_{\lambda}}\right] =$$

$$\frac{1}{2}\frac{\partial(\delta^{\mu}_{\nu}R(x))}{\partial x^{\mu}} - \frac{1}{2}\left[\frac{\partial^3 h_{\mu\nu}}{\partial x_{\mu}\partial x^{\lambda}\partial x_{\lambda}} - \frac{\partial^3 h_{\mu\nu}}{\partial x_{\mu}\partial x^{\lambda}\partial x_{\lambda}}\right] = \frac{\partial(\frac{1}{2}\delta^{\mu}_{\nu}R(x))}{\partial x^{\mu}},$$

a result which implies that,

$$\partial\left(R^{\mu}{}_{\nu}(x) - \tfrac{1}{2}\delta^{\mu}_{\nu}R(x)\right)/\partial x^{\mu} = 0,$$

whose strong-field counterpart clearly is,

$$\left(R^{\mu}{}_{\nu}(x) - \tfrac{1}{2}\delta^{\mu}_{\nu}R(x)\right)_{;\mu} = 0,$$

which is *a Bianchi identity.* Therefore we have found the curvature-related tensor that we are looking for,

$$E_{\mu\nu}(x) = R_{\mu\nu}(x) - \tfrac{1}{2}g_{\mu\nu}(x)R(x),$$

so the Einstein equation that we are looking for has the form,

$$R_{\mu\nu}(x) - \tfrac{1}{2}g_{\mu\nu}(x)R(x) = KT_{\mu\nu}(x).$$

We next obtain the value of K from this equation's Newtonian weak-field and static energy-density case.

To do so, we assume that the *only* component of the energy-momentum tensor $T_{\mu\nu}(x)$ *which doesn't vanish* is the energy density $T_{00}(\mathbf{x})$, *and that it has no time dependence,* which of course is *necessary* because *the divergence* $\partial T^{\mu}{}_{\nu}(x)/\partial x^{\mu}$ *must vanish.* We also assume that $T_{00}(\mathbf{x})$ *is weak enough to be compatible with the assumed weak-field condition* $|h_{\mu\nu}(\mathbf{x})| \ll 1$, and that the $h_{\mu\nu}(\mathbf{x})$ *also have no time dependence.* These conditions *effectively enforce Newtonian gravitational physics,* and we know that the Newtonian gravitational potential $\phi(\mathbf{x})$ satisfies the Newtonian gravitational-potential equation,

$$\nabla^2_{\mathbf{x}}\,\phi(\mathbf{x}) = 4\pi G\rho(\mathbf{x}).$$

Since *under these conditions* $h_{00}(\mathbf{x}) = 2\,\phi(\mathbf{x})/c^2$ (see Eq. (1.4j)), and the energy density $T_{00}(\mathbf{x})$ is effectively the Newtonian mass density $\rho(\mathbf{x})$ times c^2, the above Newtonian potential equation can *also* be written,

$$\nabla^2_{\mathbf{x}}\,h_{00}(\mathbf{x}) = (8\pi G/c^4)T_{00}(\mathbf{x}). \tag{1.8t}$$

These Newtonian conditions cause the above Einstein-equation form $R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) = KT_{\mu\nu}(x)$ to imply a relation *very similar to* Eq. (1.8t), which enables evaluation of the constant K. *Under these conditions,* this Einstein-equation form *becomes* $R_{\mu\nu}(\mathbf{x}) - \frac{1}{2}\eta_{\mu\nu}R(\mathbf{x}) = KT_{\mu\nu}(\mathbf{x})$. Only *four* of the components of this Einstein-equation form are needed to obtain the relation similar to Eq. (1.8t); they are,

$$R_{00}(\mathbf{x}) - \tfrac{1}{2}R(\mathbf{x}) = KT_{00}(\mathbf{x}), \tag{1.8u}$$

and,

$$R_{jj}(\mathbf{x}) + \tfrac{1}{2}R(\mathbf{x}) = 0 \ \text{ for } \ j = 1, 2 \text{ and } 3, \tag{1.8v}$$

but we *also* need *the relations of* $R_{00}(\mathbf{x})$*,* $R_{jj}(\mathbf{x})$ *and* $R(\mathbf{x})$ *to* $h_{00}(\mathbf{x})$. Since $R(\mathbf{x}) = \eta^{\mu\nu}R_{\mu\nu}(\mathbf{x}) = R_{00}(\mathbf{x}) - \sum_{j=1}^{3} R_{jj}(\mathbf{x})$, Eq. (1.8v) yields that $R(\mathbf{x}) = R_{00}(\mathbf{x}) + \frac{3}{2}R(\mathbf{x})$, so $R(\mathbf{x}) = -2R_{00}(\mathbf{x})$, which inserted into Eq. (1.8u) yields that $2R_{00}(\mathbf{x}) = KT_{00}(\mathbf{x})$. Next we use the weak-field version of the Ricci tensor,

$$R_{\mu\nu}(x) = \tfrac{1}{2}\left[\frac{\partial^2 h^{\lambda}{}_{\lambda}}{\partial x^{\nu}\partial x^{\mu}} - \frac{\partial^2 h_{\mu\lambda}}{\partial x^{\nu}\partial x_{\lambda}} - \frac{\partial^2 h_{\nu\lambda}}{\partial x^{\mu}\partial x_{\lambda}} + \frac{\partial^2 h_{\mu\nu}}{\partial x^{\lambda}\partial x_{\lambda}}\right],$$

in conjunction with the fact that $h_{\alpha\beta}(\mathbf{x})$ *is independent of time* to obtain *the result* $R_{00}(\mathbf{x}) = -\frac{1}{2}\nabla^2_{\mathbf{x}}\,h_{00}(\mathbf{x})$, that, on being inserted into the above result that $2R_{00}(\mathbf{x}) = KT_{00}(\mathbf{x})$, yields $-\nabla^2_{\mathbf{x}}\,h_{00}(\mathbf{x}) = KT_{00}(\mathbf{x})$, which on comparison with Eq. (1.8t) yields that $K = -(8\pi G/c^4)$. Insertion of this value of K into the above Einstein-equation form yields *the Einstein equation,*

$$R_{\mu\nu}(x) - \tfrac{1}{2}g_{\mu\nu}(x)R(x) = -(8\pi G/c^4)\,T_{\mu\nu}(x). \tag{1.8w}$$

There is *far less* to the Einstein equation than meets the eye because the Einstein curvature-related tensor $E_{\mu\nu}(x) = R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x)$ *doesn't determine the metric tensor* $g_{\mu\nu}(x)$; the Einstein equation *doesn't even determine a test particle's trajectory.* In fact, *with the additional requirement that* $g_{00}(x) = 1$ *everywhere,* which is fully compatible with the Einstein equation and was stipulated with delight by A. Friedmann, G. Lemaitre, R. C. Tolman, H. P. Robertson, A. G. Walker, J. R. Oppenheimer and H. Snyder, but which *is incompatible with Lorentz covariance of* $g_{\mu\nu}(x)$ (it is fully compatible with Galilean covariance of $g_{\mu\nu}(x)$) *and totally eliminates gravitational time dilation, sending c to infinity, the Einstein equation yields purely Newtonian gravitational physics, including purely Newtonian-gravitational test-particle trajectories.* The *pointlessness* of confecting *purely Newtonian gravitational physics from the Einstein equation and the Lorentz-covariance incompatible requirement that* $g_{00}(x) = 1$ *everywhere* is obvious. It is *equally* obvious that *the physical appropriateness* of the coordinate condition which accompanies the Einstein equation *is of crucial importance.* Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *is the simplest possible coordinate condition which* (1) *guarantees the existence of the inverse* $g^{\alpha\beta}(x)$ *of the metric tensor* $g_{\mu\nu}(x)$, and thereby *guarantees the existence of the affine connection,* (2) *is compatible with Lorentz covariance of the metric tensor* $g_{\mu\nu}(x)$ and (3) in the general case *supplies four additional equations* (i.e., the four equations $\Gamma^{\kappa}_{\kappa\lambda} = 0$ of Eq. (1.7h)) that *supplement* the six independent equations supplied by the Einstein equation *to fully determine the metric tensor.* These *three properties* of Einstein's coordinate condition are *unmatched* by *any* coordinate condition found in Weinberg's tome; Weinberg *favors* the *harmonic coordinate condition* $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$, which *uses* $g^{\alpha\beta}(x)$*, but fails to guarantee its existence.*

Einstein's coordinate condition *also* remarkably *reverts the form of the electrodynamics equation,*

$$F^{\mu\nu}{}_{;\mu} = 4\pi j^\nu/c, \tag{1.9a}$$

in the presence of gravitation to its form *in the absence of gravitation.* We verify this by writing out in detail the covariant divergence of $F^{\mu\nu}$, bearing in mind its antisymmetry, $F^{\mu\nu} = -F^{\nu\mu}$,

$$F^{\mu\nu}{}_{;\mu} = \frac{\partial F^{\mu\nu}}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda}F^{\lambda\nu} + \Gamma^\nu_{\mu\lambda}F^{\mu\lambda}, \tag{1.9b}$$

where, *from Einstein's coordinate condition,* $\Gamma^\mu_{\mu\lambda} = 0$ (see Eq. (1.7h)). In addition, $\Gamma^\nu_{\mu\lambda}F^{\mu\lambda} = 0$ because $\Gamma^\nu_{\mu\lambda}$ is *symmetric* under the interchange of the indices μ and λ, whereas $F^{\mu\lambda}$ is *antisymmetric* under that index interchange. Thus Eq. (1.9a) becomes,

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu} = 4\pi j^\nu/c, \tag{1.9c}$$

the form this electrodynamics equation *has in the absence of gravitation.* Einstein's coordinate condition likewise *reverts the form of the vanishing of the current density's divergence* j^ν,

$$j^\nu{}_{;\nu} = 0, \tag{1.9d}$$

in the presence of gravitation to its form *in the absence of gravitation.* The covariant divergence of j^ν is,

$$j^\nu{}_{;\nu} = \frac{\partial j^\nu}{\partial x^\nu} + \Gamma^\nu_{\nu\lambda}j^\lambda, \tag{1.9e}$$

where, *from Einstein's coordinate condition,* $\Gamma^\nu_{\nu\lambda} = 0$ (see Eq. (1.7h)). Thus Eq. (1.9d) becomes,

$$\frac{\partial j^\nu}{\partial x^\nu} = 0, \tag{1.9f}$$

the form this equation of the vanishing of the current density's divergence *has in the absence of gravitation.*

The remaining *cyclic* electrodynamics equation, $F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0$, *automatically has the same form in the presence of gravitation as in gravitation's absence* because $\Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$ and $F_{\lambda\gamma} = -F_{\gamma\lambda}$,

$$\begin{gathered} F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = \\ \left[\frac{\partial F_{\alpha\beta}}{\partial x^\gamma} - \Gamma^\lambda_{\alpha\gamma}F_{\lambda\beta} - \Gamma^\lambda_{\beta\gamma}F_{\alpha\lambda}\right] + \left[\frac{\partial F_{\beta\gamma}}{\partial x^\alpha} - \Gamma^\lambda_{\beta\alpha}F_{\lambda\gamma} - \Gamma^\lambda_{\gamma\alpha}F_{\beta\lambda}\right] + \left[\frac{\partial F_{\gamma\alpha}}{\partial x^\beta} - \Gamma^\lambda_{\gamma\beta}F_{\lambda\alpha} - \Gamma^\lambda_{\alpha\beta}F_{\gamma\lambda}\right] = \\ \frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} - \left[\Gamma^\lambda_{\alpha\gamma}F_{\lambda\beta} + \Gamma^\lambda_{\gamma\alpha}F_{\beta\lambda}\right] - \left[\Gamma^\lambda_{\beta\gamma}F_{\alpha\lambda} + \Gamma^\lambda_{\gamma\beta}F_{\lambda\alpha}\right] - \left[\Gamma^\lambda_{\beta\alpha}F_{\lambda\gamma} + \Gamma^\lambda_{\alpha\beta}F_{\gamma\lambda}\right] = \\ \frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta}, \end{gathered} \tag{1.9g}$$

so the *remaining* electrodynamics equation $F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0$ *automatically has the form,*

$$\frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} = 0, \tag{1.9h}$$

that it has in the absence of gravitation.

Although *Einstein's coordinate condition* ensures that the zero-gravity electrodynamics equations,

$$\frac{\partial F^{\mu\nu}}{\partial x^\mu} = 4\pi j^\nu/c \quad \text{and} \quad \frac{\partial F_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial F_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial x^\beta} = 0, \tag{1.9i}$$

hold even in the presence of gravitation, *nevertheless* $F_{\alpha\beta} = g_{\alpha\mu}g_{\beta\nu}F^{\mu\nu}$ *in the presence of gravitation, so it is impossible in the presence of gravitation to obtain the particular wave-type electromagnetic equation,*

$$\frac{\partial^2 F_{\alpha\beta}}{\partial x^\gamma \partial x_\gamma} = (4\pi/c)\left[\frac{\partial j_\beta}{\partial x^\alpha} - \frac{\partial j_\alpha}{\partial x^\beta}\right],$$

which holds in the *absence* of gravitation; *gravitation affects the propagation of electromagnetic waves.* Einstein applied the coordinate condition $\det(g_{\mu\nu}(x)) = -1$ to the deflection of light by the sun's gravity *for the first time on* November 18, 1915, and he thereupon *for the first time obtained the correct deflection.*

Finally, in the presence of *both* gravitation *and* an electromagnetic field, the trajectory of a test particle of mass m and charge e is governed by the equation of motion,

$$m\left(\frac{d^2x^\kappa}{d\tau^2} + \Gamma^\kappa_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = (e/c)F^\kappa{}_\lambda\frac{dx^\lambda}{d\tau}, \text{ where } F^\kappa{}_\lambda = g_{\lambda\upsilon}F^{\kappa\upsilon}. \tag{1.9j}$$

It would *sorely strain physical credibility* for the motion of such a test particle *to violate Lorentz covariance,* or for $\Gamma^\kappa_{\mu\nu}$ *to fail to exist.* Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *neatly quashes those concerns.*

We have mentioned that the space-time differential volume d^4x, which is a Lorentz invariant, *isn't a general coordinate transformation invariant;* its transformation produces the well-known factor of the absolute value of the determinant of the transformation's Jacobian matrix,

$$d^4y = \left|\det\left(\partial y^\alpha/\partial x^\mu\right)\right| d^4x. \tag{1.9k}$$

We know that $g_{\mu\nu}(x)$ transforms as a covariant second-rank tensor,

$$g_{\alpha\beta}(y) = \frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial x^\nu}{\partial y^\beta} g_{\mu\nu}(x), \tag{1.9l}$$

so, since the determinant of a product of matrices is equal to the product of their determinants,

$$\det\left(g_{\alpha\beta}(y)\right) = \left(\det\left(\partial x^\mu/\partial y^\alpha\right)\right)^2 \det\left(g_{\mu\nu}(x)\right). \tag{1.9m}$$

Applying that same product rule to the identity,

$$\frac{\partial x^\mu}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x^\nu} = \delta^\mu_\nu, \tag{1.9n}$$

yields,

$$\det\left(\partial x^\mu/\partial y^\alpha\right) = 1/\det\left(\partial y^\alpha/\partial x^\nu\right), \tag{1.9o}$$

which implies that the Eq. (1.9m) transformation relation can alternatively be written,

$$\det\left(g_{\alpha\beta}(y)\right) = \det\left(g_{\mu\nu}(x)\right)/\left(\det\left(\partial y^\alpha/\partial x^\mu\right)\right)^2, \tag{1.9p}$$

Combining Eq. (1.9p) with Eq. (1.9k) yields the transformation relation for $\sqrt{-\det(g_{\mu\nu}(x))}\, d^4x$,

$$\sqrt{-\det(g_{\alpha\beta}(y))}\, d^4y = \sqrt{-\det(g_{\mu\nu}(x))}\, d^4x. \tag{1.9q}$$

Thus $\sqrt{-\det(g_{\mu\nu}(x))}d^4x$ *is a coordinate transformation invariant* which, *in the absence of gravitation* (when $g_{\mu\nu}(x) = \eta_{\mu\nu}$), *reduces to* d^4x. Therefore Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *reverts* the coordinate transformation invariant $\sqrt{-\det(g_{\mu\nu}(x))}\, d^4x$ *to the form* d^4x *it has in the absence of gravitation.*

We have now seen that Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *reverts some key relativistic entities,* such as the electrodynamics equation $F^{\mu\nu}{}_{;\mu} = 4\pi j^\nu/c$ and the invariant differential space-time volume $\sqrt{-\det(g_{\mu\nu}(x))}d^4x$ *to the forms* $\partial F^{\mu\nu}/\partial x^\mu = 4\pi j^\nu/c$ *and* d^4x *they have in the absence of gravitation. Not a single one of the coordinate conditions to be found in Weinberg's tome,* including Weinberg's favored harmonic coordinate condition, *effects such reversions.* Clearly Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$ *infuses the nebulous Einstein equation,* $R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x) = -(8\pi G/c^4)\, T_{\mu\nu}(x)$, *with a relativistic specificity no other coordinate condition comes even close to matching.* The Einstein equation *itself is devoid of relativistic specificity,* as is quite overwhelmingly demonstrated by the addition to it of the Lorentz-covariance *incompatible* condition $g_{00}(x) = 1$ prized by Friedmann, Lemaitre, Tolman, Robertson, Walker, Oppenheimer and Snyder, which totally eliminates gravitational time dilation, sends c to infinity, and causes the Einstein equation to vent *purely Newtonian gravitational physics.* Steven Weinberg, due to his reading of a 1928 article by the mathematician K. O. Friedrichs, had some understanding *of the nebulousness inherent in the Einstein equation by itself,* an understanding *not specific enough* to adequately serve Weinberg as a theoretical physicist. On pages 92-93 of Section 4.1 in his tome Weinberg writes, "...there are generally covariant theories of gravitation that allow the construction of inertial frames at any point in a gravitational field, but that satisfy Galilean relativity rather than special relativity in these frames."

By November 18, 1915 Einstein had sufficiently progressed with his nascent relativistic gravity theory that he was able to accurately calculate the part of Mercury's perihelion shift which wasn't accounted for by that planet's gravitational interaction with the other planets. *A key factor in Einstein's success was the introduction of his coordinate condition* $\det(g_{\mu\nu}(x)) = -1$ to supplement the Einstein equation; if Einstein had, for example, *instead introduced* $g_{00}(x) = 1$, *the calculation would have yielded the purely Newtonian result of zero perihelion shift.* Einstein *in addition found that the introduction of his coordinate condition* $\det(g_{\mu\nu}(x)) = -1$ *doubled the deflection of light rays by the sun's gravity compared to his previous calculation which didn't use that coordinate condition*—his new result was verified by a 1919 solar eclipse expedition. An English translation of Einstein's November 18, 1915 paper is given within the November 21, 2021 preprint "Einstein and the Perihelion Motion of Mercury" by Michel Janssen and Jürgen Renn, which is posted on arXiv. In his November 18, 1915 paper, Einstein calculated and used *a second-order approximation to* $\Gamma^i_{00}(\mathbf{x})$ *produced by a static point mass, provided* $\det(g_{\mu\nu}(\mathbf{x})) = -1$, but by January 13, 1916 Karl Schwarzschild had worked out *the exact metric* $g_{\mu\nu}(\mathbf{x})$ *produced by a static point mass, provided* $\det(g_{\mu\nu}(\mathbf{x})) = -1$.

2. Schwarzschild's 1916 metric versus the unphysical "Schwarzschild metric" in textbooks

The metric of a static point source must, of course, be time-independent and spherically symmetric, so Schwarzschild took that metric to have the form,

$$(c\,d\tau)^2 = F(r)(c\,dt)^2 - G(r)(dr)^2 - H(r)\,r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big), \tag{2.1a}$$

on which the Einstein coordinate condition $\det(g_{\mu\nu}(r)) = -1$ *imposes the requirement,*

$$H(r) = 1/\sqrt{F(r)G(r)}, \tag{2.1b}$$

so,

$$(c\,d\tau)^2 = F(r)(c\,dt)^2 - G(r)(dr)^2 - \big(1/\sqrt{F(r)G(r)}\big)\,r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big). \tag{2.1c}$$

The Newtonian gravitational potential $\phi(r)$ produced by a point mass M located at $r = 0$ is of course,

$$\phi(r) = -GM/r. \tag{2.1d}$$

Consequently, in light of Eqs. (1.4k) and (2.1d),

$$F(r) = g_{00}(r) \approx 1 + 2\phi(r)/c^2 \;=\; 1 - r_s/r \quad\text{when}\quad r \gg r_s, \tag{2.1e}$$

where,

$$r_s \stackrel{\text{def}}{=} 2GM/c^2, \tag{2.1f}$$

is called the Schwarzschild radius. *In addition, of course,* $G(r) \to 1$ as $r \to \infty$.

The metric given by Eq. (2.1c) and constrained by the two requirements that $F(r)$ is asymptotic to $1 - r_s/r$ as $r \to \infty$ and that $G(r) \to 1$ as $r \to \infty$ *must satisfy the empty-space Einstein equation, namely,*

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = 0, \tag{2.1g}$$

at every $r > 0$. Since $g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = -R = 0$ follows from Eq. (2.1g), Eq. (2.1g) itself *is equivalent to,*

$$R_{\mu\nu} = 0, \tag{2.1h}$$

as is well known. Schwarzschild's task was to insert the Eq. (2.1c) metric form into Eq. (2.1h), and then solve it *at every* $r > 0$ *for* $F(r)$ *and* $G(r)$, *subject to the constraints that* $F(r)$ *is asymptotic to* $1 - r_s/r$ *as* $r \to \infty$ *and that* $G(r) \to 1$ *as* $r \to \infty$. Since Eq. (2.1h) *must be satisfied at every* $r > 0$, *it is crystal clear that* $F(r)$ *and* $G(r)$ *are at least twice differentiable at every* $r > 0$. The *exact solution* in Schwarzschild's January 13, 1916 paper *satisfies all of these conditions*—that paper was translated into English by S. Antoci and A. Loinger, who posted it on arXiv in 1999 (arXiv:physics/9905030v1 [physics.hist-ph] 12 May 1999).

But *before we write down* Schwarzschild's January 13, 1916 *exact metric solution, we jump ahead to* May 27, 1916, when J. Droste published *an exact metric solution for the static point source which satisfies the additional condition* $G(R) = 1/F(R)$. But *unlike Einstein's coordinate condition* $\det(g_{\mu\nu}(x)) = -1$, *this additional condition* $G(R) = 1/F(R)$ *is incompatible with Lorentz covariance,* just as the exremely widely-applied condition $g_{00}(x) = 1$ is incompatible with Lorentz covariance. *However that Lorentz-covariance incompatible additional condition* $G(R) = 1/F(R)$ *makes Droste's exact metric solution,*

$$(c\,d\tau)^2 = (1 - r_s/R)(c\,dt)^2 - (1/(1 - r_s/R))(dR)^2 - R^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big), \tag{2.1i}$$

algebraically very much simpler than Schwarzschild's January 13, 1916 exact metric solution. The famous mathematician David Hilbert *was impressed by the algebraic simplicity of the* Eq. (2.1i) *exact metric solution* (algebraic simplicity, when it is possible, is highly prized by mathematicians), so Hilbert *keenly promoted it in 1918. The upshot of Hilbert's promotional effort* is that textbooks, *including* Weinberg's tome, *prominently feature* J. Droste's May 27, 1916 Eq. (2.1i) exact metric solution for the static point mass, but astoundingly state that it was found by K. Schwarzschild. One *consequence* is that Schwarzschild's *physically-correct* January 13, 1916 exact metric solution *has been thrust into almost complete obscurity.*

The metric factor $(1/(1-r_s/R))$ in the Eq. (2.1i) Droste metric *has a severe singularity at* $R = r_s$, *which utterly contravenes the physical requirement that the metric satisfies* $R_{\mu\nu} = 0$ *at every* $R > 0$; *this physical requirement implies that such metric factors must be at least twice differentiable at every* $R > 0$, *as indeed is the case for Schwarzschild's physically-correct January 13, 1916 exact metric solution.* The Eq. (2.1i) Droste metric *in addition has a more subtle physical defect* which arises from that metric's *incompatibility with Lorentz covariance* pointed out in the paragraph above Eq. (2.1i). Although relativistic gravitational and speed time dilation has a far smaller and much less obvious effect on circular orbits than on elliptic orbits, in principle relativistic time dilation will make a circular orbit's period at a given radius *slightly longer than the Newtonian value,* and the percent of this tiny increase in the circular orbit's period above the Newtonian value should grow as the orbit's radius decreases. Eq. (8.4.25) at the top of page 188 of Weinberg's tome yields the period $T(R)$ of a circular orbit of radius R *for a metric of the type of* Eq. (2.1i),

$$T(R) = (2\pi/c)\sqrt{2R/(d(1 - r_s/R)/dR)} = 2\pi\sqrt{2R^3/(c^2 r_s)} = 2\pi\sqrt{R^3/(GM)} \text{ since } r_s = 2GM/c^2. \quad (2.1j)$$

The Eq. (2.1j) result for the period $T(R)$ of a circular orbit of radius R for the Eq. (2.1i) Droste metric *is identical to the Newtonian value; there is no time dilation whatsoever. This complete absence of time dilation for circular-orbit periods* illustrates the Eq. (2.1i) Droste metric's *incompatibility with Lorentz covariance* pointed out in the paragraph above Eq. (2.1i). Schwarzschild's *physically-correct* January 13, 1916 metric solution *contrariwise* yields a slight increase beyond the Newtonian value in the period of a circular orbit, and the percent of the increase in the period above the Newtonian value grows as the orbit's radius decreases. It is clear that the Eq. (2.1i) Droste metric (which textbooks *mistakenly* call the "Schwarzschild metric"), although an exact solution of the Einstein equation, *is unsuited to relativistic gravitation.*

Since Schwarzschild's January 13, 1916 metric *also* is an exact solution of the Einstein equation, it can be obtained from the Eq. (2.1i) Droste metric *by a transformation $R(r)$ of the Droste metric's radial coordinate R.* We have noted in the text below Eq. (2.1h) *that it is physically necessary for the metric factors $F(r)$ and $G(r)$ of the* Eq. (2.1c) *metric form to be at least twice differentiable at every* $r > 0$, whereas the Droste metric factor $(1/(1 - r_s/R))$ *has a severe singularity at* $R = r_s$. Therefore *it is absolutely essential that the transformation $R(r)$ send the point $R = r_s$ to the point* $r = 0$, which implies that the transformation $R(r)$ *satisfies* $R(r = 0) = r_s$. In *addition,* the transformation $R(r)$ *of course must be such that the transformed metric has the determinant value* -1 in accord with Einstein's coordinate condition $\det(g_{\mu\nu}(x)) = -1$. The result of applying the transformation $R(r)$ to the Eq. (2.1i) Droste metric is,

$$(c\,d\tau)^2 = (1-r_s/R(r))(c\,dt)^2 - (1/(1-r_s/R(r)))(dR(r)/dr)^2(dr)^2 - (R(r)/r)^2\, r^2\big((d\theta)^2 + (\sin\theta\, d\phi)^2\big), \quad (2.1k)$$

a metric whose determinant value is -1 when $(dR(r)/dr) = (r/R(r))^2$, an equation whose solution is readily verified to be $R(r) = (r^3 + (r_0)^3)^{\frac{1}{3}}$, where r_0 is an arbitrary constant with the dimension of length. We pointed out above that $R(r = 0) = r_s$, so the transformation we require is $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$. In his January 13, 1916 paper Schwarzschild presents the resulting metric in the wonderfully elegant, but very terse, form,

$$(c\,d\tau)^2 = (1 - r_s/R(r))(c\,dt)^2 - (1/(1 - r_s/R(r)))(dR(r))^2 - (R(r))^2\big((d\theta)^2 + (\sin\theta\, d\phi)^2\big), \quad (2.1l)$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$. A less cryptic form of Schwarzschild's January 13, 1916 metric is obtained by noting that $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$ implies that $(dR(r)/dr) = (r/R(r))^2$, which is then substituted into Eq. (2.1k),

$$(c\,d\tau)^2 = (1-r_s/R(r))(c\,dt)^2 - (1/(1-r_s/R(r)))(r/R(r))^4(dr)^2 - (R(r)/r)^2\, r^2\big((d\theta)^2 + (\sin\theta\, d\phi)^2\big), \quad (2.1m)$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$. Since $r > 0$ implies that $R(r) > r_s$, we see that when $r > 0$ Schwarzschild's January 13, 1916 metric *is free of singularities,* which we have repeatedly pointed out *is physically required.* The Eq. (2.1c) metric factors are $F(r) = (1 - r_s/R(r))$ and $G(r) = (1/(1 - r_s/R(r)))(r/R(r))^4$. David Hilbert's ill-advised 1918 promotion of J. Droste's May 27, 1916 Eq. (2.1i) metric because of its algebraic simplicity, *with insufficient scrutiny of its physical soundness,* ultimately spawned *nonexistent* "event horizons" and "wormholes", *which have long been a pernicious distraction from sound relativistic gravitational research.*

We next insert the Eq. (2.1l) Schwarzschild metric form into the Eq. (2.1j) circular-orbit period formula,

$$T(R(r)) = (2\pi/c)\sqrt{2R(r)/(d(1 - r_s/R(r))/dR(r))} = 2\pi\sqrt{(R(r))^3/(GM)} = 2\pi\sqrt{(r^3 + r_s^3)/(GM)}. \quad (2.1n)$$

Since *the Newtonian-gravity* circular-orbit period is $T_N(r) = 2\pi\sqrt{r^3/(GM)}$, the *ratio* $(T(R(r))/T_N(r)) = \sqrt{1 + (r_s/r)^3}$, which, *although extremely close to unity, is nevertheless always greater than unity and grows with decreasing radius r, exactly as one would expect.* Thus the circular-orbit period result of Schwarzschild's January 13, 1916 metric *makes relativistic-gravity sense,* but *the purely Newtonian-gravity* circular-orbit period result of the Eq. (2.1i) Droste metric *doesn't make relativistic-gravity sense.*

Finally, it is interesting to consider the *speed* $v(r)$ of the circularly-orbiting test particle; $v(r)$ of course is the *quotient* of the the circular orbit's *circumference* $2\pi r$ with the circular orbit's *period,* which in the

Newtonian-gravity case, *and also in the Droste metric case,* is $2\pi\sqrt{r^3/(GM)}$, so $v(r) = \sqrt{GM/r}$ *in those two cases.* Therefore the circular-orbit speed $v(r)$ *goes to infinity as* r *goes to zero in those two cases.* For Schwarzschild's January 13, 1916 metric, we have seen that *the circular-orbit period is increased by the factor* $\sqrt{1 + (r_s/r)^3}$ *relative to the Newtonian circular-orbit period,* so for *that* metric *the Newtonian circular-orbit speed* $\sqrt{GM/r}$ *is decreased by the factor* $\sqrt{1 + (r_s/r)^3}$. Noting that $GM = \frac{1}{2} r_s c^2$, we see that for Schwarzschild's January 13, 1916 metric *the circular-orbit speed is,*

$$v(r) = \sqrt{(GM/r)/(1 + (r_s/r)^3)} = c\sqrt{(r_s/r)/(2(1 + (r_s/r)^3))} = c\sqrt{(r/r_s)^2/(2(1 + (r/r_s)^3))}, \tag{2.1o}$$

which goes to *zero* instead of to infinity as r goes to zero, so Schwarzschild's January 13, 1916 metric *correctly captures the effect on a test-particle's speed* $v(r)$ *of surpassingly-strong gravitational time dilation.* Since the Eq. (2.1o) circular-orbit speed $v(r)$ *also goes zero as* r *goes to infinity,* $v(r)$ *is maximum at an intermediate value of* r. The function $u^2/(1 + u^3)$ has a maximum at $u = 2^{\frac{1}{3}}$, and the value of that maximum is $2^{\frac{2}{3}}/3$. Therefore the Eq. (2.1o) circular-orbit speed $v(r)$ *reaches its maximum at* $r = 2^{\frac{1}{3}} r_s = 1.25992\, r_s$, *and the value of that maximum circular-orbit speed is* $(2^{\frac{1}{3}}/\sqrt{6})\, c = 0.51436\, c$. Thus Schwarzschild's January 13, 1916 metric *also correctly captures the fact that a relativistic test-particle's speed never exceeds* c.

These facts *of relativistic gravity* of course *aren't captured at all by the purely Newtonian-gravity circular-orbit speed* $v(r) = \sqrt{GM/r}$ for the Eq. (2.1i) Droste metric. This discussion of circular-orbit speed *reconfirms* the fact that the Eq. (2.1i) Droste metric (which is *mistakenly* called the "Schwarzschild metric" by textbooks) is *unphysical* despite its being an exact solution of the Einstein equation. *A century of propagating misunderstanding of relativistic gravitation* needs to reversed forthwith *by urgent replacement in textbooks of the unphysical* Eq. (2.1i) *Droste metric* by Schwarzschild's January 13, 1916 Eq. (2.1l) metric.

Just as *the circular-orbit motion of a test particle* is rendered *purely Newtonian by the unphysical* Eq. (2.1i) *Droste metric, the radial motion of the surface of an Oppenheimer-Snyder spherical blob of zero-pressure, uniform-density perfect fluid* is rendered *purely Newtonian by the unphysical* Eq. (1.6a) *Robertson-Walker metric form,* which features the Lorentz-covariance *incompatible* condition $g_{00}(x) = 1$ that totally eliminates gravitational time dilation, sends c to infinity and causes the Einstein equation to vent *purely Newtonian gravitational physics.* The *comprehensive cure for this issue* is the replacement of the Eq. (1.6a) Robertson-Walker metric form by the Eq. (1.6e) metric form, *which is a coordinate transformation of the* Eq. (1.6a) *Robertson-Walker metric form that satisfies Einstein's coordinate condition* $\det(g_{\mu\nu}(x)) = -1$ *instead the Lorentz-covariance incompatible condition* $g_{00}(x) = 1$. This formidably cumbersome *comprehensive* approach, however, *isn't needed to deal with the radial motion of the freely-falling spherical surface of an Oppenheimer-Snyder spherical blob of zero-pressure, uniform-density perfect fluid;* the Birkhoff theorem tells us that we can proceed as if the entire conserved energy enclosed by the radially freely-falling spherical surface of the Oppenheimer-Snyder spherical blob is concentrated in a static point mass at the blob's center. Therefore in the next section we develop and analyze the equation for a test particle's freely-falling radial motion in Schwarzschild's January 13, 1916 Eq. (2.1l) metric.

3. The equation for a test particle's freely-falling radial motion in Schwarzschild's 1916 metric

To obtain a test particle's equation of freely falling radial motion in Schwarzschild's January 13, 1916 Eq. (2.1l) metric, we apply the methods of Weinberg's Section 8.4 on pages 185-188 of his tome, so it is convenient to adhere, in expressing Schwarzschild's Eq. (2.1l) metric, to Weinberg's Section 8.4 notation,

$$(c\, d\tau)^2 = B(R(r))(c\, dt)^2 - A(R(r))(dR(r))^2 - (R(r))^2\big((d\theta)^2 + (\sin\theta\, d\phi)^2\big), \tag{3.1a}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$, $B(R(r)) = 1 - r_s/R(r)$ and $A(R(r)) = 1/B(R(r))$. The Eq. (3.1a) metric itself immediately yields the particular first-order equation of freely-falling motion,

$$1 = B(R(r))(dt/d\tau)^2 - A(R(r))((1/c)dR(r)/d\tau)^2 - (R(r))^2\big(((1/c)d\theta/d\tau)^2 + (\sin\theta\,(1/c)d\phi/d\tau)^2\big). \tag{3.1b}$$

Since the freely-falling test particle we consider here is an arbitrarily small part of the radially freely-falling spherical surface of an Oppenheimer-Snyder spherical blob of zero-pressure, uniform-density perfect fluid, this test particle's motion of course *is exclusively radial,* so the Eq. (3.1b) *angular frequencies* $d\theta/d\tau$ *and* $d\phi/d\tau$ *are both equal to zero,* which *reduces* Eq. (3.1b) to a first-order equation for freely-falling *radial* motion,

$$c^2 = \big[c^2 B(R(r)) - A(R(r))(dR(r)/dt)^2\big](dt/d\tau)^2. \tag{3.1c}$$

We *can't,* of course, solve Eq. (3.1c) for the test particle's radial trajectory $r(t)$ *until we know the value of the factor* $(dt/d\tau)^2$. The Eq. (1.3) geodesic equation $d^2x^\kappa/d\tau^2 + \Gamma^\kappa_{\mu\nu}\,(dx^\mu/d\tau)\,(dx^\nu/d\tau) = 0$ *is a second-order equation of motion for* $x^\kappa(\tau) = (ct(\tau),\, \mathbf{x}(\tau))$ *for a given metric* $g_{\mu\nu}(x)$. Weinberg's Eq. (8.4.6) on his tome's page 185 *gives the geodesic equation's time component for the specific* Eq. (3.1a) *metric form,*

$$\tfrac{d^2t}{d\tau^2} + \tfrac{dt}{d\tau}\tfrac{dB(R(r))/dR(r)}{B(R(r))}\tfrac{dR(r)}{d\tau} = 0, \tag{3.2a}$$

which can be written,

$$\tfrac{1}{dt/d\tau}\tfrac{d(dt/d\tau)}{d\tau} + \tfrac{dB(R(r))/dR(r)}{B(R(r))}\tfrac{dR(r)}{d\tau} = 0, \tag{3.2b}$$

which in turn can be written,

$$d\big(\ln\big(dt/d\tau\big) + \ln\big(B(R(r))\big)\big)/d\tau = 0, \tag{3.2c}$$

which implies that,

$$\ln\big((dt/d\tau)(B(R(r)))\big) = -C, \tag{3.2d}$$

where C is an arbitrary dimensionless constant. Eq. (3.2d) implies that,

$$dt/d\tau = 1/(KB(R(r))), \tag{3.2e}$$

where $K = \exp(C)$ is an arbitrary dimensionless positive constant. Inserting Eq. (3.2e) into Eq. (3.1c) yields,

$$\big(A(R(r))/(B(R(r)))^2\big)(dR(r)/dt)^2 - \big(c^2/B(R(r))\big) = -c^2K^2. \tag{3.3a}$$

The object $dR(r)/dt$ in Eq. (3.3a) is of course equal to $(dR(r)/dr)(dr/dt)$, and we noted immediately above Eq. (2.1m) that $dR(r)/dr = (r/R(r))^2$. That follows as well from $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$, which is given immediately below Eq. (3.1a), as is $A(R(r)) = 1/B(R(r))$ and $B(R(r)) = 1 - r_s/R(r)$. Inserting all of this into Eq. (3.3a) yields,

$$\big((r/R(r))^4/(1 - r_s/R(r))^3\big)(dr/dt)^2 - \big(c^2/(1 - r_s/R(r))\big) = -c^2K^2, \tag{3.3b}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$ and $r_s = (2GM/c^2)$. We next work out the $c \to \infty$ asymptotic form of Eq. (3.3b). Taking c to infinity sends r_s to zero and $R(r)$ to r. However the $c \to \infty$ asymptotic form of the entity $(c^2/(1 - r_s/R(r)))$ is $(c^2 + (2GM/r))$. Therefore the $c \to \infty$ asymptotic form of Eq. (3.3b) is,

$$(dr/dt)^2 - (2GM/r) = c^2(1 - K^2). \tag{3.3c}$$

Upon multiplying Eq. (3.3c) through by $\frac{1}{2}m$, where m is the mass of the test particle, we obtain,

$$\tfrac{1}{2}m(dr/dt)^2 - (GMm/r) = \tfrac{1}{2}mc^2(1 - K^2). \tag{3.3d}$$

If we now assign the dimensionless constant K^2 the value $[1 - (2E/(mc^2))]$, where E *is the conserved sum* of the mass m test particle's *nonrelativistic* positive radial kinetic energy with its *nonrelativistic* negative gravitational potential energy, Eq. (3.3d) becomes,

$$\tfrac{1}{2}m(dr/dt)^2 - (GMm/r) = E, \tag{3.3e}$$

which is the standard nonrelativistic equation of radial motion of a test particle whose gravitational potential energy is $-(GMm/r)$. Thus the $c \to \infty$ asymptotic form of Eq. (3.3b) *is indeed the corresponding nonrelativistic, Newtonian-gravity equation of the test particle's radial gravitational motion.*

We next consider *the special initial condition* for Eq. (3.3e) that $dr/dt = 0$ at an initial time t_i, in which case the conserved energy E equals $-(GMm/r(t_i))$, so Eq. (3.3e) can be rewritten,

$$(dr/dt)^2 = (2GM/r(t_i))\,[(r(t_i)/r(t)) - 1]. \tag{3.3f}$$

Upon *switching from* $r(t)$ *to the dimensionless "scaled radius"* $R(t) \stackrel{\text{def}}{=} (r(t)/r(t_i))$ *which has the property that* $R(t_i) = 1$, Eq. (3.3f) assumes the form,

$$(dR/dt)^2 = (2GM/(r(t_i))^3)\,[(1/R(t)) - 1]. \tag{3.3g}$$

If $r(t_i)$, the radially-moving test particle's radial coordinate at the initial time t_i when $dr/dt = 0$, *is as well the radius of the Oppenheimer-Snyder spherical blob at that time* t_i*, then the density* $\rho(t_i)$ *of the Oppenheimer-Snyder spherical blob at that initial time* t_i *is* $\rho(t_i) = M/((4\pi/3)(r(t_i))^3)$. So in terms of the Oppenheimer-Snyder spherical blob's initial density $\rho(t_i)$, Eq. (3.3g) *is equivalently written,*

$$(dR/dt)^2 = ((8\pi G)/3)\rho(t_i)\,[(1/R(t)) - 1], \tag{3.3h}$$

which is *precisely* Eq. (11.9.24) at the bottom of page 344 of Weinberg's tome, *the central result of a very laborious 24-step Oppenheimer-Snyder calculation using the Robertson-Walker metric form.* That arriving at Eq. (3.3h)—which is nothing more than a disguised special case of Eq. (3.3e) that is very well-known to every first-year undergraduate physics student—involved complicated 24-step heroics shows the woeful lack of understanding of gravity theory which has persisted for over a century. Of course *applying the Robertson-Walker metric form,* which very prominently features the Lorentz-covariance *incompatible* condition $g_{00}(x) = 1$, thereby totally eliminating gravitational time dilation, sending c to infinity (mimicking what we have done to transition from Eq. (3.3b) to Eq. (3.3c)) and causing the Einstein equation to vent purely Newtonian gravitational physics, *transitions the Oppenheimer-Synder model* (a spherical blob of zero-pressure, uniform-density perfect fluid) *into merely first-year undergraduate physics.*

Before we leave the simple radial-motion *nonrelativistic-gravity* Eq. (3.3e) to return to its relativistic-gravity Eq. (3.3b) counterpart, we note that Eq. (3.3e) tells us that the closer the test particle is to the gravitational point source, the greater is its speed, and that *its speed in that gravitational field has no upper bound.* In the case of relativistic gravity, however, the gravitational time dilation which accompanies strong gravitation ultimately reduces a test particle's speed rather than increases it, and, in any case, *a test particle's speed must always be less than c.*

We also take note of the fact that *the gravitational acceleration* experienced by a nonrelativistic test particle governed by Eq. (3.3e) *is completely independent of its energy* E,

$$d\big(\tfrac{1}{2}m(dr/dt)^2\big)/dt = d\big((GMm/r) + E\big)/dt \ \text{ implies,}$$
$$m(dr/dt)(d^2r/dt^2) = -(GMm/r^2)(dr/dt) \ \text{ which implies, } \ d^2r/dt^2 = -(GM/r^2), \tag{3.3i}$$

irrespective of the test particle's energy E.

Returning now to the Eq. (3.3b) equation of relativistic radial motion appropriate to Schwarzschild's January 13, 1916 static point-mass metric, we rewrite it as,

$$(dr/dt)^2 = c^2(R(r)/r)^4\left[(1 - r_s/R(r))^2 - K^2(1 - r_s/R(r))^3\right] =$$
$$c^2(R(r)/r)^2\big((R(r)/r) - (r_s/r)\big)^2 - c^2K^2(R(r)/r)\big((R(r)/r) - (r_s/r)\big)^3, \tag{3.4a}$$

where $R(r) = (r^3 + r_s^3)^{\frac{1}{3}}$. An alternative way of writing Eq. (3.4a) is,

$$(dr/dt)^2 = c^2(1 + u^3)^{\frac{2}{3}}\big((1 + u^3)^{\frac{1}{3}} - u\big)^2 - c^2K^2(1 + u^3)^{\frac{1}{3}}\big((1 + u^3)^{\frac{1}{3}} - u\big)^3, \tag{3.4b}$$

where $u = (r_s/r)$. We saw from Eq. (3.3e) that in the nonrelativistic case, as $r \to 0$, $(dr/dt)^2$ *increases without bound; indeed in that Newtonian case the radial speed* $|dr/dt|$ *is asymptotic to* $\sqrt{2GM/r}$ *as* $r \to 0$. But from the Eq. (2.1o) *relativistic result for circular-orbit speed,* we expect that *in the relativistic case* the radial speed $|dr/dt|$ *instead goes to zero asymptotically as* $r \to 0$; indeed from Eq. (2.1o) we expect that $|dr/dt|$ is asymptotic to a numerical factor times $c(r/r_s)$ as $r \to 0$. The *immense change in the* $r \to 0$ *asymptotic behavior of* $|dr/dt|$ when one passes from Newtonian gravitation to relativistic gravitation *is the consequence of gravitational time dilation, which doesn't exist in Newtonian gravitation.*

To work out the asymptotic behavior of $(dr/dt)^2$ as $r \to 0$ in Eq. (3.4b), we note that as $r \to 0$, $u = (r_s/r) \to \infty$. As $u \to \infty$, we note that $(1 + u^3)^{\frac{1}{3}} = u((1/u)^3 + 1)^{\frac{1}{3}} \simeq (u + \frac{1}{3}(1/u)^2)$, which implies that $((1 + u^3)^{\frac{1}{3}} - u) \simeq \frac{1}{3}(1/u)^2$. Insertion of these $u \to \infty$ asymptotic results into Eq. (3.4b) yields $(dr/dt)^2 \simeq c^2[((1/u)^2/9) - K^2((1/u)^5/(27))] \simeq c^2(1/(3u))^2$. Since $u = (r_s/r)$, the asymptotic upshot is that,

$$\textit{the test particle's radial speed } |dr/dt| \text{ is asymptotic to } (c/(3r_s))r \text{ as } r \to 0. \tag{3.4c}$$

To work out *the upper bound of* $(dr/dt)^2$, it is convenient to reexpress Eq. (3.4b) as,

$$(dr/dt)^2 = c^2\chi(u) - c^2K^2\xi(u) \ \text{ where,}$$
$$\chi(u) \stackrel{\text{def}}{=} \big((1 + u^3)^{\frac{1}{3}}\big((1 + u^3)^{\frac{1}{3}} - u\big)\big)^2, \quad \xi(u) \stackrel{\text{def}}{=} \big(1 - \big(u/(1 + u^3)^{\frac{1}{3}}\big)\big)\,\chi(u) \ \text{ and } \ u = (r_s/r). \tag{3.4d}$$

We next show that $\chi(u)$ is a strictly decreasing function for $u \geq 0$ by verifying that $d\chi(u)/du$ is negative,

$$d\chi(u)/du = 2\big((1+u^3)^{\frac{1}{3}}\big((1+u^3)^{\frac{1}{3}} - u\big)\big)\big[2u^2/(1+u^3)^{\frac{1}{3}} - u^3/(1+u^3)^{\frac{2}{3}} - (1+u^3)^{\frac{1}{3}}\big] =$$
$$2\big((1+u^3)^{-\frac{1}{3}}\big((1+u^3)^{\frac{1}{3}} - u\big)\big)\big[2u^2(1+u^3)^{\frac{1}{3}} - u^3 - (1+u^3)\big] =$$
$$\big(1 - \big(u/(1+u^3)^{\frac{1}{3}}\big)\big)\big[4u^2\big((1+u^3)^{\frac{1}{3}} - u\big) - 2\big]. \tag{3.4e}$$

We must now *verify the inequality* $\big[4u^2\big((1+u^3)^{\frac{1}{3}} - u\big) - 2\big] < 0$ *when* $u \geq 0$. We do so by exhibiting a chain of inequalities *which are logically equivalent to it, where the final inequality in the chain is clearly valid,*

$$\big[4u^2\big((1+u^3)^{\frac{1}{3}} - u\big) - 2\big] < 0 \iff 2u^2(1+u^3)^{\frac{1}{3}} < 2u^3 + 1 \iff 8u^6(1+u^3) < (2u^3+1)^3$$
$$\iff 8u^9 + 8u^6 < 8u^9 + 12u^6 + 6u^3 + 1 \iff 4u^6 + 6u^3 + 1 > 0 \ \text{ when } \ u \geq 0. \tag{3.4f}$$

Since for $u \geq 0$, $\chi(u)$ is strictly decreasing, it follows that $\chi(u) \leq \chi(0) = 1$. It is furthermore true that $K^2 = \exp(2C) > 0$ (see Eqs. (3.2d) and (3.2e)) and that $\big(1 - \big(u/(1+u^3)^{\frac{1}{3}}\big)\big) > 0$, so we can use Eq. (3.4d) to show that for $u \geq 0$, $(dr/dt)^2 < c^2$,

$$(dr/dt)^2 = c^2\chi(u)\big[1 - K^2\big(1 - \big(u/(1+u^3)^{\frac{1}{3}}\big)\big)\big] < c^2\chi(u) \leq c^2\chi(0) = c^2, \tag{3.4g}$$

so $(dr/dt)^2 < c^2$ *under all circumstances.* Therefore the test particle, which is an infinitesimal part of the spherical surface of the spherical Oppenheimer-Snyder blob of zero-pressure uniform-density perfect fluid, *can never have a speed as great as c.* Of course *this result is an absolutely fundamental aspect of relativistic gravity theory, but it is violated in the most extreme way conceivable in the course of the "gravitational collapse" and "Big Bang" events that ensue when the purely Newtonian Robertson-Walker metric form is applied to the Oppenheimer-Snyder model.* In those "gravitational collapse" and "Big Bang" cases, the radius of the blob becomes arbitrarily small as its density becomes arbitrarily large, *so the speed of its surface becomes arbitrarily large in accord with the Newtonian-gravitational relation that* $|dr/dt|$ *is asymptotic to* $\sqrt{2GM/r}$ *as* $r \to 0$. The Einstein equation *by itself has no Lorentz-covariant specificity whatsoever;* it is fully compatible with the Galilean condition $g_{00}(x) = 1$ of the Robertson-Walker metric form that totally eliminates gravitational time dilation, sends c to infinity and causes the Einstein equation to vent *purely Newtonian gravity. It takes Einstein's coordinate condition* $\det(g_{\mu\nu}(x)) = -1$ (which is correctly built into Schwarzschild's January 13, 1916 metric that we applied to obtain the gravitational model of Eq. (3.4d)) *to self-consistently infuse the Einstein equation with physically crucial Lorentz covariance.*

In Eq. (3.4e) we worked out $d\chi(u)/du$, where $\chi(u) = \big((1+u^3)^{\frac{1}{3}}\big((1+u^3)^{\frac{1}{3}} - u\big)\big)^2$. It is useful to as well work out $d\xi(u)/du$, where $\xi(u) = \big(1 - \big(u/(1+u^3)^{\frac{1}{3}}\big)\big)\chi(u) = (1+u^3)^{\frac{1}{3}}\big((1+u^3)^{\frac{1}{3}} - u\big)^3$,

$$d\xi(u)/du = \big(u/(1+u^3)^{\frac{1}{3}}\big)^2\big((1+u^3)^{\frac{1}{3}} - u\big)^3 + (1+u^3)^{\frac{1}{3}}3\big((1+u^3)^{\frac{1}{3}} - u\big)^2\big(\big(u/(1+u^3)^{\frac{1}{3}}\big)^2 - 1\big) =$$
$$\big(\big((1+u^3)^{\frac{1}{3}} - u\big)/(1+u^3)^{\frac{1}{3}}\big)^2\big[u^2(1+u^3)^{\frac{1}{3}} - u^3 + (1+u^3)^{\frac{1}{3}}3\big(u^2 - (1+u^3)^{\frac{2}{3}}\big)\big] =$$
$$\big(1 - \big(u/(1+u^3)^{\frac{1}{3}}\big)\big)^2\big[4u^2(1+u^3)^{\frac{1}{3}} - u^3 - 3 - 3u^3\big] =$$
$$\big(1 - \big(u/(1+u^3)^{\frac{1}{3}}\big)\big)^2\big[4u^2\big((1+u^3)^{\frac{1}{3}} - u\big) - 3\big]. \tag{3.4h}$$

That $\big[4u^2\big((1+u^3)^{\frac{1}{3}} - u\big) - 3\big] < 0$ is a corollary of the Eq. (3.4f) verification, so $\xi(u)$, like $\chi(u)$, is a strictly decreasing function for $u \geq 0$, in fact $\xi(u)$ decreases more rapidly than $\chi(u)$ does.

Next we investigate the radial speed $|dr/dt|$ of the spherical surface of the Oppenheimer-Snyder blob as $r \to \infty$, which, since $u = (r_s/r)$, corresponds to $u = 0$. From Eq. (3.4d) we see that $\chi(0) = \xi(0) = 1$, so, since $(dr/dt)^2 = c^2\chi(u) - c^2K^2\xi(u)$,

$$|dr/dt| \to c\sqrt{1 - K^2} \ \text{ as } \ r \to \infty. \tag{3.5a}$$

If $K^2 > 1$, Eq. (3.5a) tells us *that the Oppenheimer-Snyder spherical blob is gravitationally bound, so its surface can't reach arbitrarily large values of* r. Conversely, *the closer* K^2 *is to zero, the closer the blob's asymptotic surface speed is to c.* We have pointed out that *in the nonrelativistic limit,* K^2 can be taken to be $1 - (2E/(mc^2))$, where m is the mass of the test particle and E *is the sum of the nonrelativistic positive kinetic energy of the test particle with the nonrelativistic negative gravitational potential energy of the test particle,* the test particle being an arbitrarily small part of the Oppenheimer-Snyder blob's spherical surface. Thus, *in the nonrelativistic limit,* $c\sqrt{1 - K^2} = \sqrt{2E/m}$. *This nonrelativistic interpretation of the relativistic entity* K^2 *being* $1 - (2E/(mc^2))$ *however is only satisfactory if* $|1 - K^2| \ll 1$, as we shall now see.

Having obtained the radial speed $|dr/dt|$ of the spherical surface of the Oppenheimer-Snyder blob as $r \to \infty$, we next investigate *its radial acceleration* d^2r/dt^2 *as* $r \to \infty$. In Eq. (3.3i) we noted that *the nonrelativistic Newtonian-gravity value of* d^2r/dt^2 *is* $-(GM/r^2)$. *The relativistic value of* d^2r/dt^2 is obtained after differentiating both sides of the Eq. (3.4d) relation $(dr/dt)^2 = c^2\chi(u) - c^2K^2\xi(u)$ with respect to t,

$$2(dr/dt)(d^2r/dt^2) = c^2\big(d\chi(u)/du - K^2 d\xi(u)/du\big)(du/dr)(dr/dt). \tag{3.5b}$$

Since $u = (r_s/r)$ and $r_s = (2GM/c^2)$, $\frac{1}{2}c^2(du/dr) = -(GM/r^2)$, so Eq. (3.5b) becomes,

$$d^2r/dt^2 = -(GM/r^2)(d\chi(u)/du - K^2 d\xi(u)/du). \tag{3.5c}$$

Since we want the asymptotic form of the acceleration d^2r/dt^2 as $r \to \infty$, we need the values of $d\chi(u)/du$ and $d\xi(u)/du$ at $u = 0$, which from Eqs. (3.4e) and (3.4h) are -2 and -3 respectively. Therefore,

the test particle's radial acceleration d^2r/dt^2 is asymptotic to $(GM/r^2)(2 - 3K^2)$ as $r \to \infty$. (3.5d)

This relativistic-gravity acceleration result *only substantially agrees with the Newtonian-gravity acceleration result* $-(GM/r^2)$ *when* $|1 - K^2| \ll 1$. In fact, when $0 < K^2 < \frac{2}{3}$, which makes the asymptotic speed $c\sqrt{1 - K^2}$ of the spherical blob's surface *a substantial fraction of c, the expected negative acceleration becomes positive acceleration.* Exactly such an unexpected positive acceleration of the expansion of the universe has been very reliably observed, and its discoverers awarded a Nobel prize. Because cosmological models, of which the Oppenheimer-Snyder spherical blob of zero-pressure uniform-density perfect fluid is the simplest, have up to now *always been solved using the Robertson-Walker metric form, which rigidly enforces the nonrelativistic purely Newtonian-gravity solution,* such a gross deviation from "normal" Newtonian-gravity negative acceleration *seemed as inexplicable and physically unattainable as falling upwards in the earth's gravity would be. Nonrelativistic Newtonian gravity fails to include gravitational time dilation,* a phenomenon which, *when it is strong enough,* provokes motion *opposite to that associated with Newtonian gravity.*

Unwilling to admit that *the purely Newtonian-gravity cosmological solutions which the Robertson-Walker metric form rigidly enforces are an issue which must be addressed,* the gurus of gravity *instead* chose to postulate a completely ad hoc epicycle that incredibly took the form of adding a cosmological constant to the Einstein equation, a maneuver which Einstein had emphatically characterized as "my biggest mistake". Adding a $\lambda g_{\mu\nu}$ term to the Einstein equation *scuppers the verification that the Einstein equation reproduces Newtonian gravity in the weak-field and static energy-density case* (see the long paragraph accompanying Eqs. (1.8t) through (1.8w)). *Discarding the untenable* $\lambda g_{\mu\nu}$ *term for the second time cannot occur too soon.*

We next elucidate *the possible time dependences* $r(t)$ of the spherical blob's radius r as $r \to 0$. We *first* treat *Newtonian gravity,* where $(dr/dt)^2 \simeq (2GM/r)$ as $r \to 0$. The two possible $r \to 0$ time dependences *follow from solutions of the two differential equations* $dr_B/dt = +\sqrt{2GM/r_B}$ *and* $dr_C/dt = -\sqrt{2GM/r_C}$ *that also satisfy* $r_B(t) \to 0$ *and* $r_C(t) \to 0$. Therefore $r_B(t)$, dr_B/dt, $r_C(t)$ and dr_C/dt are given by,

$$r_B(t) = \big((9/2)GM(t - t_B)^2\big)^{\frac{1}{3}} \;\text{ and }\; dr_B/dt = \big((4/3)GM/(t - t_B)\big)^{\frac{1}{3}} \;\text{ as }\; (t - t_B) \to 0+, \;\text{ and,}$$

$$r_C(t) = \big((9/2)GM(t_C - t)^2\big)^{\frac{1}{3}} \;\text{ and }\; dr_C/dt = -\big((4/3)GM/(t_C - t)\big)^{\frac{1}{3}} \;\text{ as }\; (t_C - t) \to 0+, \tag{3.6a}$$

where $r_B(t)$ and dr_B/dt *give the* $r_B(t) \to 0$ *"Big Bang" asymptotic time behavior as* $(t - t_B) \to 0+$, *where* t_B *is the time earlier than* t *when the "Big Bang" occurred,* while $r_C(t)$ and dr_C/dt *give the* $r_C(t) \to 0$ *"gravitational collapse" asymptotic time behavior as* $(t_C - t) \to 0+$, *where* t_C *is the time later than* t *when the "gravitational collapse" will occur.* Both of these time dependences accord with $r \to 0$, *but they both as well accord with* $|dr/dt| \to \infty$, *which is the most extreme violation of* $|dr/dt| < c$ *conceivable.* Moreover, *all initial conditions for this model, when solved using purely Newtonian gravity, exhibit either a past "Big Bang" or a future "gravitational collapse" or both.* In a nutshell, *rigidly enforcing purely Newtonian gravity by using the Robertson-Walker metric form for cosmological models is a disastrous physics mistake.*

We now treat *the relativistic-gravity version* of this model, where $(dr/dt)^2 \simeq ((c/(3r_s))r)^2$ as $r \to 0$ (see Eq. (3.4c)). Therefore we find solutions of the two differential equations $dr_I/dt = +(c/(3r_s))r_I$ and $dr_D/dt = -(c/(3r_s))r_D$ that also satisfy $r_I(t) \to 0$ and $r_D(t) \to 0$,

$$r_I(t) = r_I(0)\exp(ct/(3r_s)) \;\text{ and }\; dr_I/dt = c(r_I(0)/(3r_s))\exp(ct/(3r_s)) \;\text{ for }\; t \le 0 \;\&\; r_I(0) \to 0, \;\text{ and,}$$

$$r_D(t) = r_D(0)\exp(-ct/(3r_s)) \;\text{ and }\; dr_D/dt = -c(r_D(0)/(3r_s))\exp(-ct/(3r_s)) \;\text{ for }\; t \ge 0 \;\&\; r_D(0) \to 0, \tag{3.6b}$$

where the inflationary $r_I(t) = r_I(0)\exp(ct/(3r_s))$ *for* $t \le 0$ *increases exponentially with time* t*, and the deflationary* $r_D(t) = r_D(0)\exp(-ct/(3r_s))$ *for* $t \ge 0$ *decreases exponentially with time* t. Since $|dr_I/dt| \le c(r_I(0)/(3r_s))$ and $r_I(0) \to 0$, $|dr_I/dt| < c$, and since $|dr_D/dt| \le c(r_D(0)/(3r_s))$ and $r_D(0) \to 0$, $|dr_D/dt| < c$.

We also note that both the inflationary and the deflationary $r \to 0$ *asymptotic forms* $r_I(t)$ *and* $r_D(t)$ *manifest entirely positive acceleration.* This is *a prime example* of the fact that *gravitational time dilation, when it is strong enough, provokes motion opposite to that associated with Newtonian gravity.* It is apparent that proper understanding of the universe's inflationary era can't be attained without appreciation of the fundamental underlying role of gravitational time dilation.

At the same time, *it is trivially obvious* that the metric condition $g_{00}(x) = 1$ for all x, first introduced by Friedmann in 1922, *and an absolutely integral part of the Robertson-Walker metric form,*

$$(c\,d\tau)^2 = (c\,dt)^2 - (R(t))^2\big[(1-kr^2)^{-1}(dr)^2 + r^2\big((d\theta)^2 + (\sin\theta\,d\phi)^2\big)\big],$$

(see Eq. (1.6a)), *utterly and completely eliminates gravitational time dilation,* which is,

$$\big[(\text{the tick rate of the clock at } x_2)/(\text{the tick rate of the clock at } x_1)\big] = \sqrt{g_{00}(x_2)/g_{00}(x_1)}\,,$$

(see Eq. (1.5d)). *Therefore there can be no fundamental understanding of the universe's inflationary era,* among other fascinating relativistic-gravity facts which are closely related to gravitational time dilation, *until the Robertson-Walker metric form ceases to be applied to cosmological models.*

It is mind-boggling that *nowhere* in Weinberg's 657-page tome *is there any mention whatsoever of the trivially-obvious fact that the metric condition* $g_{00}(x) = 1$ *utterly and completely eliminates gravitational time dilation.* Of course *recognition* of this trivially-obvious fact *invalidates entire chapters of Weinberg's tome,* particularly those concerned with cosmology, so there has to be *more than a minor suspicion* that this "oversight" in Weinberg's 657-page tome *occurred on purpose.*

Another strange "oversight" in Weinberg's tome *is the failure to exhibit the result of applying the tome's* page-188, Eq. (8.4.25) *formula for the angular frequency of circular orbits of a class of metrics to the simplest metric of that class,* which is the Droste metric (*mistakenly* called the "Schwarzschild metric" by virtually all textbooks, *including* Weinberg's tome). *That result reveals that the Droste-metric circular-orbit angular frequency is identical to the Newtonian-gravity circular-orbit angular frequency, which obviously casts doubt on the physical validity of the Droste metric.* There has to be *more than a minor suspicion* that Weinberg *purposely* didn't exhibit this result in order *not to be displaying evidence that the Droste metric is unphysical* (which it definitely is, see the discussion in the paragraphs containing Eqs. (2.1i) through (2.1n)).

Finally, when Weinberg presents the evidence in Subsection C of Section 13.5 on page 403 of his tome that the Robertson-Walker metric form solves the Einstein equation for spherically-symmetric and spatially-homogeneous energy-momentum sources, *he "conveniently forgets" to remind the reader that every coordinate transformation of a metric which solves the Einstein equation is also a metric which solves the Einstein equation,* a property which makes the Einstein equation *by itself* all but *useless* for obtaining *definite physical results.* It is, however, straightforward to work out a coordinate transformation of the Robertson-Walker metric form which satisfies Einstein's coordinate condition $\det(g_{\mu\nu}(x) = -1$ *instead of the disastrous Lorentz-covariance incompatible Friedmann condition* $g_{00}(x) = 1$ *satisfied by the Robertson-Walker metric form itself that eliminates gravitational time dilation, sends c to infinity and causes the Einstein equation to vent purely Newtonian gravity.* See Eq. (1.6e) *for the coordinate-transformed Robertson-Walker metric form which satisfies Einstein's coordinate condition* $\det(g_{\mu\nu}(x) = -1$. In a nutshell, Weinberg apparently *purposely leaves the reader with the utterly false impression that the Robertson-Walker metric form alone satisfies the Einstein equation for spherically-symmetric and spatially-homogeneous energy-momentum sources.*

Weinberg's seeming efforts *to hide simple, straightforward facts from the reader's view which could cause the reader to seriously question some of what Weinberg presents* are the utter antithesis of what science is.

In this section we have learned far more relevant facts about cosmology models than are to be found in textbooks such as Weinberg's tome. To begin with, we have learned that the Newtonian-gravity Big Bang *is disastrously physically untenable because* $|dr/dt|$ *is unbounded, which is the most extreme violation of* $|dr/dt| < c$ *conceivable* (see below Eq. (3.6a)). This fact *is apparent as well in the combination of* Eqs. (15.1.20) *and* (15.1.22) on page 472 of Weinberg's tome, but as Weinberg *repeatedly* does with regard *to inconvenient pertinent facts* (such as the trivially-obvious fact that Friedmann's $g_{00}(x) = 1$ metric condition completely eliminates gravitational time dilation) *Weinberg copes by simply turning a blind eye.*

In *contrast,* we have been able to show that the radial speed $|dr/dt|$ of the simple Oppenheimer-Snyder blob's spherical surface satisfies $|dr/dt| < c$ *under all circumstances* when treated using the Birkhoff theorem *and the physically-correct relativistic metric published by Schwarzschild on January 13, 1916.* Denoting the total conserved energy enclosed by the blob's radially freely-falling surface as Mc^2, and the blob's radius as $r(t)$, we also showed that as that radius goes to infinity, $|dr/dt| \to c\sqrt{1-K^2}$, where $K = \exp(C)$ is a dimensionless positive constant of integration. However, when $K^2 > 1$, the blob is gravitationally bound and its radius cannot attain arbitrarily large values. When $|1-K^2| \ll 1$, we can interpret an arbitrarily small part of the blob's surface as *a nonrelativistic particle of mass m and total nonrelativistic energy E* (E is the sum of that infinitesimal particle's nonrelativistic positive kinetic energy and its nonrelativistic negative gravitational potential energy), where $K^2 = 1 - (2E/(mc^2))$, so that $c\sqrt{1-K^2} = \sqrt{2E/m}$.

In addition to the radial *speed* $|dr/dt| = c\sqrt{1-K^2}$ of this relativistic blob's spherical surface in the limiting case of arbitrarily large values of its radius $r(t)$, we have obtained that the radial *acceleration* d^2r/dt^2 of this relativistic blob's spherical surface in the limiting case of arbitrarily large values of its radius $r(t)$ is $d^2r/dt^2 = (GM/r^2)(2-3K^2)$, *which doesn't agree with the well-known nonrelativistic negative acceleration* $d^2r/dt^2 = -(GM/r^2)$ *unless* $|1-K^2| \ll 1$. In fact, when $0 < K^2 < \frac{2}{3}$, *which makes the blob's radial surface speed $c\sqrt{1-K^2}$ a considerable fraction of c, the negative acceleration turns positive, a classic consequence of sufficiently strong gravitational time dilation.* This result is very interesting indeed, since it is now known that the acceleration of the universe *is actually positive,* and the current "explanation" of that observation *is a completely ad hoc epicycle* that, *even worse,* postulates a $\lambda g_{\mu\nu}$ term in the Einstein equation which scuppers the verification that the Einstein equation reproduces Newtonian gravity in the weak-field and static energy-density case.

We have furthermore obtained from this relativistic-gravity model that in the limiting case of arbitrarily small values of the blob's radius $r(t)$, $(dr/dt)^2$ is asymptotic to $((c/(3r_s))r(t))^2$ (where the Schwarzschild radius r_s is $(2GM/c^2)$), which implies the two differential equations $dr_I/dt = (c/(3r_s))r_I$ and $dr_D/dt = -(c/(3r_s))r_D$ for asymptotically small $r_I(t)$ and $r_D(t)$. The asymptotically small solution for $r_I(t)$ is $r_I(t) = r_I(0)\exp(ct/(3r_s))$ for $t \leq 0$ and $r_I(0) \to 0$, and the asymptotically small solution for $r_D(t)$ is $r_D(t) = r_D(0)\exp(-ct/(3r_s))$ for $t \geq 0$ and $r_D(0) \to 0$. The *inflationary* asymptotically small solution $r_I(t)$ *increases exponentially with increasing time,* while the *deflationary* asymptotically small solution $r_D(t)$ *decreases exponentially with increasing time. Both $r_I(t)$ and $r_D(t)$ have positive acceleration due to very strong gravitational time dilation.* It is of course *well known* that the early universe was *inflationary* when it was sufficiently small, and these solutions of *the relativistic-gravity* Oppenheimer-Snyder model show that *this was a consequence of very strong gravitational time dilation.*

This *relativistic-gravity* Oppenheimer-Snyder model shows the supreme importance of gravitational time dilation both to the observed acceleration of the expansion of the universe and to the inflationary character of the universe when it was sufficiently small. Of course this *relativistic-gravity* model *only permits speeds strictly less than c under all circumstances, so nothing remotely like a Big Bang, with its unbounded speeds, could ever occur.* Also, *the relativistic-gravity universe has existed forever,* albeit in a state of extreme gravitational time dilation "suspended animation" when it was far smaller than its Schwarzschild radius $r_s = (2GM/c^2)$. With no Big Bang whatsoever, and its having existed forever, there is absolutely no reason why the universe should not have a surplus of particles over antiparticles. In the next, very short section we round out this picture with additional broad-brush ideas of a more speculative nature.

4. Further broad-brush ideas of a more speculative nature about the universe's evolution

A key property of the universe is that it is expanding, so it presumably was arbitrarily compact and dense in the sufficiently remote past; in particular it was far inside its Schwarzschild radius $r_s = (2GM/c^2)$, where Mc^2 is the universe's conserved energy. In that era its behavior would have been dominated by gravitational time dilation, so all physical processes would have been greatly slowed and radiation frequencies greatly reduced; it would have been dark and cold with almost paralyzed physical processes, even its expansion rate would have been greatly reduced. *Going further back in time only further accentuates its "suspended animation" character.* Going *forward* in time eventually brings it to a radius of the order of r_s. The accompanying *decrease in gravitational time dilation* would have allowed *its expansion rate to increase,* which would have *still further reduced gravitational time dilation,* causing its expansion rate *to increase still further,* etc.

Thus when the universe reached a radius of the order of r_s it was on the cusp *of a rapid increase in its expansion rate.* Physical process rates in that era *would have also rapidly increased* as the dead hand of extreme gravitational time dilation fell away. *Notwithstanding its rapid expansion, the universe would still have been vastly, vastly more compact and dense than today's universe,* which has undergone billions of years *of additional expansion.* So *dense* a universe, *which was liberated from extreme gravitational time dilation,* would have been able to give birth to every conceivable kind of young star at an utterly enormous rate, with particular emphasis on immensely massive, extremely hot and short-lived giants. However *considering how much even denser than that the universe was when it neared the liberating radius* r_s, only a quite small fraction of its matter *would have been able to participate directly in those fireworks; by far the greatest part of its matter would have been compelled to take the form of primordial black holes* (but do bear in mind that black holes *absolutely do not have event horizons*). However those primordial black holes *profoundly modulated* the spectacular star-formation fireworks underway by, for example, becoming the active nuclei of galaxies, with the primordial black holes of lesser mass *being utterly crucial to galaxy formation by supplying the necessary cold, dark gravitational "glue".* When the compact, dense universe's star-formation fireworks was at its zenith, the universe was obviously extremely hot, so the black-body cosmic microwave background is the frequency-downshifted remnant of the universe's immense black-body radiation of that intense star-formation era. With its continued expansion, the universe's density of course diminished, diminishing its rate of star and galaxy formation. The James Webb Space Telescope may possibly be registering evidence of rapid galaxy formation in the early universe.