xxxxxxxxxx Volume xx, Issue x (2024)

Incenter-Orthocenter-Centroid Triangle Operator

Yuly Shipilevsky[1]

1 Summary

We consider a mapping from the set of triangles on the same plane onto its-elf, wherein each triangle is being mapped to the triangle, having vertices, which are the orthocenter, the centroid and the incenter of the parent triangle and we consider the corresponding inverse mapping as well.

2 Introduction

In geometry, the Euler line is a line, passes through several important points, determined from any not equilateral triangle, including the orthocenter, the centroid, the circumcenter, the Exeter point and the center of the nine-point circle of the triangle.

Recall that an altitude of a triangle is a line segment through a vertex and p-erpendicular to a line containing the side opposite the vertex. The three pos-sibly extended altitudes intersect in a single point, called the orthocenter of the triangle. An median of a triangle is a line segment joining a vertex to the midpoint of the opposite side. The centroid of a triangle is the intersection of the three medians of the triangle. The incenter of a triangle is the intersection point of all the three interior angle bisectors of the triangle. An angle bisect-or divides the angle into two angles with equal measures. Together with the centroid, circumcenter and orthocenter, the incenter is the only one of the fo-ur that does not in general lie on the Euler line.

Problems, related to the set of triple: incenter, orthocenter and centroid, as well as related to the other sets of characteristic points of triangle, are consi-dered since ancient times.

In 1982, William Wernick published a list of 139 problems, each regarding construction of a triangle from triples of located points, selected from 16 ch-aracteristic points on the Euclidean plane.

It is supposed, that construction must be done using straightedge and compa-

[1] Correspondence to: Yuly Shipilevsky (Email: yulysh2000@yahoo.ca)

ss only.
Wernick divided the problems into four categories:
Redundant problems: if there is a point in the given triple such that it is uniquely determined and constructible from the remaining two points we say that the problem is redundant (denoted by **R**).
Locus dependent problems: if there exists the required triangle ABC(not a way to construct it, but the triangle itself) only for given points meeting certain constraints, then we say that the problem is locus dependent (denoted by **L**). All such problems in Wernick's list have infinitely many solutions.
Solvable problems: if there is a construction of the required triangle ABC (whenever it exists, while it does not exist only in some special cases) starting with the given points, we say that the problem is solvable or constructible (denoted by **S**).
Unsolvable problems: if for some given points the required triangle ABC exists, but it is not constructible, then we say that the problem is unsolvable or unconstructible(denoted by **U**).
Wernick's list comprises a problem number 121, asking for construction of a triangle from its incenter, centroid and orthocenter.
This problem was claimed resolved in 1995 by Leroy F. Meyers together with other 29 problems from Wernick's list. He claimed a proof that problem 121 from Wernick's list is unsolvable: denoted by **U**. In his paper he gives as example the solution of the problem number 115 from Wernick's list and regarging solutions of other 29 problems, L. F. Meyers recommended to ask him in person to obtain the details of the proofs. However, L. F. Meyers died suddenly in 1995, and his paper was published just in 1996.
More details can be found in the paper: P. Schreck et al., "Wernick's list: A Final Update", Forum Geometricorum, Volume 16, pp. 69-80, 2016.
In contrast, we are going to consider triangles, having incenter, orthocenter and centroid of parent triangles as their vertices, instead of sets of separated triples. We will show the advantages of this concept.

3 Triangle Mapping

Recall that an equilateral triangle has three sides of the same length. An isosceles triangle has two sides of equal length. A scalene triangle has all its sides of different lengths.

The incenter lies on Euler line only for an isosceles triangle. That is why incenter, orthocenter and centroid of a scalene triangle form some triangle.
Thus, we can define a mapping: D: $T_S \rightarrow T$, $T_S \subset T$, wherein operator D assign to each scalene triangle ABC from the domain T_S exactly one triangle PQR from the codomain T of general triangles. PQR has the following vertices: incenter, orthocenter and centroid of said scalene triangle ABC.
Furthermore, we can consider an inverse operator $D^{-1}: T \rightarrow T_S$ that sends each triangle PQR $\in$ T to the corresponding anti-triangle ABC $\in T_S$, such that D(ABC) = PQR, D^{-1}(PQR) = ABC.
The existence of an inverse operator D^{-1} follows from P. Schreck's et al. coments, given above regarding unsolvable problems: "... if for some given points the required triangle ABC exists, but it is not constructible, then we say that the problem is unsolvable or unconstructible(denoted by **U**) ...".
Problems, having non-unique solutions are specifically commented by P. Schreck et al. : "... All such [Locus dependent] problems in Wernick's list have infinitely many solutions ...". In addition, P. Schreck et al. notice:
" ... Let us also give a more precise meaning of the labels annotating the problems in Wernick's corpus. A problem has status **S** or **U** if it has solutions in the Euclidean plane, regardless contractibility using straightedge and compass: it has label **S** if it is straightedge and compass constructible, and label **U** (unconstructible) otherwise...".
Accordingly, can be considered the nth powers of operators D and D^{-1}:
$D^n := D(D^{n-1})$, $D^{-n} := (D^{-1})^n$, n = 2, ...
Recall that a right triangle has one of its interior angles measuring 90°.
A triangle with all interior angles measuring less than 90° is an acute triangle . A triangle with one interior angle measuring more than 90° is an obtuse triangle.
Note that for acute and right triangles ABC, the corresponding triangle PQR $\subset$ conv { A, B, C }, - convex hull of points A, B, C, which is not true for the obtuse parent anti-triangles ABC.

4 Six invertible operators

On the other hand, each of the three vertices of the triangle PQR can be considered as any one of the following points: or as incenter I, or as centroid C or as orthocenter H of the triangle ABC. The following combinations are possible:

1. P(I), Q(C), R(H),
2. P(I), Q(H), R(C),
3. P(C), Q(I), R(H),
4. P(C), Q(H), R(I),
5. P(H), Q(C), R(I),
6. P(H), Q(I), R(C).

Thus, for each triangle $PQR \in T$ we can consider 6 parent anti-triangles like ABC and correspondingly 6 operators like operators D and D^{-1} as well.

5 Conclusions

Thus, a concept of triangles, derived from the parent triangles and having incenter, orthocenter and centroid of the parent triangles as their vertices, gives a possibility to define nth powers of operator D for positive and negative "n". Similarly, instead of each or both: orthocenter and centroid any other triangle's characteristic points can be considered.

\