Differenceless derivatives with equidistant steps

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Abstract— Differenceless derivatives with equidistant steps have been developed, which makes it possible to calculate an unlimited number of derivatives. One application of this algorithm is the extrapolation and prediction of functions over a wide range of arguments, which can be used in various fields of science and technology.

Derivatives and especially numerical derivatives are widely used in mathematics, physics, applied sciences and almost anywhere else where there is a need for it. Differenceless derivatives with non-equidistant steps were presented in work [1]. Using similar mathematical methods, we developed differenceless derivatives with equidistant steps. One of the best approximations of functions $f(x)$ are Fourier series, using generally accepted notation, we write them in the form:

$$f(x+h)=f(x)+f'(x)h+\frac{1}{2!}f''(x)h^2+\frac{1}{3!}f'''(x)h^3+\frac{1}{4!}f^{(4)}(x)h^4+\cdots\frac{1}{n!}f^{(n)}(x)h^n+\mathrm{O}(h^{n+1}) \qquad (1)$$

Here we want to make a very important remark, Fourier series describe functions well not only in a small neighborhood of point x, but also at any arbitrarily large distance h from point x, it all depends on how many derivatives we know. For the cake of convenience and simplification of the subsequent presentation, we will rewrite (1) in a different form, hinting that the unknowns here are the derivatives $f', f'', \ldots, f^{(n)}$:

$$hf'+\frac{1}{2!}h^2f''+\frac{1}{3!}h^3f'''+\ldots+\frac{1}{n!}h^nf^{(n)}=\Delta f(h) \qquad (2)$$

Consider a set of m equidistant points: $h, 2h, 3h, \ldots, ih, \ldots, mh$. In fact, these are points $x+h, x+2h, \ldots, ih, \ldots, x+mh$, for convenience, we will omit the designation x, where this does not cause misunderstandings. Then the system of equations for calculating derivatives can be written in matrix form as:

$$\begin{pmatrix} (\sum i^2)h & \frac{1}{2!}(\sum i^3)h^2 & \frac{1}{3!}(\sum i^4)h^3 & \ldots & \frac{1}{n!}(\sum i^{n+1})h^n \\ (\sum i^3)h & \frac{1}{2!}(\sum i^4)h^2 & \frac{1}{3!}(\sum i^5)h^3 & \ldots & \frac{1}{n!}(\sum i^{n+2})h^n \\ (\sum i^4)h & \frac{1}{2!}(\sum i^5)h^2 & \frac{1}{3!}(\sum i^6)h^3 & \ldots & \frac{1}{n!}(\sum i^{n+3})h^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (\sum i^{n+1})h & \frac{1}{2!}(\sum i^{n+2})h^2 & \frac{1}{3!}(\sum i^{n+3})h^3 & \ldots & \frac{1}{n!}(\sum i^{n+n})h^n \end{pmatrix} \begin{pmatrix} f' \\ f'' \\ f''' \\ \ldots \\ f^{(n)} \end{pmatrix} = \begin{pmatrix} \sum (i\Delta f(ih)) \\ \sum (i^2\Delta f(ih)) \\ \sum (i^3\Delta f(ih)) \\ \ldots \\ \sum (i^n\Delta f(ih)) \end{pmatrix} \qquad (3)$$

Were $\sum() \rightarrow \sum_{i=1}^{m}()$ and $m \geq n$. For small values $n=1,2,3,4$, one can use analytical methods for solving systems of linear equations, but in the general case it is necessary to use numerical methods. If one are targeting 64-bit arithmetic, then some difficulties may arise in cases where n > 8, then you need to use the GNU Multiple Precision Arithmetic Library for Linux platforms, for example.

Now let's look at the formulas for calculating the first derivative for small values of n, expect surprises. Let's put $m=n=1$ in equation (3), we get the well-known formula:

$$f^{'}(x)=\frac{f(x+h)-f(x)}{h} \tag{4}$$

This is a forward difference approximation of the first derivative of the first order of accuracy, the worst accuracy that is possible. To obtain the formula for the first derivative of the second order of accuracy, set $m=n=2$ in expression (3). Then the system of equations is transformed to the form: $\begin{pmatrix} 5h & \frac{9}{2}h^2 \\ 9h & \frac{17}{2}h^2 \end{pmatrix}\begin{pmatrix} f^{'} \\ f^{''} \end{pmatrix}=\begin{pmatrix} \Delta f(h)+2\Delta f(2h) \\ \Delta f(h)+4\Delta f(2h) \end{pmatrix}$, one of the solutions of which is:

$$f^{'}(x)=\frac{1}{2h}(-f(x+2h)+4f(x+h)-3f(x)) \tag{5}$$

This is also a well-known expression. Sometimes in practice some values of the function $f(x)$ may be unknown or unavailable, for example we have the values of the function at points $(h,3h)$ but not at point $2h$, in this case we can also use equation (3). To do this we define the sign of the sum $\sum()$ as $\sum_{\substack{i=1\\ i\neq 2}}^{3}()$ and substituting the values $m=3$ and $n=2$, the first derivative is calculated as:

$$f^{'}(x)=\frac{1}{18h}(-3f(x+3h)+27f(x+h)-24f(x)) \tag{6}$$

We went further in this computational experiment and obtained the following results. If points $(h,4h)$ are given, then:

$$f^{'}(x)=\frac{1}{72h}(-6f(x+4h)+96f(x+h)-90f(x)) \tag{7}$$

If $(h,5h)$:

$$f^{'}(x)=\frac{1}{20h}(-f(x+5h)+25f(x+h)-24f(x)) \tag{8}$$

This method can be used to derive a lot of formulas for the first derivative of the second order of accuracy, but we will limit ourselves to the case $(h,10h)$, then

$$f^{'}(x)=\frac{1}{810h}(-9f(x+10h)+900f(x+h)-891f(x)) \tag{9}$$

To compare all the above formulas for calculating the first derivative, we calculated the error in calculating the derivative for the function e^x at point 0, where the derivative is equal to 1. The calculation results are presented in Figure 1. As one would expect, the first-order derivative, the black curve, shows lower computational accuracy compared to the second-order derivatives. All derivatives of the second order of accuracy run in parallel, but some degradation of accuracy is observed with increasing distance between two points. At first glance, it may seem that formulas (5), (6), (7), (8) and (9) for calculating the first derivative are of only academic interest, but this is far from the case, they are of practical interest. Let's look at just one example: a GPS navigation system. One of the capabilities of GPS devices is the ability to calculate the speed of a moving object using GPS coordinates measured at certain time intervals, for example every 1 second. Everything works fine as long as the GPS signal is available, problems arise if the GPS signal is not available or is suppressed by electronic warfare. In this case, to calculate the speed, one can use formulas (5), (6), (7), (8), (9) and the like, but this is inconvenient from a practical point of view, there can be 100, 1000 such formulas, or even more. Differenceless derivatives of the second order of accuracy [1] for arbitrary points h_1

and h_2 come to the rescue: $f'(x)=\frac{h_1 f(x+h_2)}{h_2(h_1-h_2)}+\frac{h_2 f(x+h_1)}{h_1(h_2-h_1)}+\frac{(h_1+h_2)f(x)}{h_1 h_2}$. This expression really justifies the name "differenceless" when compared with forward difference or backward difference formulas. It's easy to verify, if one substitute h and $2h$ into this formula, one get formula (5), if h and $3h$ then (6), and so on.

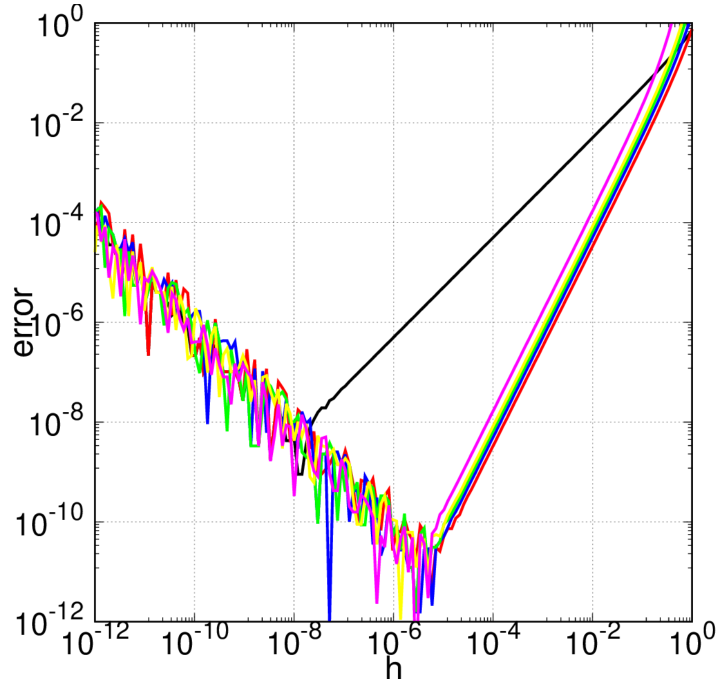

Figure 1

Historical note. Finite differences were invented more than three hundred years ago, and since then scientists know how to calculate the first, second, third, maybe fourth derivative and how to use them, but no one has thought about how to calculate the thousandth derivative. The simplest answer to this is, the thousandth derivative is not needed. This is really the correct answer if you do not know how to calculate this derivative, and how and why to use this derivative. For this reason, people have not even thought about using thousands of derivatives, but in fact there are many such uses in the modern world. Let's consider one of the exciting issues of our time: global warming. Suppose there is a weather station that has been operating for a hundred years. Every day the station determined the average daily temperature, and we have approximately 36,500 measured points. Defining today as $x=0$ and all past days as $(h,2h,3h,\ldots,36500h), were \; h=-1$, one can calculate 36500 derivatives using equation (3). Fourier series (1) will model the average daily temperature for a hundred years with zero approximation accuracy, which will allow, according to our estimates, to predict the average daily temperature for a hundred years in advance for a given station. Of course, if some data are missing or there are data measured at arbitrary points in time, it is necessary to use differenceless derivatives with non-equidistant steps.

REFERENCES

1. Y. Mahotin, Estimation of derivatives by the method of differenceless derivatives, [3259] viXra:2304.0033, 2023