More Than ‘The Chromatic Number of the Plane’

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Abstract

We generalize the famous ‘Chromatic Number of the Plane’. For every finite metric space we define a similar question. We show that 15 colors suffice to generate a coloring of the plane without monochromatic distances 1 or 2.

1 Introduction

Assume two points. What is the least number of colors to color $\mathbb{R}^2$ to ensure that the points have different colors, if we put them anywhere on the plane always with the same distance? That is the well-known problem of ‘The Chromatic Number of the Plane’. It is also known under the name of ‘Hadwiger-Nelson’. Recently it was discovered that this number is at least 5. See [1]. It is easy to demonstrate that 7 colors suffice. See [2].
Here we show a way to generalize this problem to each finite metric space. Further, we prove that 15 colors suffice for a coloring of the plane in the case of the domain$\{1,2,3\}$.

2 Generalization

We need some definitions.

Definition 1. We call f a *distance preserving function* if and only if (X,d_X) and (Y,d_Y) are metric spaces, $\mathsf{f}: X \to Y$, and it holds $d_X(a,b) = d_Y(\mathsf{f}(a),\mathsf{f}(b))$ for all $a,b \in X$.

We call every finite metric space a *set of vertices.*

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Definition 2. Let V be a set of vertices. Let $\mathcal{X}$ (V) be the minimum number of needed colors such that every element of $\mathsf{f}(\mathsf{V})$ have a different color, if $(\mathsf{f} : (\mathsf{V}, d_{\mathsf{V}}) \to (\mathbb{R}^2, ||))$, and f runs through all distance preserving functions, where d_{V} is a metric on V. In the case that there is no such number, or if there is no distance preserving function f, we would write $\mathcal{X}(\mathsf{V}) = \infty$.

Let us define a subset of natural numbers. Let n be a natural number.

Definition 3. We call $\mathbb{N}_n$ the subset of the first n natural numbers, i.e.
$\mathbb{N}_n := \{1, 2, 3, \ldots n-2, n-1, n\}$.

We generate some coloring constants.

Let $\mathcal{X}_n$ be the least number of colors to color the plane such that every element of $\mathsf{f}(\mathbb{N}_n)$ have a different color, where $(\mathsf{f} : (\mathbb{N}_n, |) \to (\mathbb{R}^2, ||))$, and f runs through all distance preserving functions. In the case that there is no such number we would write $\mathcal{X}_n = \infty$.

Therefore, we have generated a monotone increasing sequence $(\chi_n)(n \in \mathbb{N})$.

Remark 1. Note that for a set of vertices which has exactly two elements, it is the problem of the 'Chromatic Number of the Plane', i.e. $\mathcal{X}_2 = \mathcal{X}$.

Proposition 1.

$$\text{If } \mathsf{V} := \{1, 2, 3\} = \mathbb{N}_3 \subset \mathbb{R} \quad \text{it holds} \quad 5 \leq \mathcal{X}(\mathsf{V}) \leq 15$$

Proof. The left inequality is true since our problem contains 'The Chromatic Number of the Plane'. The right inequality is proved now. Please see Figure 1. There we show a coloring of the plane with 15 colors.

3 The Proof

Figure 1:

We see a coloring of the plane with 15 colors.

											1								
1															1				
				1															1
								1											
												1							
	1															1			
					1														
									1										
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14	15	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	1	2	3
						1													
										1									

Above we show a cut-out of a coloring of $\mathbb{R}^2$ without monochromatic distances one or two. We name the colors by $1,2,3,\dots,14,15$. We start with one arbitrary square, and color it with 1. The following squares are colored with $2,3,\dots,14,15$. Then we start with 1 again. The squares have sidelengths of $\frac{1}{2}\cdot\sqrt{2}$ (i.e. the diagonals have lengths 1). The upper side and the right side of each square (including the corner) are colored like the interior. It holds that three sidelengths of a square are larger than 2 by Lemma 1.

Lemma 1. $\frac{3}{2}\cdot\sqrt{2} > 2$.

Proof. It holds $\frac{3}{2}\cdot\sqrt{2} > 2 \Leftrightarrow \frac{9}{4}\cdot 2 > 4$. □

The coloring of one row, which we call 'row Nr. 1', is completely shown. In the other rows, only the squares with color 1 are named. Above row Nr. 1, in row Nr. 2 we shift to the left the coloring of row Nr. 1 by 4, in row Nr. 3 we shift it by 8, in row Nr. 4 by 12, in row Nr. 5 by 1, in row Nr. 6 by 5, in row Nr. 7 by 9, etc.
In the row below row Nr. 1, the colorings are shifted to the left by 11, in the next row they are shifted by 7, etc. .
Also we see a sloped line between two squares, both are colored with color 1. Their distance is two. The endpoints and the midpoint have different colors due to our coloring.
By this way, we color the entire plane with 15 colors without monochromatic distances one or two. □

The following conjecture in the case of correctness may be difficult to prove.

Conjecture 1. Let V be a set of vertices. Then $\mathcal{X}(\mathsf{V})$ is a natural number.

The following conjecture is included by the previous one, and it seems to be true.

Conjecture 2. For all numbers n, χ_n is a natural number.

The next conjecture will be easy to prove, if it is true at all.

Conjecture 3. $\mathcal{X}(\{1,2,3\})$ is less than 15.

Question 1. We ask for the values of χ_n, for $n > 1$.

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References

[1] Aubrey de Grey: *The Chromatic Number is at least 5*, Geombinatorics XXVIII, Issue 1 (2018)

[2] Alexander Soifer: *The Mathematical Coloring Book. Mathematics of Coloring and the Colorful Life of its Creators*, New York Springer (2009)