On Symmetries of Geometric Algebra $Cl(3,1)$ for Space-Time

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Abstract. From viewpoints of crystallography and of elementary particles, we explore symmetries of multivectors in the geometric algebra $Cl(3,1)$ that can be used to describe space-time.

Keywords. Geometric algebra, Clifford algebra, symmetry, multivector symmetry, crystallography, elementary particles.

1. Introduction

Recently, V. Gopalan [10] and P. Fabrykiewicz [9] classified multivectors based on their symmetries[1] under space inversion (main grade involution in geometric algebra) $\hat{1}$, time reversal $1'$ and reversion (called wedge reversion by them) $\tilde{1}$. V. Gopalan [10] notes in the conclusions that *One could perhaps explore charge reversal ($\hat{C}$), parity reversal ($\hat{P}$) and time reversal ($\hat{T}$) in the relativistic context [14].* In the standard model of elementary particle physics many experiments have confirmed violation of parity symmetry by the weak interaction, and of $\hat{C}\hat{P}$ symmetry. However, strong interactions by themselves do preserve $\hat{C}\hat{P}$ symmetry [6].

Therefore one aim of this paper is to work in this direction by looking at the effect of these three symmetries on multivectors of $Cl(3,1)$, a (geometric) algebra that can be used to express space-time physics, with $\hat{C}$, $\hat{P}$ and $\hat{T}$ symmetries, e.g., defined by C. Doran and A. Lasenby [7], but here we only focus on the effect of these transformations on the 16 basis blades that constitute the multivector basis of $Cl(3,1)$, and ignore any functional dependence of coefficients in linear combinations that might express spinors or other

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Soli Deo Gloria. This work is dedicated to peace for Israel: *Pray for the peace of Jerusalem: May those who love you be secure. May there be peace within your walls and security within your citadels.* (Ps. 122:6+7, NIV).

[1] Note that V. Gopalan [10] and P. Fabrykiewicz [9] use for space inversion $\bar{1}$, and for (wedge) reversion $1^{\dagger}$.

physical quantities. This means that rotations and other transformations are treated as active rather than passive transformations, that is, physical objects and fields are transformed and not coordinate systems.[2] Furthermore, D. Hestenes [14] and C. Doran and A. Lasenby [7] have a clear preference for the use of $Cl(1,3)$, while in this work we prefer[3] to use $Cl(3,1)$ because its volume-time subalgebra $\{1, e_0, e_{123}, e_{0123}\}$ is isomorphic to quaternions, where e_0 expresses the time direction, at the foundation of the theory of space-time Fourier transforms [20].

V. Gopalan [10] and P. Fabrykiewicz [9] work signature independent for all Clifford algebras of quadratic spaces. We work in an algebra of specific signature and want to take advantage of the principal reverse[4] (see e.g. [20] (2.1.12)), which applied to $Cl(3,1)$ acts like the conventional reverse, but in addition changes the sign of the time vector $e_0 \to -e_0$. So we focus on the group of eight symmetries generated by grade involution $\hat{1}$, reversion $\tilde{1}$ and principal reverse[5] $1'$.

An introduction to geometric algebra can be found in [19]. A mathematically very thorough introductory textbook is [27]. The use of geometric algebra in physics can be found in [14] and more recently in [7]. Computations, like the ones performed in this paper can be checked with computer algebra software, e.g. with the MATLAB package [28]. In the field of computer science, the textbook [8] is a standard reference, and [15] shows how to optimize geometric algebra computations. Extensive surveys of applications can be found in [3, 17, 23]. Applications of geometric algebra to crystallography can be found in [13, 16, 22].

This paper is structured as follows. Section 2 studies the symmetries of $Cl(3,1)$ multivectors under space-time inversion, reversion and principal reverse. Section 3 is devoted to aspects of charge conjugation, parity reversal and time reversal, when $Cl(3,1)$ is applied in the description of elementary particle physics. For ease of reference, four tables are included that show the application of space-time inversion, reversion and principle reverse to the multivector basis of $Cl(3,1)$ in Table 1, the composition of the symmetries $\hat{C}, \hat{P}$ and $\hat{T}$ in Table 2, the application of the symmetries $\hat{C}, \hat{P}$ and $\hat{T}$ to the multivector basis of $Cl(3,1)$ in Table 3, and a reordered version of that in Table 4 in Appendix A.

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[2] I thank an anonymous reviewer for alerting me to this fact.

[3] Another notable work using $Cl(3,1)$ in elementary particle physics is, e.g., B. Schmeikal [29]. Nevertheless, we point out that currently we are not aware that our results presented in this work for $\hat{C}$, $\hat{P}$ and $\hat{T}$ symmetries, would depend on the choice of $Cl(3,1)$ over $Cl(1,3)$.

[4] The principal reverse is in geometric algebra the equivalent of matrix transposition, see [1].

[5] The reader should be aware that therefore in this work we do not use a priori the notion of time reversal of V. Gopalan [10] and P. Fabrykiewicz [9], which there also has the symbol $1'$. Although we do obtain it for multivectors that have e_0 as a factor, by the product of reversion $\tilde{1}$ and principal reverse $1'$ (our notation).

2. Symmetries of $Cl(3,1)$ multivectors generated by space-time inversion, reversion and principal reverse

V. Gopalan [10] and P. Fabrykiewicz [9] correctly turn to Clifford algebra in order to generalize the notion of cross product that only exists in three dimensions to arbitrary dimensions. In this context [21], for crystallographers it may be of interest to know that J. G. Grassmann (Justus G. was the father of Hermann G. Grassmann) originally introduced the characterization of crystal planes by orthogonal vectors, now commonly denoted with Miller indices (see Erhard Scholz [30], pp. 37–46). J. G. Grassmann's work, including his mathematical school textbooks, provided H. G. Grassmann with fertile ideas for his new concepts of algebra (including exterior algebra), solely defined by the relations of its elements, from which G. Peano distilled the modern concept of vectors. Grassmann's pioneering approach was so far ahead of its time that only a few bright minds (like R. W. Hamilton, F. Klein and S. Lie) recognized its genius during his lifetime, late in Grassmann's life. But the young Cambridge-educated genius W. K. Clifford was truly exceptional, and published in 1878 (one year after Grassmann's death) his seminal paper 'Applications of Grassmann's Extensive Algebra' in Am. J. Math. [5]. It elegantly unified the earlier works of Hamilton on quaternions and Grassmann's metric-free algebra of extension to geometric algebras (now known as Clifford algebras), by simply adding in the Clifford (or geometric) product the inner product of vectors (necessary for measurements) and the outer product of Grassmann. Unknowingly perhaps, V. Gopalan [10] and P. Fabrykiewicz [9] thus return to the origins of both crystallography and modern algebra in their search for a dimension independent mathematical framework.

The unit blade basis of the geometric algebra $Cl(3,1)$ is given by one scalar, four vectors, six bivectors, four trivectors and one pseudoscalar quadvector I,

$$\{1, e_0, e_1, e_2, e_3, e_{01}, e_{02}, e_{03}, e_{23}, e_{31}, e_{12}, e_{023}, e_{031}, e_{012}, e_{123}, I = e_{0123}\}, \quad (1)$$

with

$$e_0^2 = -1,\ \ e_1^2 = e_2^2 = e_3^2 = 1,\ \ e_j \cdot e_k = 0\ \ \forall\ j \neq k. \qquad (2)$$

We then have for $j, k \in \{1, 2, 3\}$, $j \neq k$,

$$e_{0j}^2 = 1,\ \ e_{jk}^2 = -1,\ \ e_{0jk}^2 = 1,\ \ e_{123}^2 = -1,\ \ e_{0123}^2 = -1. \qquad (3)$$

We define the main grade involution (full space-time inversion) for $M \in Cl(3,1)$ by

$$\hat{1}M = \widehat{M} = \sum_{k=0}^{4} (-1)^k \langle M \rangle_k, \qquad (4)$$

where $\langle M \rangle_k$ is the grade k part of M. This is equivalent to reversing the direction of every vector $e_i \to -e_i$, $i \in \{0, 1, 2, 3\}$ in the expression for M.

The reversion (reversing the geometric product order of all geometric products of vectors) of M is defined as

$$\tilde{1}M = \widetilde{M} = \sum_{k=0}^{4}(-1)^{\frac{1}{2}k(k-1)}\langle M\rangle_k. \tag{5}$$

The product of grade involution and reversion leads to Clifford conjugation

$$\bar{1}M = \overline{M} = \hat{1}\tilde{1}M = \tilde{1}\hat{1}M = \sum_{k=0}^{4}(-1)^{\frac{1}{2}k(k+1)}\langle M\rangle_k. \tag{6}$$

Finally, the principal reverse $1'M = M'$ of $M \in Cl(3,1)$ is defined identical to the reversion with additionally changing each occurrence of the time basis vector e_0 to $-e_0$. In a general Clifford algebra after reversion simply every unit basis vector is multiplied by its own square.[6]

These involutions generate by composition the following Abelian group of involutions

$$G = \{1, \hat{1}, \tilde{1}, \bar{1}, 1', \hat{1}', \tilde{1}', \bar{1}'\}. \tag{7}$$

In Table 1 all seven involutions (apart from the identity) of the group (7) are applied to the blade basis (1) of $Cl(3,1)$. Since the involutions only involve sign changes, the letter e for even indicates no sign change and the letter o for odd indicates a sign change.

The eight principal types $S, V_0, V, B_0, B, T_0, T$ and Q denoted in Table 1 are all uniquely characterized by the action of the elements of the group of involutions (7). It is now of course possible to follow the pattern established by V. Gopalan [10] and P. Fabrykiewicz [9] and regard linear combinations of principal types as new types, for which the action of the the group of involutions (7) would then be called *mixed m*. For example if we add scalars and quadvectors we would get the type $SQ = S + Q$ with the group action given by a new line in Table 1 that has the seven entries (in the same order as in the table)

$$e \quad e \quad e \quad m \quad m \quad m \quad m. \tag{8}$$

Another example is that a combination of SB_0B or of SB_0BQ has the seven group action entries

$$e \quad m \quad m \quad m \quad m \quad m \quad m. \tag{9}$$

[6]See also Footnote 4. An early reference is also [26], where it is called *principal involution.* I thank an anonymous reviewer for pointing out that applied to basis blades of an orthogonal basis, the principal reverse is related to computing the inverse (or the Hermitian conjugate), as can, e.g., be found in [12]. I thank the same reviewer for a recent reference [2] which also applies the Hermitian conjugate, which writes in the introduction: *For each multi-index that represents the basis vector with $e_J^2 = +1$ the Hermitian conjugation does nothing but changes signs if $e_J^2 = -1$. Therefore, the basis elements e_J and $e_J^\dagger$ can differ by sign only.* We note that our definition of the principal reverse provides an explicit method for obtaining the correct sign even without computing the square e_J^2 explicitly.

Table 1. Action of involutions of group (7) on all 16 basis elements (1) of $Cl(3,1)$. Tp. = type with scalar S, time vector V_0 multiple of e_0, space vector V, bivector B_0 with e_0 factor, space bivector B, trivector T_0 with e_0 factor, space trivector T and pseudoscalar quadvector Q. Bas. = basis element, e = even (no sign change), o = odd (sign change).

Tp.	Bas.	$\hat{1}$	$\tilde{1}$	$\bar{1}$	$1'$	$\hat{1}'$	$\tilde{1}'$	$\bar{1}'$
S	1	e	e	e	e	e	e	e
V_0	e_0	o	e	o	o	e	o	e
	e_1	o	e	o	e	o	e	o
V	e_2	o	e	o	e	o	e	o
	e_3	o	e	o	e	o	e	o
	e_{01}	e	o	o	e	e	o	o
B_0	e_{02}	e	o	o	e	e	o	o
	e_{03}	e	o	o	e	e	o	o
	e_{23}	e	o	o	o	o	e	e
B	e_{31}	e	o	o	o	o	e	e
	e_{12}	e	o	o	o	o	e	e
	e_{023}	o	o	e	e	o	o	e
T_0	e_{031}	o	o	e	e	o	o	e
	e_{012}	o	o	e	e	o	o	e
T	e_{123}	o	o	e	o	e	e	o
Q	e_{0123}	e	e	e	o	o	o	o

This method leads, similar to Table 3 of P. Fabrykiewicz [9], again to distinguish exactly 51 types of multivectors characterized by the action of the group of involutions (7).

The results on the eight principal multivector types and 43 further multivector types of [9] can be fully transferred to $Cl(3,1)$ with the following map, where the index 31 stands for $Cl(3,1)$, and the index GF for the authors V. Gopalan and P. Fabrykiewicz of [10] and [9], respectively. First we state the map of the seven involutions

$$\begin{aligned} &\bar{1}_{GF} \to \hat{1}_{31}, \qquad 1'_{GF} \to \tilde{1}'_{31}, \qquad 1^{\dagger}_{GF} \to \tilde{1}_{31}, \qquad 1'^{\dagger}_{GF} \to 1'_{31}, \\ &\bar{1}'_{GF} \to \bar{1}'_{31}, \qquad \bar{1}^{\dagger}_{GF} \to \bar{1}_{31}, \qquad \bar{1}'^{\dagger}_{GF} \to \hat{1}'_{31}. \end{aligned} \tag{10}$$

Next the map for the multivector type labels

$$\begin{aligned} &S'_{GF} \to S_{31}, \qquad V_{GF} \to V_{0\,31}, \qquad V'_{GF} \to V_{31}, \qquad B_{GF} \to B_{0\,31}, \\ &B'_{GF} \to B_{31}, \qquad T_{GF} \to T_{0\,31}, \qquad T'_{GF} \to T_{31}, \qquad S_{GF} \to Q_{31}. \end{aligned} \tag{11}$$

With the two maps (10) and (11) all results of Table 3 of page 383 of [9] can now be transferred to a classification of $Cl(3,1)$ multivectors into a total of 51 types, including the eight principal types (which appear in the first column of Table 1). The grades in Table 3 of page 383 of [9] are then of course restricted to $\{0, 1, 2, 3, 4\}$, S having grade zero and Q having grade four. For example, the label $S'VBT'$ of No. 43 in Table 3 of page 383 of [9] is mapped to SV_0B_0T, etc.

3. On symmetries of $Cl(3, 1)$ related to elementary particles: charge conjugation, parity reversal and time reversal

In this section we apply the symmetry operations of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ expressed in the geometric algebra $Cl(3, 1)$ for the description of space-time as they can, e.g., be found in C. Doran and A. Lasenby [7], page 283. There the application is to spinors (even grade valued multivector functions $\mathbb{R}^{1,3} \to Cl_+(1, 3)$) including reflection at the time axis e_0 of the argument of the spinor. Here we prefer to work instead with $Cl(3, 1)$ as explained in the introduction. And we restrict ourselves to only study the action of the three symmetry operations on the constant basis elements[7] (1) of $Cl(3, 1)$. Following [7], page 283, we therefore define for multivectors $M \in Cl(3, 1)$

$$\hat{C}M = Me_1e_0, \qquad \hat{P}M = e_0Me_0, \qquad \hat{T}M = Ie_0Me_1, \tag{12}$$

where M is not restricted to the even grade subalgebra. The associativity of the geometric product has as immediate consequence that the composition of these three symmetry operations is also associative, e.g.,

$$\hat{C}(\hat{P}(\hat{T}M)) = (\hat{C}\hat{P})\hat{T}M = \hat{C}(\hat{P}\hat{T}M) = \hat{C}\hat{P}\hat{T}M, \tag{13}$$

so it is not necessary to write brackets to indicate the order of composition, and we generally drop brackets when composing the symmetry operations as in the last expression above. We further find that

$$\begin{aligned} &\hat{C}\hat{C}M = Me_{10}e_{10} = M, \qquad \hat{P}\hat{P}M = e_0^2Me_0^2 = M, \\ &\hat{T}\hat{T}M = Ie_0Ie_0Me_1^2 = e_{123}^2M = -M, \end{aligned} \tag{14}$$

which shows that[8]

$$\hat{C}^2 = 1, \qquad \hat{P}^2 = 1, \qquad \hat{T}^2 = -1. \tag{15}$$

Computation further shows the following commutation relations

$$\begin{aligned} &\hat{P}\hat{T}M = \hat{T}\hat{P}M = I\hat{C}M, \qquad \hat{T}\hat{C}M = -\hat{C}\hat{T}M = -I\hat{P}M, \\ &\hat{C}\hat{P}M = -\hat{P}\hat{C}M = -I\hat{T}M = e_0Me_1 \quad \Rightarrow \quad \hat{T}M = I\hat{C}\hat{P}M. \end{aligned} \tag{16}$$

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[7]In real applications in physics spinors and fields expressed with multivector functions will of course depend on their space-time positions. Our current study is preliminary, because we only look at the constant basis elements and do not take into account changes due to transformations of the basis coefficient functions.

[8]We note that $\hat{T}^2 = -1$ is not specific to the choice of $Cl(3, 1)$. The same would hold true when choosing $Cl(1, 3)$ instead. It is a simple consequence of the algebraic operations applied in the second line of (14).

Table 2. Table of all compositions of symmetry operators $\hat{C},\hat{P}$ and $\hat{T}$, where operations in the top row are applied first to M followed by an operation from the first column. For example: combining $\hat{T}\hat{C}$ from the top row with $\hat{C}\hat{P}$ from the first column (6th row) shows that $\hat{C}\hat{P}\hat{T}\hat{C}M = \hat{P}\hat{T}M$.

1st: 2nd:	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
$\hat{C}$	$\hat{C}$	1	$\hat{C}\hat{P}$	$-\hat{T}$	$\hat{P}$	$-\hat{T}\hat{C}$	$\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$
$\hat{P}$	$\hat{P}$	$-\hat{C}\hat{P}$	1	$\hat{C}\hat{P}\hat{T}$	$-\hat{C}$	$\hat{P}\hat{T}$	$\hat{T}\hat{C}$	$\hat{T}$
$\hat{T}\hat{C}$	$\hat{T}\hat{C}$	$\hat{T}$	$-\hat{C}\hat{P}\hat{T}$	1	$\hat{P}\hat{T}$	$\hat{C}$	$-\hat{P}$	$\hat{C}\hat{P}$
$\hat{C}\hat{P}$	$\hat{C}\hat{P}$	$-\hat{P}$	$\hat{C}$	$\hat{P}\hat{T}$	-1	$\hat{C}\hat{P}\hat{T}$	$-\hat{T}$	$-\hat{T}\hat{C}$
$\hat{T}$	$\hat{T}$	$\hat{T}\hat{C}$	$\hat{P}\hat{T}$	$-\hat{C}$	$-\hat{C}\hat{P}\hat{T}$	-1	$\hat{C}\hat{P}$	$-\hat{P}$
$\hat{C}\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$	$-\hat{T}\hat{C}$	$\hat{P}$	$\hat{T}$	$-\hat{C}\hat{P}$	-1	$-\hat{C}$
$\hat{P}\hat{T}$	$\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{T}$	$\hat{C}\hat{P}$	$-\hat{T}\hat{C}$	$-\hat{P}$	$-\hat{C}$	-1

Moreover,

$$\hat{C}\hat{P}\hat{T}M = IM, \qquad \hat{C}\hat{P}\hat{T}\hat{C}\hat{P}\hat{T}M = -M. \tag{17}$$

and applying the above commutation relations leads to

$$\hat{C}\hat{P}\hat{T} = \hat{P}\hat{T}\hat{C} = -\hat{P}\hat{C}\hat{T} = \hat{C}\hat{T}\hat{P} = -\hat{T}\hat{C}\hat{P} = \hat{T}\hat{P}\hat{C}, \tag{18}$$

and

$$\hat{C}\hat{C}\hat{P}\hat{T}M = \hat{P}\hat{T}M, \quad \hat{P}\hat{C}\hat{P}\hat{T}M = -\hat{C}\hat{T}M, \quad \hat{T}\hat{C}\hat{P}\hat{T}M = \hat{C}\hat{P}M. \tag{19}$$

Putting all this information together we can represent all possible compositions of the symmetry operators $\hat{C},\hat{P}$ and $\hat{T}$ in Table 2, where the symmetry operations in the top row are applied first to $M \in Cl(3,1)$, followed by the symmetry operations in the first column.

Inspection of the table shows that under the following map from the three symmetry operations and their compositions to the basis elements of the geometric algebra of space $Cl(3,0)$, which itself is isomorphic to $Cl_+(3,1)$, the even subalgebra of $Cl(3,1)$, the $\hat{C},\hat{P},\hat{T}$ composition table Table 2 is seen to be isomorphic to the multiplication table of the basis elements $\{1, e_1, e_2, e_3, e_{12}, e_{31}, e_{23}, e_{123}\}$ of $Cl(3,0)$ itself.

$$\begin{aligned} &\hat{C} \to e_1, \quad \hat{P} \to e_2, \quad \hat{T}\hat{C} \to e_3, \\ &\hat{C}\hat{P} \to e_{12}, \quad \hat{T} \to e_{31}, \quad \hat{C}\hat{P}\hat{T} \to e_{23}, \quad \hat{P}\hat{T} \to e_{123}. \end{aligned} \tag{20}$$

The consequence is that the composition of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ forms a non-Abelian group that is isomorphic to the 16-dimensional group $\{\pm 1, \pm e_1, \pm e_2, \pm e_3, \pm e_{23}, \pm e_{31}, \pm e_{12}, \pm e_{123}\}$, formed by products of the basis elements of $Cl(3,0)$ or that of $Cl_+(3,1) \cong Cl(3,0)$. As for $Cl(3,0)$, the group is generated by the products of the basis vectors $\{e_1, e_2, e_3\}$ of three-dimensional Euclidean space $\mathbb{R}^3$.

TABLE 3. Application of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ (top row) defined in (12), to all elements of the basis (first column) of $Cl(3,1)$ given in (1).

Basis	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$-e_{01}$	-1	e_{0123}	e_{01}	e_{23}	e_{0123}	$-e_{23}$
e_0	e_0	e_1	$-e_0$	e_{123}	$-e_1$	$-e_{023}$	e_{123}	e_{023}
e_1	e_1	e_0	e_1	$-e_{023}$	e_0	e_{123}	e_{023}	e_{123}
e_2	e_2	$-e_{012}$	e_2	$-e_{031}$	$-e_{012}$	e_3	e_{031}	e_3
e_3	e_3	e_{031}	e_3	$-e_{012}$	e_{031}	$-e_2$	e_{012}	$-e_2$
e_{01}	e_{01}	-1	e_{01}	$-e_{23}$	-1	$-e_{0123}$	e_{23}	$-e_{0123}$
e_{02}	e_{02}	e_{12}	e_{02}	$-e_{31}$	e_{12}	$-e_{03}$	e_{31}	e_{03}
e_{03}	e_{03}	$-e_{31}$	e_{03}	$-e_{12}$	$-e_{31}$	e_{02}	e_{12}	$-e_{02}$
e_{23}	e_{23}	$-e_{0123}$	$-e_{23}$	$-e_{01}$	e_{0123}	-1	$-e_{01}$	1
e_{31}	e_{31}	$-e_{03}$	$-e_{31}$	$-e_{02}$	e_{03}	e_{12}	$-e_{02}$	$-e_{12}$
e_{12}	e_{12}	e_{02}	$-e_{12}$	$-e_{03}$	$-e_{02}$	$-e_{31}$	$-e_{03}$	e_{31}
e_{023}	e_{023}	e_{123}	$-e_{023}$	$-e_1$	$-e_{123}$	e_0	$-e_1$	$-e_0$
e_{031}	e_{031}	e_3	$-e_{031}$	$-e_2$	$-e_3$	$-e_{012}$	$-e_2$	e_{012}
e_{012}	e_{012}	$-e_2$	$-e_{012}$	$-e_3$	e_2	e_{031}	$-e_3$	$-e_{031}$
e_{123}	e_{123}	e_{023}	e_{123}	e_0	e_{023}	$-e_1$	$-e_0$	$-e_1$
e_{0123}	e_{0123}	$-e_{23}$	e_{0123}	1	$-e_{23}$	e_{01}	-1	e_{01}

In this context we note the work of V. V. Varlamov ([31] and references cited therein), as pointed out by an anonymous reviewer. He also treats parity reversal, time reversal and charge conjugation in the context of Clifford algebras. But he starts with complex Clifford algebras $\mathbb{C}_n$ and introduces the space $\mathbb{R}^{p,q}$ with the help of complex unit imaginary i, $i^2 = -1$, Wick rotations $e_k \to ie_k$ applied to the basis vectors e_{p+1} to e_{p+q}, which leads to a subalgebra $Cl_{p,q}$ in $\mathbb{C}_n$. Then he uses complex conjugation on complex multivectors to define a one-to-one mapping which he identifies with charge conjugation. The resulting product table (e.g. Table 1 in [31]) is clearly different from our Table 2, note, e.g., that in Table 2 the main diagonal is $(+,+,+,+,-,-,-,-)$, whereas Table 1 in [31] has only plus signs. The isomorphism found in [31] in a Clifford algebra over the field $\mathbb{C}$ between an extended automorphsim group of $\mathbb{C}_n$, $\mathrm{Ext}(\mathbb{C}_n)$, and the products of parity reversal, time reversal and charge conjugation is expressed in terms of (real or complex) matrix representations of the units of the Clifford algebra. Though Varlamov's findings are very likely related to ours, we use in comparison a minimalistic algebraic framework of only the real Clifford algebra $Cl(3,1)$ with no need for a real or complex matrix representation.

Finally, we add a table Table 3 showing the application of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ to all elements of the basis of $Cl(3,1)$ given in (1).

We state a handful of immediate observations about Table 3.

- The maps $\hat{C}, \hat{P}$ and $\hat{T}$ map even grade elements to even grade elements and odd grade elements to odd grade elements, i.e., they preserve even and odd grade multivector subspaces of $Cl(3,1)$.
- The map $\hat{P}$ only leads to sign changes.
- The rows for the even basis elements $\{1, e_{01}, e_{23}, e_{0123}\}$ all contain these four elements twice, they form together a commutative subalgebra generated by $\{e_{01}, e_{23}\}$. The operators $\hat{C}$, $\hat{T}$ and their composition $\hat{T}\hat{C}$, applied to any of the four elements $\{1, e_{01}, e_{23}, e_{0123}\}$, generate the other three.
- The rows for the other even basis blades, i.e., the four bivectors $\{e_{02}, e_{03}, e_{31}, e_{12}\}$ all contain precisely these bivectors twice, i.e., they exclude the bivectors $\{e_{01}, e_{23}\}$, and they obviously do not form a subalgebra. The operators $\hat{C}$, $\hat{T}$ and their composition $\hat{T}\hat{C}$, applied to any of the four bivectors, generate the other three.
- The rows for the four odd basis blades $\{e_0, e_1, e_{023}, e_{123}\}$ all contain these four elements twice. The operators $\hat{C}$, $\hat{T}$ and their composition $\hat{T}\hat{C}$, applied to any of the four elements $\{e_0, e_1, e_{023}, e_{123}\}$, generate the other three.
- The rows for the other four odd basis blades $\{e_2, e_3, e_{031}, e_{012}\}$ all contain these four elements twice. The operators $\hat{C}, \hat{T}$ and their composition $\hat{T}\hat{C}$, applied to any of the four elements $\{e_2, e_3, e_{031}, e_{012}\}$, generate the other three.
- Thus Table 3 has four groups (two with even blades and two with odd blades, respectively) of four rows, and inside each group each of the four rows contains the same set of elements twice in different positions. Within each group of four, the operators $\hat{C}$, $\hat{T}$ and their composition $\hat{T}\hat{C}$, applied to any of the four elements present in that group, generate the other three elements.
- The four groups can be clustered together by reordering Table 3, see Table 4 in the appendix. This reveals that each group of four contains two pairs of dual elements (dual with respect to multiplication with $\pm I$), where the duality is element by element from left to right in each pair of rows.
- The reordered table Table 4 also reveals that (up to a sign ± 1) every row can be obtained from the first row (starting with 1) by multiplication with the first element of each row. The same applies to the relation of the first column with every other column (using multiplication of the first column with the elements in the top row of each column).

4. Discussion

It may be interesting to apply both approaches in Clifford space gravity [4], and the study of elementary particles using a new embedding of octonions in geometric algebra [24,25]. Clifford space gravity uses 16-dimensional Clifford algebra valued coordinates and 16-dimensional poly-vector valued momenta.

The line element therefore also has 16 square terms. It may be interesting to study what the effect of the $\hat{C}, \hat{P}$ and $\hat{T}$ maps is on these coordinates and momenta and if it may reveal hidden structure in the theory. In particular what would physically correspond to the isomorphism with the basis element multiplication table of $Cl(3,0)$ revealed in Table 2?

Equally interesting may be to think of the new embedding of octonions in geometric algebra [24, 25]. Octonions can thus be in a minimalistic way embedded in $Cl(3,0)$. This $Cl(3,0)$ is now related to the composition of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$. Elementary particle theory [25] formulated in this setting may therefore have from the outset deep connections with the very symmetries observed in nature for the fundamental forces described in the standard model.

Even without the use of octonions as suggested in [25], the formulation of standard model physics in geometric algebra found in [7] reveals that the composition of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ is intimately related to the geometric algebra of three-dimensional Euclidean space $Cl(3,0)$. Here we may remember that D. Hestenes in [14] showed that the fundamental sigma matrix Pauli algebra is isomorphic to $Cl(3,0)$ and naturally appears as the even subalgebra of $Cl(1,3)$ (or $Cl(3,1)$). This had the major benefit of giving a real geometric interpretation to relativistic quantum mechanics. Therefore, it is of course interesting to find the geometric algebra of three-dimensional Euclidean space $Cl(3,0)$ again as describing the structure of composing charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$. Could it be that the geometric structure of the three-dimensional space we live in, in some yet to be clarified way even determines the symmetry laws that govern the most fundamental forces of nature as we know them in terms of the standard model?

The fact that only time reversal $\hat{T}$ squares to -1 in (15), but parity reversal $\hat{P}$ and charge conjugation $\hat{C}$ both square to $+1$ is a simple algebraic consequence. But only this way can time reversal $\hat{T}$ be seen as isomorphic to a Euclidean bivector (here chosen as e_{31}) in (20). If this has in physics a deeper meaning remains to be explored, also taking into account that at least one other related formalism, i.e., that of V. V. Varlamov [31], has instead $\hat{T}^2 = +1$, compare Table 2 of [31].

At the moment the discussion of symmetry of multivectors in Section 2 and the application of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ symmetries to multivectors in Section 3 simply stand side by side. They are related by the fact that we chose from the outset to exclusively use the multivector algebra $Cl(3,1)$ for space-time, due to its obvious relevance for physics. In further investigation it would of course be possible to combine both approaches which may be relevant for the crystallography of small clusters, high energy scattering, relativistic crystals [11], or wherever elementary particle physics draws on models from solid state physics, etc.

5. Conclusion

In this work we have pursued the application of elementary symmetries of the geometric algebra $Cl(3,1)$ that can describe space-time. Inspired by V. Gopalan [10] and P. Fabrykiewicz [9], we chose three involutions of space-time inversion, reverse and *principal reverse* and studied the Abelian group thus generated and its action on the multivectors of $Cl(3,1)$. We found that similar to [9], a classification in eight principal and further 43 types of multivectors is thus possible, leading to a total of 51 types. Then we looked at algebraic aspects of applying charge conjugation, parity reversal and time reversal to the multivector basis of $Cl(3,1)$. We found that the composition of the symmetry operations $\hat{C}, \hat{P}$ and $\hat{T}$ form a non-Abelian group isomorphic to the multiplicative group $\{\pm 1, \pm e_1, \pm e_2, \pm e_3, \pm e_{23}, \pm e_{31}, \pm e_{12}, \pm e_{123}\}$ of basis elements of $Cl(3,0)$ and $Cl_+(3,1) \cong Cl(3,0)$, and we commented on the structures found when $\hat{C}, \hat{P}$ and $\hat{T}$ are applied to the complete set of basis blades of $Cl(3,1)$.

Acknowledgments

The author wishes to thank God: *Do not conform to the pattern of this world, but be transformed by the renewing of your mind. Then you will be able to test and approve what God's will is – his good, pleasing and perfect will.* (Paul's recommendation in Romans 12:2, NIV). He further thanks his colleagues C. Perwass, D. Proserpio, S. Sangwine, the organizers of the ICCA 2023 conference in Holon, Israel, and the anonymous reviewers of this paper.

Data Availability Statement Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflict of interest The author declares that he has no conflict of interest.

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Appendix A. Reordered table of $\hat{C}, \hat{P}$ and $\hat{T}$ application to multivector basis of $Cl(3,1)$

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Table 4. Reordered table (compare Table 3) of application of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ (top row) defined in (12), to all elements of the basis (first column) of $Cl(3,1)$ given in (1). Double rows contain dual elements (one above the other). The top half contains only even elements, the bottom half only odd elements.

Basis	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$-e_{01}$	-1	e_{0123}	e_{01}	e_{23}	e_{0123}	$-e_{23}$
e_{0123}	e_{0123}	$-e_{23}$	e_{0123}	1	$-e_{23}$	e_{01}	-1	e_{01}
e_{01}	e_{01}	-1	e_{01}	$-e_{23}$	-1	$-e_{0123}$	e_{23}	$-e_{0123}$
e_{23}	e_{23}	$-e_{0123}$	$-e_{23}$	$-e_{01}$	e_{0123}	-1	$-e_{01}$	1
e_{02}	e_{02}	e_{12}	e_{02}	$-e_{31}$	e_{12}	$-e_{03}$	e_{31}	e_{03}
e_{31}	e_{31}	$-e_{03}$	$-e_{31}$	$-e_{02}$	e_{03}	e_{12}	$-e_{02}$	$-e_{12}$
e_{03}	e_{03}	$-e_{31}$	e_{03}	$-e_{12}$	$-e_{31}$	e_{02}	e_{12}	$-e_{02}$
e_{12}	e_{12}	e_{02}	$-e_{12}$	$-e_{03}$	$-e_{02}$	$-e_{31}$	$-e_{03}$	e_{31}
e_0	e_0	e_1	$-e_0$	e_{123}	$-e_1$	$-e_{023}$	e_{123}	e_{023}
e_{123}	e_{123}	e_{023}	e_{123}	e_0	e_{023}	$-e_1$	$-e_0$	$-e_1$
e_1	e_1	e_0	e_1	$-e_{023}$	e_0	e_{123}	e_{023}	e_{123}
e_{023}	e_{023}	e_{123}	$-e_{023}$	$-e_1$	$-e_{123}$	e_0	$-e_1$	$-e_0$
e_2	e_2	$-e_{012}$	e_2	$-e_{031}$	$-e_{012}$	e_3	e_{031}	e_3
e_{031}	e_{031}	e_3	$-e_{031}$	$-e_2$	$-e_3$	$-e_{012}$	$-e_2$	e_{012}
e_3	e_3	e_{031}	e_3	$-e_{012}$	e_{031}	$-e_2$	e_{012}	$-e_2$
e_{012}	e_{012}	$-e_2$	$-e_{012}$	$-e_3$	e_2	e_{031}	$-e_3$	$-e_{031}$