On the quantum description of inertia

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Abstract

By extending the Gaussian gravitational flux theory, it is shown that *the inertia of an accelerating object is proportional to the flux of its positional uncertainty in space.* The constant of proportionality is found to be $\rho c/A$ where ρ is the density of the object, c is the speed of light and A is the *diameter of the smallest possible black hole in nature* - itself another constant of nature. In the case of a rectilinear acceleration, the proposed quantum formulation leads to $F_I = -Ma$. In the case of a rotational acceleration, the formulation leads to the rotational inertia $T_I = -I\,\dot{\Omega}$; both cases consistent with the predictions of the classical mechanics. In the case of a disc spinning under a *constant* rotational velocity Ω, the inertia resulting from the centripetal uncertainty of its constituents is found to reduce to $T_I = -\frac{2}{3}MR\,\Omega^2$, again identical to that of the classical mechanics. The latter, however, is found to be an underestimation of the actual quantum-relativistic solution wherein $r\Omega$ is non-negligible compared to c.

1 Background

The c-SRQM theory is developed by considering the special case of a particle having a definite momentum p in vacuum. The theory begins with the postulation that the particle velocity v' and the magnitude of its spatial uncertainty ϵ along the axis x' of an inertial frame $I'(x', t')$ were related as follows [1]:

$$\epsilon = \frac{A}{\sqrt{c^2/v'^2 - 1}} \tag{1}$$

where A is the invariant interval of the spacetime coordinate uncertainty four-vector in spacetime. The wavefunction $\psi(x)$ of the particle was then defined such that the square of its magnitude $|\psi(x)|^2$ had the following character:

$$|\psi(x')|^2 = \begin{cases} 1/\epsilon & |x'| < \epsilon/2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

By examining Eqn 1, it clear that as the velocity $v' \to 0$, i.e. as the particle gets closer and closer to a stationary condition in I', the uncertainty in the position of the particle in that frame also approaches to zero, i.e. $\epsilon \to 0$; and subsequently from Eqn 2, its spatial probability density distribution in that frame approaches to infinity, i.e. $|\psi(x')|^2 \to \infty$. Hence, as expected from the spatial probability density distribution of a stationary particle,

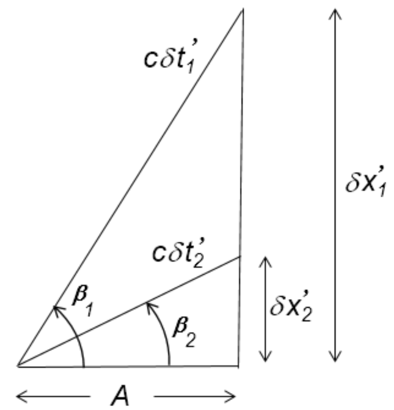

Figure 1: Invariant A of the uncertainty four-vector of two particles with velocities: $v_1' > v_2'$

it peaks where the particle *happens to be* and vanishes anywhere else on x'. On the other hand, as the speed of the particle increases, the length interval ϵ, representing the uncertainty in spatial coordinate of the particle increases progressively in length, such that at the limit velocity $v' = c$, the spatial uncertainty becomes infinite, i.e. $\epsilon \to \infty$, and subsequently, $|\psi(x')|^2 \to 0$. For all other conditions, where velocity of the particle is between two extreme limits, $0 < v' < c$, the spatial uncertainty ϵ and the magnitude of the wavefunction $|\psi(x')|^2$ assume some non-extreme values. From Special Relativity [2], the intervals of proper-time $d\tau$ and coordinate-time dt' are related by the Lorentz transformation as follows:

$$\frac{d\tau}{dt'} = \sqrt{1 - \frac{v'^2}{c^2}} \tag{3}$$

where τ is the proper time of the particle. By re-arrange Eqn 3 we have:

$$\frac{d\tau}{dt'} = \frac{v'}{c}\sqrt{\frac{c^2}{v'^2} - 1} \tag{4}$$

substituting for $\sqrt{c^2/v'^2 - 1}$ from Eqn 1 we get:

$$\frac{d\tau}{dt'} = \frac{v'}{c}\frac{A}{\epsilon} \tag{5}$$

and further by substituting for $1/\epsilon$ from Eqn 2 we arrive at a fundamental relationship between the *time dilation* of theory of Relativity and the *wavefunction* of Quantum Mechanics as follows:

$$\frac{d\tau}{dt'} = \frac{v'}{c}A|\psi(x')|^2 \tag{6}$$

As shown in Fig 1, the inherent spacetime coordinate uncertainties $c\delta t'$ and $\delta x'$ (or ϵ) of such particle in the reference frame I' was shown to constitute a *time-like* four-vector that obey Lorentz transformation and has the invariant interval A:

$$(c\delta t')^2 - (\delta x')^2 = A^2 \tag{7}$$

The locus of uncertainties, as given by Eqn 7, represents a north-opening hyperbola shown in Fig 2. The higher the coordinate velocity v' , the higher the inherent uncertainties $c\delta t'$ and $\delta x'$ in the spacetime coordinates ct' and x' of the particle in the reference frame I'; hence,

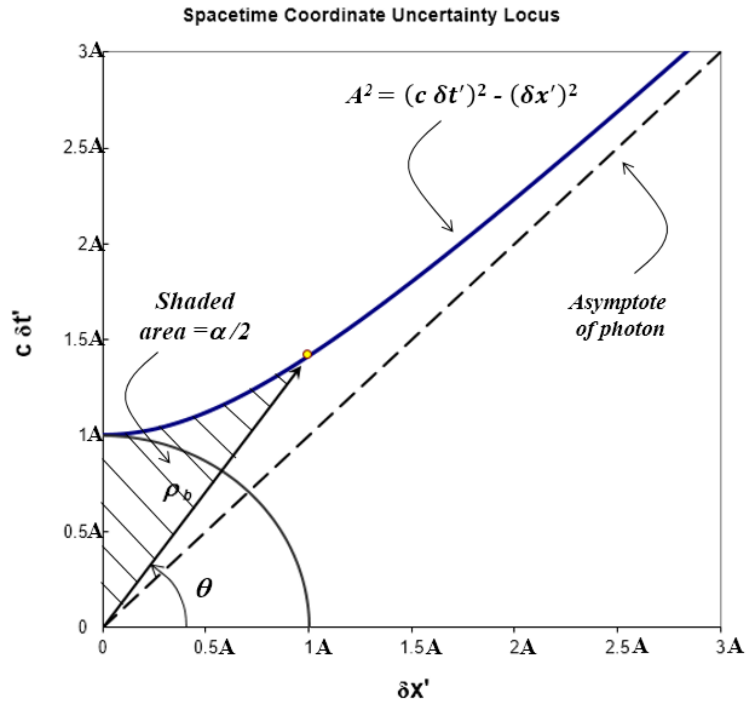

Figure 2: The locus of inherent uncertainties of a point particle in its spacetime coordinate

pushing the level of coordinate uncertainties upward on the locus. The locus asymptotically approaches to the 45° asymptote when $v' \to c$. Eqn 7 can be equivalently written as:

$$(A \cosh \alpha)^2 - (A \sinh \alpha)^2 = A^2 \tag{8}$$

where the hyperbola parameter α is *twice the area under the locus and an intersecting ray from the origin* - as shown in Fig 2. Moreover, it was shown that:

$$\begin{aligned} \delta x' &= \rho_b \cos(\theta) = A \sinh(\alpha) = \frac{Av'}{\sqrt{c^2 - v'^2}} \\ c\delta t' &= \rho_b \sin(\theta) = A \cosh(\alpha) = \frac{Ac}{\sqrt{c^2 - v'^2}} \end{aligned} \tag{9}$$

where the magnitude of ray ρ_b was given by:

$$\rho_b = A\sqrt{\frac{c^2 + v'^2}{c^2 - v'^2}} \tag{10}$$

In addition to being an invariant, the length interval A was shown to have two important features. First, it represented the *diameter of the event horizon of the smallest possible black hole in nature* called Unit Black Hole (UBH), whose mass is $M_0 = Ac^2/(4G)$. Second, it represented the Compton wavelength $A = h/(\bar{m}c)$ of a mass limit $\bar{m}$ below which a quantum particle is physically treated as massless.

The intersection of the ray ρ_b with the *unit circle* is given by the coordinates:

$$\begin{aligned} \cos(\theta) &= \frac{v'}{\sqrt{c^2 + v'^2}} \\ \sin(\theta) &= \frac{c}{\sqrt{c^2 + v'^2}} \end{aligned} \tag{11}$$

The phase angle θ, representing the *instantaneous slope of the world line* of the particle in the spacetime (ct', x'), is given by:

$$\tan\theta = \frac{cdt'}{dx'} = \frac{c}{v'} \tag{12}$$

As shown in Fig 2, the phase angle θ varies between the limits $\pi/4 < \theta < \pi/2$, where the upper limit $\pi/2$ corresponds to the condition of a stationary particle (whose coordinate uncertainty is a minimum) and the lower limit $\pi/4$ corresponds to that of the light particle (whose coordinate uncertainty is infinite prior to being observed). Derivation of the quantized form of these equations can be found in [1].

The c-SRQM theory also constrained the *local acceleration* a of an accelerating particle to the limit:

$$a_u = \frac{c^2}{A} \tag{13}$$

and using above, it also constrained the mass density of the UBH singularity ρ_s to the limit:

$$\rho_s = \frac{3}{2\pi}\left(\frac{l_p}{A}\right)^2 \rho_p \tag{14}$$

where $l_p = \sqrt{G\,h/c^3}$ is the Planck length and $\rho_p = c^5/G^2h$ is the Planck density. Starting from those of the UBH, the mass M_b, event horizon diameter D_b and Hawking temperature T_b of black holes were subsequently quantized as follows [1]:

$$\begin{aligned} M_b &= M_0 + \frac{1}{4}\,b\,m_p \\ D_b &= A + b\,l_p \\ T_b &= \frac{l_p}{2\pi D_b}\,T_p \end{aligned} \tag{15}$$

where $m_p = \sqrt{c\,h/G}$ is the Planck mass, $T_p = m_p c^2/\kappa$ is Planck temperature, κ is the Boltzmann constant and $b = 0, 1, 2, ...$ is the *quantum index of black holes.*

In this article it will be shown that the *probability* of matter ubiety in a region of space is not a mere *mathematical concept*, but rather a *physical entity* which has indeed an upper limit to its rate of variation. More specifically, it will be shown that the *flux* of uncertainty in position of the quantum constituents of an accelerating object is responsible to the force of inertia $F_I = -Ma$, or in the case of accelerated rotation $T_I = -I\dot{\Omega}$.

The flux of uncertainty $\dot{\epsilon}.d\sigma$ defined in this article has some parallel to the Gaussian gravitational flux $d\phi = g.d\sigma$. In the latter, the gravitational acceleration g is integrated over a surface area surrounding a large gravitating body of mass $\mathcal{M}$ to arrive at the celebrated expression $\Phi = 4\pi G\mathcal{M}$ of the gravitational flux. In our theory, similarly, the local acceleration a, which is defined in terms of the rate of change of the positional uncertainty $\dot{\epsilon}$, is integrated over the cross sectional areas of an accelerating body.

2 The physical nature of positional uncertainty

The equation of motion of an accelerating particle, as viewed by an inertial observer in the frame $I'(x', t')$, is given by the derivative of its momentum p. Using the equation of momentum from Special Relativity we then have:

$$\frac{dp}{dt'} = m\,a = \frac{d}{dt'}\left(\frac{mv'}{\sqrt{1-\frac{v'^2}{c^2}}}\right) \tag{16}$$

where a is the local-acceleration of the particle as discussed earlier. Solving for the acceleration a from above, while eliminating m and multiplying the right hand side of the equation by $\frac{c}{A} \times \frac{A}{c}$, we then have:

$$a = \frac{c}{A}\frac{d}{dt'}\left(\frac{Av'}{\sqrt{c^2 - v'^2}}\right) \tag{17}$$

Comparing with the equation of spatial uncertainty $\delta x'$ from Eqn 9, we arrive at the following relation between the rate of variation of the spatial uncertainty and the local acceleration a as follows:

$$a = \frac{c}{A}\frac{d}{dt'}(\delta x') \tag{18}$$

Accordingly, *the local acceleration of an object along a coordinate direction x' is proportional to the rate of change of its spatial uncertainty in that direction.* The constant of proportionality is found to be the ratio of two physical constants c and A. As mentioned earlier, according to the c-SRQM theory the upper limit of the local acceleration a is $a_u = c^2/A$. From Eqn 18 we therefore conclude:

$$\lim_{a \to a_u} \frac{d}{dt'}(\delta x') = c \tag{19}$$

This indicates that *no physical object can accelerate so rapidly that the length interval corresponding to its positional uncertainty expands superluminally.* Therefore, *speed of light is the physical limit to the rate at which the span of positional uncertainties of physical objects in space can alter.* From this we also conclude that the *probabilistic ubiety of matter in space is not a mathematical concept only, but rather a physical entity which has an upper limit to its rate of variation.* Lastly, since acceleration is a vector quantity, from Eqn 18 we conclude that *the rate of change of positional uncertainty is a vector quantity as well* - defined by its magnitude and the direction along which the variation is occurring. The magnitude of the spatial uncertainty vector is given by Eqn 9 and its direction by the unit vector $v'/|v'|$. Several applications of the theory will be discussed in the following sections.

3 Uncertainty variation in perpetual free fall

As shown in Fig 3, consider a satellite on a stable orbital radius R from Earth $\mathcal{M}$. We now want to demonstrate that the circular motion of the satellite leads to a varying uncertainty in its radial position $d(\delta x'_r)/dt'$ thereby generating a gravitational field, equal and opposite to that of Earth $g = G\mathcal{M}/R^2$. The orbital velocity of the satellite in a circular orbit R is given by $v' = \sqrt{G\mathcal{M}/R}$. Since the latter is constant in a stable circular orbital motion, therefore, the time variation of the positional uncertainties in the *tangential* direction $\delta x'_t$ from Eqn 18 is zero:

$$\frac{d}{dt'}\left(\frac{Av'}{\sqrt{c^2 - v'^2}}\right) = 0 \tag{20}$$

Recalling the spatial uncertainties were found to be vector quantities, as shown in Fig 3b,

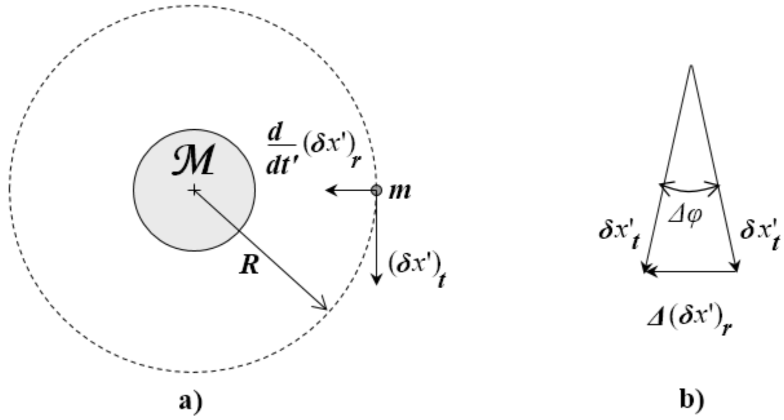

Figure 3: Rate of uncertainty variation in radial position of a satellite

a change in the *direction* of the tangential uncertainties $\delta x'_t$, from one instant to the other, results in a variation of the spatial uncertainties in the *radial* direction $\delta x'_r$ as follows:

$$\frac{d}{dt'}(\delta x'_r) = \lim_{\Delta t' \to 0} \frac{\Delta\phi}{\Delta t'}\, \delta x'_t = \frac{v'}{R}\, \delta x'_t = \frac{Av'^2}{R\sqrt{c^2 - v'^2}} \tag{21}$$

Substituting in Eqn 18 for $\frac{d}{dt'}(\delta x'_r)$ from above we then have:

$$a = \frac{cv'^2}{R\sqrt{c^2 - v'2}} \tag{22}$$

substituting for v' in Eqn 22 from $v' = \sqrt{G\mathcal{M}/R}$, the local acceleration a of the satellite in terms of the gravitational acceleration g at radius R will be given by the following:

$$a = \frac{g}{\sqrt{1 - \frac{G\mathcal{M}}{c^2 R}}} \tag{23}$$

It is clear that in a weak gravitational field where the term $G\mathcal{M}/c^2R << 1$, the local gravitational field induced by the uncertainty in the radial direction will be closely equal to that of the gravitational field of the body $\mathcal{M}$. For example, for a satellite at orbital radius of 500(km) from Earth, the term $G\mathcal{M}/c^2R \approx 6.45E - 10$, meaning local gravity induced by uncertainty in the quantum constituents of the satellite is $a \approx g$. Lastly, note that for a non-relativistic condition where $v' << c$, Eqn 22 reduces to its classical form $a = v'^2/R$.

4 Gaussian gravitational flux

Gauss's flux theorem for gravity [3] states that the flux of the gravitational field of a large gravitating body $\mathcal{M}$ over any surface area enclosing the body has a fixed value proportional to the body mass. As shown in Fig 4, the expression for the flux Φ of the gravitational field $g = G\mathcal{M}/R^2$ of a body of mass $\mathcal{M}$ is obtained by taking the surface integral of the gravitational flux $d\phi = g.d\sigma$ over the enclosing surface Σ as:

$$\Phi = \int_\Sigma g.d\sigma = -4\pi G\mathcal{M} \tag{24}$$

In this article we have hypothesized that the Gaussian gravitational flux theorem can be extended to the state of accelerating objects in space. According to our extension of Gauss theorem, *the flux of uncertainty in position of an accelerating body of density ρ over comoving cross sections of its volume is proportional to its inertia*. The constant of proportionality is $\rho c/A$. As will be shown next, in the proposed extension of Gauss law for inertia, the gravitational acceleration g is replaced with the local acceleration a of the object. Moreover, the closed surface Σ is replaced with a set of comoving cross sectional areas Σ_i which sweep the entire volume of the accelerating object. The cross sectional areas Σ_i are *instantaneously at rest* with the accelerating object and oriented such that their normal is along the direction of acceleration.

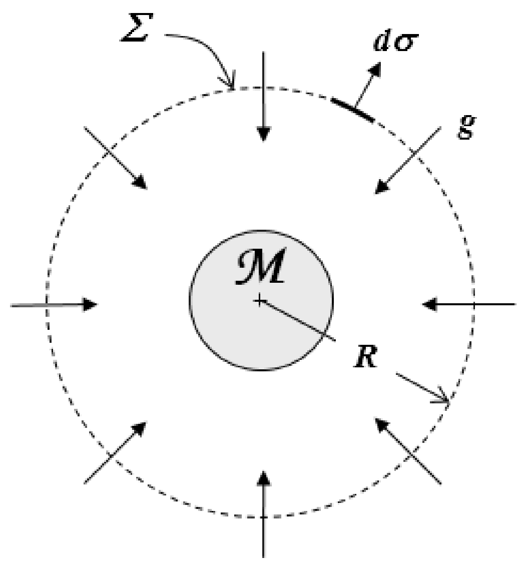

Figure 4: Gaussian gravitational flux around body of mass $\mathcal{M}$

5 Uncertainty flux and force of inertia

Recall that according to Eqn 18 any acceleration (or deceleration) of an object along a direction in space results in a *change* in the positional uncertainty of its constituents along that direction in space. Now, as shown in Fig 5, consider an object of mass M and uniform density ρ *being accelerated* along the x' axis of an inertial frame I'. Some cross section Σ_i which is normal to the local acceleration a is also shown in Fig 5. As mentioned earlier, the set of cross sections which sweep the entire volume are obtained by cutting the object with a set of comoving *inertial cut planes that are instantaneously at rest with the object.* Due to the object's acceleration a, according to Eqn 18 there will be a *flux* of uncertainty through each and every cross section Σ_i of the object, inducing a body force in the opposite direction of the flux. It is the summation of these cross sectional internal forces over the entire volume of the body that is experienced as the force of inertia by the object. It is now trivial to show that the *uncertainty flux* $\dot{\epsilon}.d\sigma$ of the object's constituents, integrated over the cross section Σ_i, integrated over all cross sections along the entire length L of the object will arrive at the force of inertia F_I as follows:

$$F_I = -\rho\frac{c}{A}\int_0^L dl \int_{\Sigma_i} \dot{\epsilon}.d\sigma \tag{25}$$

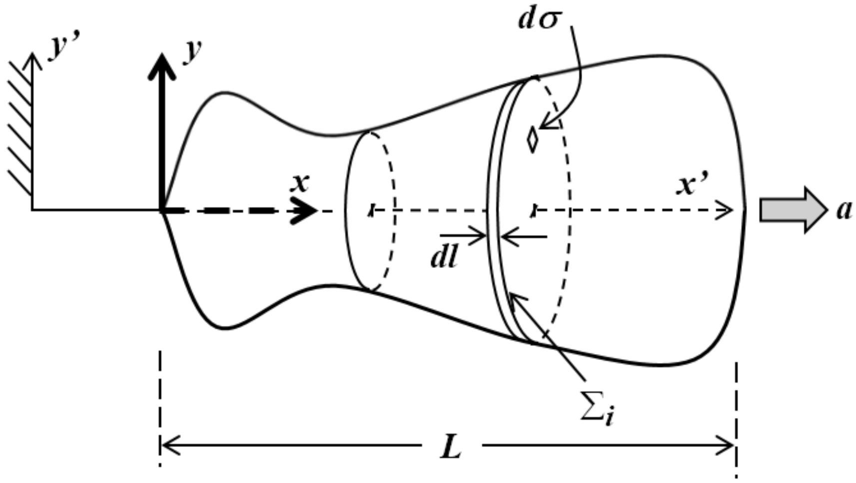

Figure 5: Comoving cross sections Σ_i are instantaneously at rest with the accelerating object

The negative sign is to indicate that the force of inertia is in the opposite direction of the uncertainty flux. To realize this, from Eqn 1 for $\dot{\epsilon}$ we have:

$$\dot{\epsilon} = \frac{d\epsilon}{dt'} = A\frac{a'c^2}{(c^2 - v'^2)^{3/2}} \tag{26}$$

where $a' = dv'/dt'$ is the *coordinate acceleration* of the object. From Special Relativity, the coordinate acceleration a' in terms of local acceleration a is given as [2]:

$$a' = (1 - \frac{v'^2}{c^2})^{3/2} a \tag{27}$$

Substituting in Eqn 26 for a' from Eqn 27 we then have:

$$\dot{\epsilon} = \frac{A}{c}\, a \tag{28}$$

Finally, substituting in Eqn 25 for $\dot{\epsilon}$ from Eqn 28 and integrating over the entire volume V knowing $dV = dl\, d\sigma$ we arrive at:

$$F_I = -a \int_V \rho\, dV = -Ma \tag{29}$$

It is evident that if an object of mass M fall freely in a gravitational field of strength $g = a$, then according to Eqn 29 the inertial force acting on it would be $F_I = -Mg$, i.e equal and opposite to its weight $F = Mg$. This, therefore, clearly shows why the mass of inertia, that of Newton's second law and that of gravity are all the same. Lastly, when the coordinate velocity v' is constant, then $a' = a = 0$, and with that $F_I = 0$, as expected.

6 Uncertainty flux in an accelerating disc

As shown in Fig 6, the classical rotational inertia $T_I = -I\dot{\Omega}$ of a disc with the moment of inertia I and rotational acceleration of $\dot{\Omega}$ can be arrived by considering the flux of uncertainty in the *tangential* position of its constituents through $d\sigma = Ldr$, integrated over cross section Σ_i, integrated along the circumference. Accordingly:

$$T_I = -\rho\frac{c}{A}\int_0^{2\pi} r d\theta \int_{\Sigma_i} r\dot{\epsilon}_t . d\sigma \tag{30}$$

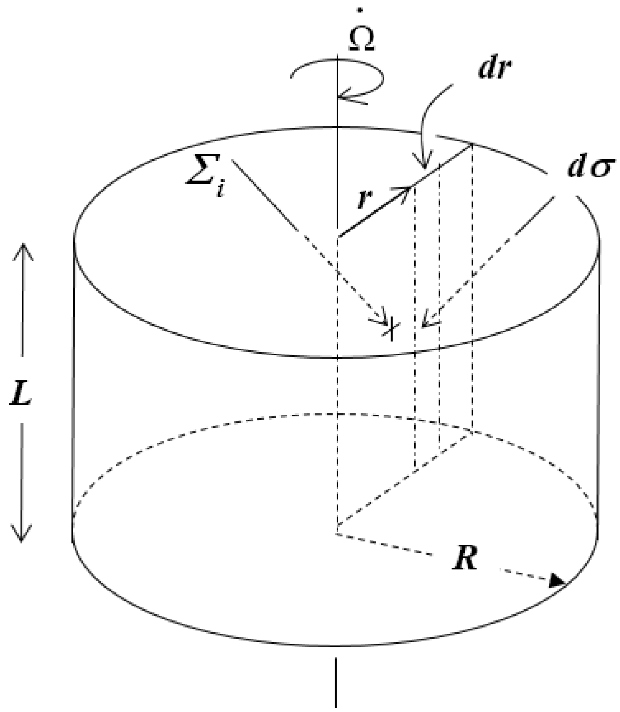

Figure 6: Tangential uncertainty flux through elemental area $d\sigma = Ldr$

Note that in the equation above, the element of body force acting on the elemental area $d\sigma$ is multiplied by its distance r to give the element of torque around the axis of rotation. Substituting for the local tangential acceleration $a = r\dot{\Omega}$ in Eqn 28, the uncertainty in the tangential direction $\dot{\epsilon}_t$ of the constituents on the elemental area $d\sigma$ will be given by:

$$\dot{\epsilon}_t = \frac{A}{c} r\dot{\Omega} \tag{31}$$

Substituting for $\dot{\epsilon}_t$ in Eqn 30 from Eqn 31 and integrating we arrive at:

$$T_I = -\frac{1}{2}\rho\pi R^4 L\,\dot{\Omega} = -I\,\dot{\Omega} \tag{32}$$

as $M = \rho(\pi R^2)L$ and the moment inertia $I = \frac{1}{2}MR^2$, for a disc of mass M and radius R. In this case too, the cross section Σ_i is produced by a cut plane that is instantaneously at rest with the accelerating dics.

7 Radial inertia of a disc of constant rotational velocity

While there is no force of inertia involved in the linear motion with constant velocity, i.e. $a = 0$, there is still a *radial inertia* involved with the rotational motion even if the acceleration $\dot{\Omega} = 0$. The latter scenario is discuss in this section. To that aim, Fig 7 shows a cylindrical disc of mass M, density ρ, outer radius R and length L spinning with *constant* velocity Ω. An element area $d\sigma = L\,rd\theta$ is also shown within the disc where the flux of uncertainty in the *radial* direction is crossing through. The equation of inertia resulting from the flux of uncertainty in the radial position $\dot{\epsilon}_r$ can then be written as:

$$F_I = -\rho\frac{c}{A}\int_0^R dr \int_{\Sigma_i} \dot{\epsilon}_r . d\sigma \tag{33}$$

At a given radius r, where $v' = r\Omega$, the rate of uncertainty $\dot{\epsilon_r}$ in the radial position of the local disc constituents from Eqn 21 is given by:

$$\dot{\epsilon_r} = \frac{Ar\Omega^2}{\sqrt{c^2 - r^2\Omega^2}} \tag{34}$$

Substituting in Eqn 33 for $\dot{\epsilon_r}$ from above and then integrating over the cross section Σ_i we will have:

$$F_I = -2\pi\Omega^2 \rho L c \int_0^R \frac{r^2}{\sqrt{c^2 - r^2\Omega^2}}\, dr \tag{35}$$

Integrating Eqn 35, for a non-relativistic case where $r\Omega << c$, we then finally have the following relationship for the force of radial inertia of a spinning disc:

$$F_I = -\frac{2}{3} M R\,\Omega^2 \tag{36}$$

which is identical to that of the classical mechanics. According to Eqn 36, with the increase of the rotational velocity Ω the force of the radial inertia would eventually exceed the structural strength of the material and lead to the disc burst. This condition would be similar to a case wherein a linearly accelerating object gets crushed structurally under its own force of inertia.

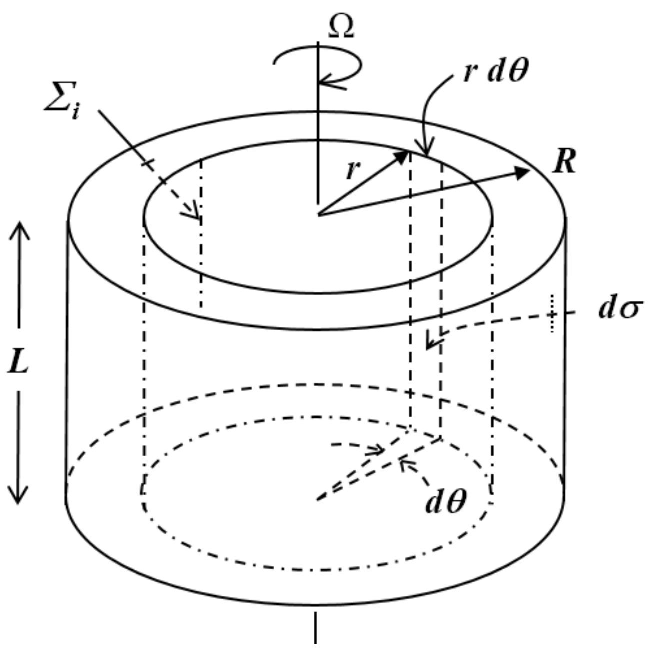

Figure 7: Radial uncertainty flux through elemental area $d\sigma = Lrd\theta$

Integrating Eqn 35 under the condition that the magnitude of tangential speed $r\Omega$ is not negligible compared to the speed of light c (and therefore $\Omega \neq 0$ in the equation below) we then have:

$$F_I = Mc\,\Omega\left(\sqrt{\left(\frac{c}{R\Omega}\right)^2 - 1} - \left(\frac{c}{R\Omega}\right)^2 \sin^{-1}\left(\frac{R\Omega}{c}\right)\right) \tag{37}$$

Comparing Eqn's 36 and 37, the former (classical form) is found to be underestimating the actual cumulative radial force of very high speeds.

8 Conclusion

According to the combined theory of Special Relativity and Quantum Mechanics, c-SRQM, the local acceleration of a physical object is proportional to the rate of variation of uncertainty in its position in space. By extending Gaussian theory of the gravitational flux to the accelerating objects, the c-SRQM theory, offers a means to express the inertia of physical objects using the quantum uncertainty in position of their constituents in space. It is shown that the quantum based formulation of inertial force reduces to those of the classical mechanics under both the linear and rotational acceleration. In the case of a disc of a constant rotational velocity, it is shown that the continuous centripetal acceleration of its constituents generate a radially inward flux of uncertainty. The resulting centrifugal force, under the relativistic conditions, is found to be higher than that estimated by the classical mechanics.

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