A Short Note on the de Broglie Wavelengths of Composite Objects

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ABSTRACT

The de Broglie wavelength of an object is inversely proportional to the object's mass and relative velocity multiplied by Planck's constant. A composite object, such as an atom composed of a proton and electron, possesses a total mass and therefore a composite de Broglie wavelength. Mass is simply additive, although the relation for combining de Broglie wavelengths does not seem to have been stated anywhere in the literature. In this paper, we derive the relation for combining the de Broglie wavelengths of an object's component parts into the composite de Broglie wavelength of the object as a whole. In so doing, we discover an interesting fact concerning de Broglie waves – that they do not undergo ordinary wave interference. In general, when we combine two waves to form a composite wave, the composite wave is the algebraic sum of the two original waves, point by point in space (Superposition Principle). The wavelength of the resultant wave is ordinarily that of the shorter wavelength wave, measured from adjacent crest to trough. De Broglie waves behave differently, the wavelength of the resultant de Broglie wave being the reciprocal of the sum of the inverses of the component wavelengths, multiplied by a factor to correct for relative velocity. De Broglie wave interference is shown to have a geometric analogue in the form of the Crossed Ladders Theorem. The macroscopic conservation of mass is partly accounted for, in particular its additivity, by interpreting the reciprocal sum formula for the resultant wavelength as the truest value.

Introduction

In 1924, Louis de Broglie, then a French graduate student, proposed in his doctoral dissertation that the wave-particle duality, then known to exist for radiation, was also a characteristic of matter. At the time, his suggestion was highly speculative, since there was yet no experimental evidence for wave-like behavior of electrons or any other particles. De Broglie described his realization with these words:

> After the end of World War I, I gave a great deal of thought to the theory of quanta and to the wave-particle dualism ... It was then that I had a sudden inspiration. Einstein's wave-particle dualism was an absolutely general phenomenon extending to all physical nature.

Since the universe consistes entirely of matter and radiation, de Broglie's hypothesis is a fundamental statement about the grand symmetry of nature.

The de Broglie Hypothesis

De Broglie stated his proposal with the following simple equations for the frequency and wavelength of electron waves, which are referred to as the *de Broglie relations*:

$$f = \frac{E}{h}$$

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

where E is the total energy, h is Planck's constant, p is the momentum, and λ is called the *de Broglie wavelength* of the particle. For photons, these same equations result directly from Einstein's quantization of radiation $E = hf$ and the equation $E^2 = (pc)^2 + (mc^2)^2$ for a particle of zero rest energy $E = pc$ as follows:

$$E = pc = hf = \frac{hc}{\lambda}$$

By a more indirect approach using relativistic mechanics, de Broglie was able to demonstrate that the de Broglie relations also apply to particles with mass.

Particle-Wave Interference

In a brief note in the August 14, 1925 issue of the journal *Naturwissenschaften*, Walter Elsasser, at the time a student of Franck's (of the Frank-Hertz experiment), proposed that the wave effects of low-velocity electrons might be detected by scattering them from single crystals. The first such measurements of the wavelengths of electrons were made in 1927 by Davisson and Germer, who were studying electron reflection from a nickel target at Bell Telephone Laboratories.

The wave properties of neutral atoms and molecules were first demonstrated by Stern and Estermann in 1930 with beams of helium atoms and hydrogen molecules diffracted from a lithium fluoride crystal.

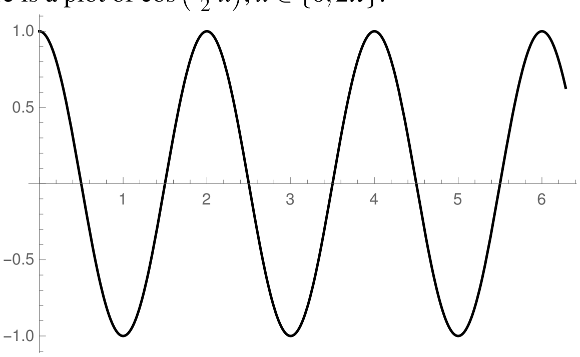
The constructive and destructive interference exhibited in particle beams may lead one to jump to the conclusion that de Broglie waves combine in composite objects through ordinary wave interference. But a straightforward calculation shows that that cannot be the case. Component de Broglie waves couple, but do not superimpose, in the formation of resultant de Broglie waves.

An Observation on Ordinary Wave Addition

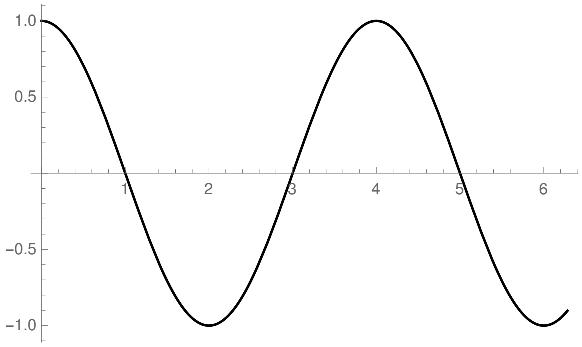
It's usually completely ignored or glossed over in elementary physics courses how wavelengths interact during wave addition. Let's simply look at the addition of two static sinusoidal waves of wavelength λ using the equation

$$y(x) = A\cos\left(\frac{2\pi}{\lambda}x\right)$$

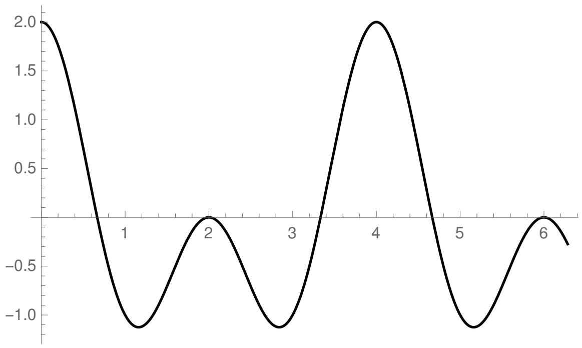
Here is a plot of $\cos\left(\frac{2\pi}{2}x\right)$, $x \in \{0, 2\pi\}$:



Here is a plot of the longer wavelength $\cos\left(\frac{2\pi}{4}x\right)$, $x \in \{0, 2\pi\}$:



And here is a plot of their sum, $\cos\left(\frac{2\pi}{2}x\right) + \cos\left(\frac{2\pi}{4}x\right)$, $x \in \{0, 2\pi\}$:



Notice that the wavelength of the resultant wave *is* the wavelength of the shorter component wave, though the crests and troughs now occur at variable heights, a pattern which quickly becomes more complex as more component waves are added. But for our purposes, the key observation is that the wavelength is that of the shortest wavelength component wave, measured strictly from crest to trough (*not* measured by *period* of the entire wave form). However, measured by the wave form's period (repeat unit of the entire wave), then the "wavelength" could be considered that of the longer wavelength component wave. The point of this is that if we were to combine masses m_1 and $m_2 = 2m_1$ into a single object, and if the de Broglie waves added that way, one would expect the resultant mass to be extremely close to one of the component objects' masses, depending on how one interprets "wavelength", a violation of mass conservation either way. Alternatively, one could consider some type of average of these wavelengths as the "effective wavelength", but that would also violate conservation of mass. None of the above is the case. In fact, *the wavelength must get shorter than any contributing component wavelength*. Thus, de Broglie waves clearly do not combine through ordinary wave addition. The actual way that de Broglie wavelengths combine is easily derived using straightforward algebra from relations given in any modern physics textbook, as is shown next.

The Two-Body Case

Geometrically combining the parts of an object into a whole, the total mass is the simple addition of the component masses:

$$m_c = m_1 + m_2$$

where m_c is the composite or total mass. The de Broglie relations for each part can be given:

$$m_1 = \frac{h}{\lambda_1 v} \qquad m_2 = \frac{h}{\lambda_2 v}$$

whereas for the composite de Broglie wavelength we have

$$\lambda_c = \frac{h}{(m_1 + m_2)v} = \frac{h}{(\frac{h}{\lambda_1 v} + \frac{h}{\lambda_2 v})v} = \frac{1}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}} = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$$

The Three-Body Case

In solving for the 3-body case, we assume that 2 parts have combined as before, then combine with a third part, i.e.,

$$\lambda_c = \frac{\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}\cdot\lambda_3}{\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}+\lambda_3} = \frac{\lambda_1\lambda_2\lambda_3}{\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3}$$

The General Case

Arguing inductively, the general relation for the composition of de Broglie wavelengths for an arbitrary number of component wavelengths is:

$$\lambda_c = \frac{\prod_{k=1}^{N}\lambda_k}{\sum_{\substack{j=1 \\ i_1<i_2<\cdots<i_{N-1}}}^{\binom{N}{N-1}} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{N-1}}} = \frac{\prod_{k=1}^{N}\lambda_k}{\sum_{\substack{j=1 \\ i_1<i_2<\cdots<i_{N-1}}}^{N} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{N-1}}} = \frac{1}{\frac{1}{\lambda_{i_1}}+\frac{1}{\lambda_{i_2}}+\cdots+\frac{1}{\lambda_{i_N}}}$$

This analysis shows that the wavelength of the resultant de Broglie wave is the reciprocal of the sum of the inverses of the component wavelengths. This relation holds deep implications for the meaning of mass. Notice that the primitive concept of mass has been eliminated, and the problem of explaining composite mass is thus shifted to the problem of explaining why de Broglie waves combine the way they do. Why do de Broglie waves combine *this* way? That is the same as asking why mass is (classically) simply additive or even what total mass *is*.

There is more discussion of the functional form of the result in reference 4. It turns out that a fairly common name for it is *reciprocal addition*. The most important property of the operation is that its application to any set of real numbers of the same sign creates an output which is less in absolute value than the originals. The only other place that I personally have seen an equation like it in physics is in the calculation of resistance across a circuit of parallel resistors in electronics.

The First Refinement

Now there is one nagging issue to be addressed. It would be desirable that the negative correlation between mass and the de Broglie wavelength be maintained for all velocities. That is a glaring deficiency in the de Broglie relation given as $\lambda = h/mv$. You can see that the case $v = 0 \Rightarrow \lambda = \infty$, and therefore the mass could have *any* value, and a particle with nonzero rest mass would have no wavelength, thus the correlation is broken. It would be desirable that $\lambda = \infty$ only when $m = 0$, and be finite when $v = 0$ and we have nonzero rest mass. Here we'll derive a modified form of our original equation that fits those specifications. As you will see, it turns out that our original equation only applies to particles at rest – one small modification is needed to make it applicable to moving particles.

For low energies where relativistic effects can be ignored, the de Broglie relation can be rewritten in terms of the kinetic energy $E_k = \frac{1}{2}mv^2 = p^2/2m$ as follows:

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}}$$

To find the equivalent expression that covers both relativistic and nonrelativistic speeds, begin with the relativistic equation relating the total energy to the momentum:

$$E^2 = (pc)^2 + (mc^2)^2$$

Writing E_0 for the rest energy mc^2 of the particle, this becomes

$$E^2 = (pc)^2 + E_0^2$$

Since the total energy $E = E_0 + E_k$, this becomes

$$(E_0 + E_k)^2 = (pc)^2 + E_0^2$$

which, when solved for p, yields

$$p = \frac{(2E_0E_k + E_k^2)^{1/2}}{c}$$

from which $\lambda = h/p$ gives

$$\lambda = \frac{hc}{\sqrt{2E_0E_k + E_k^2}}$$

This equation is superior to $\lambda = h/mv$ simply for its ability to resolve the rest mass issue. Next our strategy is to substitute back in expressions for E_0 and E_k that involve mass $(m_1 + m_2)$ and then perform the same trick we did in the derivation of our original result. Proceeding:

$$\begin{aligned}
\lambda &= \frac{hc}{\sqrt{2E_0E_k + E_k^2}} \\
&= \frac{hc}{\sqrt{2(m_1+m_2)c^2\frac{1}{2}(m_1+m_2)v^2 + \left[\frac{1}{2}(m_1+m_2)v^2\right]^2}} \\
&= \frac{hc}{\sqrt{(m_1+m_2)^2v^2c^2 + \frac{1}{4}(m_1+m_2)^2v^4}} \\
&= \frac{hc}{(m_1+m_2)v\sqrt{c^2 + \frac{1}{4}v^2}} \\
&= \frac{h}{(m_1+m_2)v\sqrt{1 + (\frac{v}{2c})^2}}
\end{aligned}$$

Now using the equivalences

$$\frac{h}{(m_1+m_2)v} = \frac{h}{(\frac{h}{\lambda_1 v} + \frac{h}{\lambda_2 v})v} = \frac{1}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}}$$

we can rewrite our original result thus:

$$\boxed{\lambda_c = \frac{1}{(\frac{1}{\lambda_1} + \frac{1}{\lambda_2})\sqrt{1 + \left(\frac{v}{2c}\right)^2}}}$$

The above equation accounts for cases in which the composite masses are moving and for those at rest. When $v = 0$, we have $\lambda \neq \infty$, but rather some finite number. This is much more pleasing. In addition, when $v = 0$, the equation reduces to our original result, demonstrating that we had derived the correct solution for masses at rest. It also generalizes to N component wavelengths simply by adding more terms to the reciprocal addition.

It's interesting to note that in the hypothetical case of a composite mass moving at light speed, this equation predicts

$$\lambda_{v=c} = \sqrt{\frac{4}{5}}\frac{1}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_N}}$$

A Word on Notation

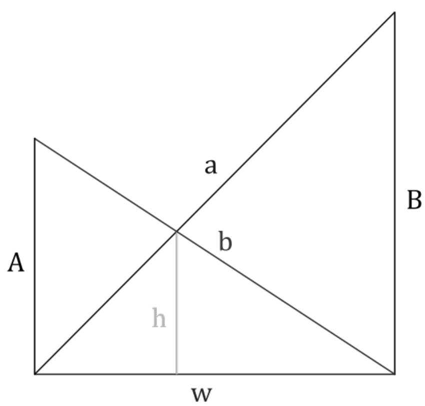
Drawing from the work given in reference 4, let's use the symbol Λ to denote the composite wavelength, and the symbol $\bigoplus$ to denote a reciprocal sum over an arbitrary number of N terms such that $\bigoplus(\lambda_1, \lambda_2, \ldots, \lambda_N) = \lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_N$. Then our relation for the resultant de Broglie wavelength can be stated as follows:

$$\boxed{\Lambda(\lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{\bigoplus(\lambda_1, \lambda_2, \ldots, \lambda_N)}{\sqrt{1 + \left(\frac{v}{2c}\right)^2}} = \frac{1}{\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_N}\right)\sqrt{1 + \left(\frac{v}{2c}\right)^2}} \quad \text{where } v = N\bigoplus(v_1, v_2, \ldots, v_N)}$$

A Geometric Interpretation of the Reciproal Sum Operation

A geometrical interpretation can be made of the reciprocal sum operation $A \oplus B$, as it is the solution for the height of the point where the ladders cross in the Crossed Ladders Theorem (do not confuse this with the crossed ladders problem, which makes use of the crossed ladders theorem in its solution).

Crossed Ladders Theorem: Two ladders of lengths a and b lie oppositely across an alley, as shown in the figure. The ladders cross at a height h above the alley floor. What is the value of h in terms of A and B, A being the height at which b contacts its opposing wall and B being the height at which a contacts its opposing wall? It turns out that $h = A \oplus B$.



To prove this, divide the baseline into two parts at the point where it meets h, and call the left and right parts w_1 and w_2, respectively. The angle where a meets w is common to two similar triangles with bases w and w_1. The angle where b meets w is common to two similar triangles with bases w and w_2. This tells us that,

$$\frac{w_1}{h} = \frac{w}{B} \quad \Rightarrow \quad \frac{w_1}{w} = \frac{h}{B}$$
$$\frac{w_2}{h} = \frac{w}{A} \quad \Rightarrow \quad \frac{w_2}{w} = \frac{h}{A}$$

Next, since we note that since $w = w_1 + w_2$, we have

$$\frac{w_1}{w} + \frac{w_2}{w} = 1$$

Now by simple substitution we have that

$$\frac{h}{B} + \frac{h}{A} = 1$$

$$\frac{1}{h} = \frac{1}{B} + \frac{1}{A}$$

$$h = \frac{1}{\frac{1}{A} + \frac{1}{B}}$$

■

Interpretation of the Reciprocal Sum as Truest Value

So far no explanation has been given as to *why* de Broglie waves sum as they do. One possible way to answer the question as to why, is to explain why it ought be so.

The reciprocal sum is closely related to the concept of harmonic mean. The harmonic mean is one of several kinds of average. Typically, it is appropriate for certain situations when the average of rates is desired. The harmonic mean can be expressed as the "reciprocal of the arithmetic mean of the reciprocals of the given set of observations".

The harmonic mean H of the positive real numbers $x_1, x_2, \ldots, x_2$ is defined to be

$$H = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_2}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}} = \left(\frac{\sum_{i=1}^{n} x_i^{-1}}{n} \right)^{-1}$$

The third formula in the above equation expresses the harmonic mean as the reciprocal of the arithmetic mean of the reciprocals.

In certain situations, especially many situations involving rates and ratios, the harmonic mean provides the *truest average*. For instance, if a vehicle travels a certain *distance* at a speed x and then the same *distance* again at a speed y, then its average speed is the harmonic mean of x and y, and its *total travel time is the same as if it had traveled the whole distance at that average speed*. However, if the vehicle travels for a certain amount of *time* at a speed x and then the same amount of *time* at a speed y, then its average speed is the arithmetic mean of x and y, and the *total distance traveled is the same as if it had traveled the whole time at that average speed*. A key realization is that the concept of "object" is defined by proximity in space of its parts, i.e., distance, not proximity in time. And for this reason the harmonic mean would provide the truest average speed of a multipartite object (with parts moving at arbitrary, possibly different, speeds).

If we leave the division by n out of the definition of the harmonic mean, we are left with the reciprocal sum. So in this sense, as the harmonic mean provides the truest average speed, the reciprocal sum itself provides the truest de Broglie wavelength of the composite object. The reciprocal sum does not provide the truest velocity. If a bipartite object has two parts moving at the same speed, the speed of the object is not twice the speed of its parts. However, if a bipartite object has two parts that have the same mass, the object has twice the mass of its parts. And the de Broglie relation relates mass to wavelength. So we have

$$\lambda_{\text{true}} = \frac{1}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_N}}$$

Now this begs the question, why don't ordinary waves sum such that this "truest wavelength" is the resultant? It has to do with the fact that that there are no parts left after ordinary wave addition. The component waves no longer exist. On the contrary, in a hydrogen atom, an electron and a proton still exist, there is no "electron-proton", though the atom as a whole has a total mass.

De Broglie waves do not add, and in their pairing the truest value of the resultant wavelength is observed.

Davisson and Germer's interference experiments were a reflection of this phenomenon, not ordinary wave addition. The necessity of the reciprocal sum is responsible for the classical additivity of mass, given that the de Broglie relation is true, which negatively correlates mass with de Broglie wavelength.

The mass of an object, with the exception of fundamental particles, is a vibration of nothing that possesses the truest wavelength representing the set of de Broglie wavelengths of the component parts of the object.

This theory applies down to the level of fundamental particles, outwards. Though it accounts for the additivity of mass in systems of fundamental particles, it does not account for mass conservation in the existential sense, nor does it account for the mass of a fundamental particle. The theory squarely places mass, in the existential sense, on fundamental particles, a situation which is reminiscent of the René Magritte painting "The Treachery of Images", also known as "This is not a Pipe".

How Many de Broglie Waves Does a Multipartite Object Possess?

It would be presumptuous to assert that any multipartite object possesses just one de Broglie wavelength. As any subset of fundamental particles composing the object also has a total mass, each subset must have an associated de Broglie wave. The total number of de Broglie waves the object possesses (and the total number of submasses) is given by:

$$T = \sum_{k=1}^{N} \binom{N}{k} = 2^N - 1$$

where the object possesses N fundamental particles chosen k at a time. This recognizes the fact that every subpart has a longer de Broglie wavelength than any of its associated superparts. Of the T de Broglie waves, N of them belong to fundamental particles, while $2^N - N - 1$ of them have a virtual character.

References

1. Paul Tipler and Ralph Llewellyn. *Modern Physics*. W. H. Freeman and Company, New York, 1999.
2. De Broglie, Louis. *Matter and Light: The New Physics*. Dover, New York, 1939.
3. Resnick, R. and D. Halliday. *Basic Concepts in Relativity and Early Quantum Theory*. 2d ed., Wiley, New York, 1992.
4. I found an interesting thread on the internet concerning the functional form of my result. I've tried my best to reproduce the thread here. The author appears to have remained anonymous. But his results are quite interesting:

   I have recently found some interesting properties of the function/operation:

   $x \oplus y = \frac{1}{\frac{1}{x}+\frac{1}{y}} = \frac{xy}{x+y}$ where $x, y \neq 0$.

This is a formula often used in physics, for calculating equivalent resistance/capacitance in circuits. I've heard this referred to by my teacher as "reciprocal addition" but have not been able to find much significant mention of this operation outside of its direct connection to circuits and an isolated youtube video.

Since there is no common name/use for this operation, I'm asking for help in finding properties of this function.

to denote that the operations ("o-plus" and "o-minus" — name from the youtube channel 3Blue1Brown) and proven a few simple properties of it, including many similarities, but also differences, with addition.

For example, the $\oplus$ ("o-plus") operation is:

1. Associative (EDIT: this only seems fully true if we extend the domain to the projective reals, and say ∞ is the identity and $a \oplus 0 = 0$ while $a \oplus -a = \infty$)
2. Commutative
3. Distributive with multiplication

However, it does not have an identity (such as zero for addition or 1 for multiplication) at least among the real numbers. EDIT: adding ∞ to the domain and working under the projective reals gives ∞ as the identity.

I've done a lot of playing around with simple properties of $\oplus$ and found many interesting things about it, such as ways to do arithmetic using it – for instance:

$\frac{a}{x} \oplus \frac{a}{y} = \frac{a}{x+y}$

Which means one would "o-plus" fractions by finding a common numerator instead of a common denominator.

Also, just as repeated addition is a form of multiplication($a+a+a+a=4a$), repeated "o-plussing" is a form of division.

$a \oplus a \oplus a \oplus a = \frac{a}{4}$

There are a multitude of cool properties that come out of the definition of this operation that I've defined here

$\sqrt[a]{x}\sqrt[b]{x} = \sqrt[a \oplus b]{x}$

One other thing I found was the derivative of two "o-plussed" functions:

$\frac{d}{dx}(f(x) \oplus g(x)) = \frac{f'(x)g(x)^2+g'(x)f(x)^2}{(f(x)+g(x))^2}$

which looks over-complicated, and I think it is. I'd like to find a way to describe this function that actually includes the $\oplus$ operation itself, but have been unable to find one. However, there is a nice parallel which I found which makes more sense of this derivative rule:

$f \oplus g = \frac{1}{\frac{1}{f}+\frac{1}{g}} = \frac{fg}{f+g} = \frac{fg}{f+g}\frac{(f+g)}{(f+g)} = \frac{fg^2+gf^2}{(f+g)^2}$

which parallels perfectly with what I said before,

$(f \oplus g)' = \frac{f'g^2+g'f^2}{(f+g)^2}$.

I've also found that $a \oplus b$ is the solution to the "crossed ladder problem" The Crossed Ladder Problem is to find the height, h, given a and b, Which asks to find h, given A and B as heights to "cross ladders" over.

This leads me to believe it would be natural to define $a \oplus 0$ to be zero, since the lines given those heights would intersect at a height of 0. This is not a formal 'definition', just pattern-continuation. One might also define $a \oplus -a$ to be 'infinite', since that seems to solve the associativity problem in some cases, though this may not rigorously work...

I have found a really cool connection between this operation and another area: Since **the oplussing of two real numbers creates an output which is less than both of the originals**, I wanted to see what conditions were necessary for the following to be true:

$x \oplus y = x - y$

and the result was fantastic (derivation below)!

$x \oplus y = x - y$

$\frac{1}{\frac{1}{x}+\frac{1}{y}} = x - y$

$\frac{xy}{x+y} = x - y$

$xy = x^2 - y^2$

$0 = x^2 - xy - y^2$

and, using the quadratic formula,

$x = \frac{y \pm \sqrt{y^2+4y^2}}{2}$

$x = y\frac{1\pm\sqrt{5}}{2}$

which means that the ratio between x and y is $\varphi \approx 1.618$, the golden ratio!

$x = \varphi y$

And through this, I saw that only the quadratic formula was necessary to find this ratio, and wondered if there was a number (clearly complex) which might satisfy:

$x \oplus y = x + y$

and there is!

$\frac{1}{\frac{1}{x}+\frac{1}{y}} = x + y$

$\frac{xy}{x+y} = x + y$

$xy = x^2 + 2xy + y^2$

$0 = x^2 + xy + y^2$

$x = \frac{-y \pm \sqrt{y^2-4y^2}}{2}$

$x = y\frac{1\pm\sqrt{-3}}{2}$

$x = y\frac{1\pm\sqrt{3}i}{2}$

And this is an extremely interesting number – I call it (and I found one internet forum from 2011 which has also named it this) the "imaginary golden ratio".

$\varphi_i = \frac{1+\sqrt{3}i}{2}$

It has a ton of interesting properties similar to the real golden ratio, and some of its own. I was wondering if anybody could help me uncover a bit of this mystery. Once I am able to write a question which asks more in depth about this number, because I'd like to reignite a discussion that seems to have been left unmentioned since the other forum in 2011.

Here are some properties of the imaginary golden ratio.

$\varphi_i = \cos(\pi/3) + i\sin(\pi/3) = e^{i\pi/3}$

$|\varphi_i| = 1$

$\varphi_i^2 = \varphi_i - 1$

which parallels

$\varphi^2 = \varphi + 1$