The Building Block of Quantized Dirac Field

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Abstract

We learn how to quantize a two dimensional constrained system that functions as the building block of Dirac field.

Suppose you are interested in quantizing the system defined by the Lagrangian

$$L(\phi, \partial_t \phi) \ = \ \frac{1}{2} \int\limits_{\mathbb{R}^3} \Big((\partial_t \phi(\mathbf{x}))^2 \ - \ \| \nabla \phi(\mathbf{x}) \|^2 \ - \ m^2 (\phi(\mathbf{x}))^2 \Big) d^3 x,$$

where $\phi : \mathbb{R}^3 \to \mathbb{R}$ and $\partial_t \phi : \mathbb{R}^3 \to \mathbb{R}$ are fields, and $m \in \mathbb{R}$ is some constant. The equation of motion for this system is Klein-Gordon equation, and the field ϕ appearing here can be called a Klein-Gordon field. If you know the basic principles of quantum mechanics, you might consider the approach that maybe we first solve the Hamiltonian $H(\phi, \Pi)$, where Π is the canonical momentum field of the system, and then turn this Hamiltonian into an operator with the substitution

$$\Pi(\mathbf{x}) \ \leftarrow \ -i\hbar \frac{\delta}{\delta \phi(\mathbf{x})}.$$

The Hamiltonian operator can then be used to write down a Schrödinger equation, which in this case can be seen to be an infinite dimensional partial differential equation. The problem that we then encounter is that the infinite dimensional partial differential equation can seem intimidating, and it is not immediately obvious how to study its solutions.

If you take a Fourier transform of the Klein-Gordon field, you can find the result that actually the system can be seen to be an infinite product of one dimensional systems defined by a Lagrangian of the form

$$L(x, \dot{x}) \ = \ \frac{1}{2} \dot{x}^2 - \frac{\omega^2}{2} x^2.$$

Here $x, \dot{x} \in \mathbb{R}$ are coordinates, and $\omega \in \mathbb{R}$ is some constant. This Lagrangian defines a one dimensional harmonic oscillator. We see that Klein-Gordon field can be seen to be an infinite dimensional harmonic oscillator. Mainstream physicists have realized that if you quantize the one dimensional

harmonic oscillator, that quantized system can then be used as a building block to construct the full quantized Klein-Gordon field. If we assume that our equations make sense, the quantized Klein-Gordon field found in this way should also provide solutions to the original infinite dimensional partial differential equation.

Next, suppose you are interested in quantizing the system defined by the Lagrangian

$$L(\psi, \partial_t \psi) \;=\; \int_{\mathbb{R}^3} \mathrm{Re}\Big(\overline{\psi}(\mathbf{x})\big(i(\gamma^0 \partial_t + \boldsymbol{\gamma} \cdot \nabla) \;-\; m\big)\psi(\mathbf{x})\Big) d^3x,$$

where $\psi : \mathbb{R}^3 \to \mathbb{C}^4$ and $\partial_t \psi : \mathbb{R}^3 \to \mathbb{C}^4$ are fields, and $m \in \mathbb{R}$ is some constant. Here we used a notation $\overline{\psi}(\mathbf{x}) = \psi^\dagger(\mathbf{x})\gamma^0$, and $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are the gamma matrices of Dirac equation. The symbol "Re" means the real part of a complex number. The equation of motion for this system is Dirac equation, and the field ψ appearing here can be called a Dirac field. In this system the field ψ and the canonical momentum field Π must satisfy a primary constraint $\Pi = -i\psi$. The presence of this constraint means that the system cannot be quantized by simply substituting a differential operator in the place of Π. Those who have studied Dirac's 1964 book *Lectures on Quantum Mechanics* have some idea of what maybe should work: We should find the Dirac-Poisson brackets of this system, and then find operators with the same commutation structure. The problem that we then encounter is that the Dirac-Poisson brackets define an infinite set of commutation relations that can seem intimidating, and it is not obvious what kind of ansatz could be used find the correct operators.

If we recall how the quantization of Klein-Gordon field was made to work, we get an obvious idea that maybe it is possible to write the Dirac field system as an infinite product of some simpler finite dimensional systems too. The answer is that yes this does work. If you take a Fourier transform of the Dirac field, and if you find the correct 8-dimensional real linear transforms, you can find the result that actually the system can be seen to be an infinite product of two dimensional systems defined by a Lagrangian of the form

$$L(x, y, \dot{x}, \dot{y}) \;=\; \dot{x}y - \dot{y}x - a(x^2 + y^2). \tag{1}$$

Here $x, y, \dot{x}, \dot{y} \in \mathbb{R}$ are coordinates, and $a \in \mathbb{R}$ is some constant. Both positive and negative values for a are relevant for Dirac field. So a strategy to quantize Dirac field will be that first we learn how to quantize the two dimensional system defined by Equation (1), and then that quantized system will be used as a building block to construct the full quantized Dirac field. The quantization of the system defined by Equation (1) is the main topic of this article.

Let's first have a look at the classical behavior of the system defined by Equation (1). The partial derivatives of this Lagrangian are

$$\partial_x L \;=\; -\dot{y} - 2ax, \qquad \partial_y L \;=\; \dot{x} - 2ay,$$

$$\partial_{\dot{x}} L \;=\; y \qquad \text{and} \qquad \partial_{\dot{y}} L \;=\; -x.$$

In this case the Euler-Lagrange equations

$$D_t \partial_{\dot{x}} L(x(t), y(t), \dot{x}(t), \dot{y}(t)) \;=\; \partial_x L(x(t), y(t), \dot{x}(t), \dot{y}(t))$$

and

$$D_t \partial_{\dot{y}} L(x(t), y(t), \dot{x}(t), \dot{y}(t)) \;=\; \partial_y L(x(t), y(t), \dot{x}(t), \dot{y}(t))$$

can be simplified into the form

$$\dot{x}(t) \;=\; ay(t) \qquad \text{and} \qquad \dot{y}(t) \;=\; -ax(t).$$

If initial values $x(0)$ and $y(0)$ are fixed, the solution to the time evolution equations will then be given by the formula

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \;=\; \begin{pmatrix} \cos(at) & \sin(at) \\ -\sin(at) & \cos(at) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

We see that the time evolution is simple rotation around the origin, and the angular speed does not depend on the vector's distance to the origin. The direction of the rotation is determined by the sign of a: If $a > 0$, the solutions rotate clockwise, and if $a < 0$, then counterclockwise.

The canonical momenta of this system are given by formulas

$$p_x \;=\; y \qquad \text{and} \qquad p_y \;=\; -x.$$

We see that it is not possible to write the velocities $\dot{x}$ and $\dot{y}$ as functions of x, y, p_x, p_y (without using the time evolution equations), which is what happens with constrained systems.

We can write down constraint functions

$$\phi_1(x, y, p_x, p_y) \;=\; p_x - y \qquad \text{and} \qquad \phi_2(x, y, p_x, p_y) \;=\; p_y + x,$$

and state that the system must obey the primary constraints

$$\phi_m(x, y, p_x, p_y) \;=\; 0 \qquad \text{for } m \in \{1, 2\}.$$

The values of a Hamiltonian are supposed to be equal to the quantity

$$p_x \dot{x} \;+\; p_y \dot{y} \;-\; L(x, y, \dot{x}, \dot{y}).$$

By using the contraints this naturally simplifies to

$$H(x, y, p_x, p_y) \;=\; a(x^2 + y^2).$$

Since this is a contrained system, there does not exist a unique Hamiltonian for all x, y, p_x, p_y. The different Hamiltonians give unique values for only those x, y, p_x, p_y that satisfy the constraints $\phi_m(x, y, p_x, p_y) = 0$ for $m \in \{1, 2\}$. The Hamiltonian we found now above is probably the simplest one. We see that if $a > 0$, the energies are positive and bounded from below, but if $a < 0$, the energies are negative and bounded from above. This is related to the fact that the energies of Dirac field can reach arbitrarily low values (approach $-\infty$), which is one of the confusing features of Dirac field.

The partial derivatives of the chosen Hamiltonian are

$$\partial_x H \;=\; 2ax, \qquad \partial_y H \;=\; 2ay, \qquad \partial_{p_x} H \;=\; 0 \qquad \text{and} \qquad \partial_{p_y} H \;=\; 0.$$

If f and g are some functions $(x, y, p_x, p_y) \mapsto f(x, y, p_x, p_y)$ and $(x, y, p_x, p_y) \mapsto g(x, y, p_x, p_y)$, their Poisson bracket is defined as

$$[f, g] \;=\; \partial_x f \partial_{p_x} g \;+\; \partial_y f \partial_{p_y} g \;-\; \partial_{p_x} f \partial_x g \;-\; \partial_{p_y} f \partial_y g.$$

To apply the Dirac's constraint theory, we must define a matrix $[\phi]$ as the matrix whose elements are $[\phi_m, \phi_{m'}]$, where $m, m' \in \{1, 2\}$. The matrix $[\phi]$ turns out to be

$$[\phi] \;=\; \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

Its inverse is

$$[\phi]^{-1} \;=\; \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

The Poisson brackets between the constraint functions and the Hamiltonian are

$$[\phi_1, H] \;=\; -2ax \qquad \text{and} \qquad [\phi_2, H] \;=\; -2ay.$$

According to the Dirac's constraint theory the time evolution of the system is supposed to come from the equations

$$\begin{aligned}
\dot{x}(t) \;&=\; \partial_{p_x} H \;-\; \sum_{m,m'=1}^{2} \partial_{p_x} \phi_m ([\phi]^{-1})_{m,m'} [\phi_{m'}, H], \\
\dot{y}(t) \;&=\; \partial_{p_y} H \;-\; \sum_{m,m'=1}^{2} \partial_{p_y} \phi_m ([\phi]^{-1})_{m,m'} [\phi_{m'}, H], \\
\dot{p}_x(t) \;&=\; -\partial_x H \;+\; \sum_{m,m'=1}^{2} \partial_x \phi_m ([\phi]^{-1})_{m,m'} [\phi_{m'}, H], \\
\dot{p}_y(t) \;&=\; -\partial_y H \;+\; \sum_{m,m'=1}^{2} \partial_y \phi_m ([\phi]^{-1})_{m,m'} [\phi_{m'}, H].
\end{aligned}$$

If we substitute the solved results into these equations, they turn out to be

$$\begin{aligned}\dot{x}(t) &= ay(t),\\ \dot{y}(t) &= -ax(t),\\ \dot{p}_x(t) &= -ax(t),\\ \dot{p}_y(t) &= -ay(t).\end{aligned}$$

The two first equations are the same as those that we got from the Euler-Lagrange equations, so they are right. The two latter ones we haven't yet seen above, but we can check that they are right too. If a path $t \mapsto (x(t), y(t), p_x(t), p_y(t))$ evolves according to these equations, it has a consequence that

$$D_t\phi_1\big(x(t), y(t), p_x(t), p_y(t)\big) \;=\; \dot{p}_x(t) - \dot{y}(t) \;=\; -ax(t) - (-ax(t)) \;=\; 0$$

and

$$D_t\phi_2\big(x(t), y(t), p_x(t), p_y(t)\big) \;=\; \dot{p}_y(t) + \dot{x}(t) \;=\; -ay(t) + ay(t) \;=\; 0.$$

This means that if the inital values $x(0), y(0), p_x(0), p_y(0)$ have been chosen so that the constraints $\phi_m(x(0), y(0), p_x(0), p_y(0)) = 0$ for $m \in \{1, 2\}$ are satisfied, then the constraints $\phi_m(x(t), y(t), p_x(t), p_y(t)) = 0$ for $m \in \{1, 2\}$ remain satisfied for all $t > 0$ under this time evolution.

The Dirac-Poisson bracket for this system is defined as

$$\begin{aligned}[f, g]^* &= [f, g] \;-\; \sum_{m,m'=1}^{2} [f, \phi_m]([\phi]^{-1})_{m,m'}[\phi_{m'}, g]\\ &= [f, g] \;-\; \frac{1}{2}[f, \phi_1][\phi_2, g] \;+\; \frac{1}{2}[f, \phi_2][\phi_1, g].\end{aligned}$$

It could be a good idea to first write down

$$[x, \phi_1] \;=\; 1, \qquad [x, \phi_2] \;=\; 0, \qquad [y, \phi_1] \;=\; 0, \qquad [y, \phi_2] \;=\; 1,$$

$$[p_x, \phi_1] \;=\; 0, \qquad [p_x, \phi_2] \;=\; -1, \qquad [p_y, \phi_1] \;=\; 1 \qquad \text{and} \qquad [p_y, \phi_2] \;=\; 0$$

somewhere. Using these we can then easily calculate the Dirac-Poisson brackets

$$\begin{aligned}&[x, y]^* \;=\; \frac{1}{2}, \qquad [x, p_x]^* \;=\; \frac{1}{2}, \qquad [x, p_y]^* \;=\; 0,\\ &[y, p_x]^* \;=\; 0, \qquad [y, p_y]^* \;=\; \frac{1}{2} \qquad \text{and} \qquad [p_x, p_y]^* \;=\; \frac{1}{2}.\end{aligned} \tag{2}$$

The rest of the Dirac-Poisson brackets between the quantities x, y, p_x, p_y follow from the identities $[g, f]^* = -[f, g]^*$ and $[f, f]^* = 0$.

The key idea in our quantization of the system defined by Equation (1) is to start studying a regularized system defined by a Lagrangian

$$L_\varepsilon(x,y,\dot{x},\dot{y}) \;=\; \varepsilon(\dot{x}^2+\dot{y}^2)+\dot{x}y-\dot{y}x-a(x^2+y^2),$$

where $\varepsilon > 0$. No matter how small ε is, as long as it is not precisely zero, this system will not be a constrained system, and this system can be quantized in the ordinary way with differential operators.

The partial derivatives of this regularized Lagrangian are

$$\partial_x L_\varepsilon \;=\; -\dot{y}-2ax, \qquad \partial_y L_\varepsilon \;=\; \dot{x}-2ay,$$

$$\partial_{\dot{x}} L_\varepsilon \;=\; 2\varepsilon\dot{x}+y \qquad \text{and} \qquad \partial_{\dot{y}} L_\varepsilon \;=\; 2\varepsilon\dot{y}-x,$$

and the Euler-Lagrange equations are

$$\ddot{x}(t) \;=\; -\frac{1}{\varepsilon}\big(\dot{y}(t)+ax(t)\big) \qquad \text{and} \qquad \ddot{y}(t) \;=\; \frac{1}{\varepsilon}\big(\dot{x}(t)-ay(t)\big).$$

It is not impossible to solve these via ansatz that uses trigonometric functions, but next we will solve the time evolution via Hamilton's equations of motion.

The canonical momenta of this system are

$$p_x \;=\; 2\varepsilon\dot{x}+y \qquad \text{and} \qquad p_y \;=\; 2\varepsilon\dot{y}-x.$$

The velocities can be solved to be

$$\dot{x} \;=\; \frac{1}{2\varepsilon}(p_x-y) \qquad \text{and} \qquad \dot{y} \;=\; \frac{1}{2\varepsilon}(p_y+x).$$

By substituting these into the quantity

$$p_x\dot{x} \;+\; p_y\dot{y} \;-\; L_\varepsilon(x,y,\dot{x},\dot{y})$$

we get a Hamiltonian

$$H_\varepsilon(x,y,p_x,p_y) \;=\; \frac{1}{4\varepsilon}(p_x^2+p_y^2) \;+\; \frac{1}{2\varepsilon}(p_yx-p_xy) \;+\; \Big(\frac{1}{4\varepsilon}+a\Big)(x^2+y^2).$$

This Hamiltonian can also be written in the form

$$H_\varepsilon(x,y,p_x,p_y) \;=\; \frac{1}{4\varepsilon}(x,y,p_x,p_y)\begin{pmatrix} 1+4a\varepsilon & 0 & 0 & 1 \\ 0 & 1+4a\varepsilon & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}$$

which can sometimes be useful. The partial derivatives of this Hamiltonian are

$$\partial_x H_\varepsilon \;=\; \frac{1}{2\varepsilon}p_y \;+\; \Big(\frac{1}{2\varepsilon}+2a\Big)x, \qquad \partial_y H_\varepsilon \;=\; -\frac{1}{2\varepsilon}p_x \;+\; \Big(\frac{1}{2\varepsilon}+2a\Big)y,$$

$$\partial_{p_x} H_\varepsilon \;=\; \frac{1}{2\varepsilon}(p_x - y) \qquad \text{and} \qquad \partial_{p_y} H_\varepsilon \;=\; \frac{1}{2\varepsilon}(p_y + x).$$

In this case the Hamilton's equations of motion

$$\dot{x}(t) \;=\; \partial_{p_x} H_\varepsilon, \qquad \dot{y}(t) \;=\; \partial_{p_y} H_\varepsilon,$$

$$\dot{p}_x(t) \;=\; -\partial_x H_\varepsilon \qquad \text{and} \qquad \dot{p}_y(t) \;=\; -\partial_y H_\varepsilon$$

can be written in the form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{p}_x(t) \\ \dot{p}_y(t) \end{pmatrix} \;=\; \frac{1}{2\varepsilon} \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ -1-4a\varepsilon & 0 & 0 & -1 \\ 0 & -1-4a\varepsilon & 1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ p_x(t) \\ p_y(t) \end{pmatrix}.$$

We can find the solutions to this by diagonalizing the encountered matrix. Let's find the eigenvalues with Gaussian elimination technique:

$$\begin{aligned}
&\det \begin{pmatrix} -\lambda & -1 & 1 & 0 \\ 1 & -\lambda & 0 & 1 \\ -1-4a\varepsilon & 0 & -\lambda & -1 \\ 0 & -1-4a\varepsilon & 1 & -\lambda \end{pmatrix} \\
&= \det \begin{pmatrix} -\lambda & -1 & 1 & 0 \\ 0 & -\frac{\lambda^2+1}{\lambda} & \frac{1}{\lambda} & 1 \\ 0 & \frac{1+4a\varepsilon}{\lambda} & -\frac{\lambda^2+1+4a\varepsilon}{\lambda} & -1 \\ 0 & -1-4a\varepsilon & 1 & -\lambda \end{pmatrix} \\
&= \det \begin{pmatrix} -\lambda & -1 & 1 & 0 \\ 0 & -\frac{\lambda^2+1}{\lambda} & \frac{1}{\lambda} & 1 \\ 0 & 0 & -\frac{\lambda(\lambda^2+2+4a\varepsilon)}{\lambda^2+1} & \frac{4a\varepsilon-\lambda^2}{\lambda^2+1} \\ 0 & 0 & \frac{\lambda^2-4a\varepsilon}{\lambda^2+1} & -\frac{\lambda(\lambda^2+2+4a\varepsilon)}{\lambda^2+1} \end{pmatrix} \\
&= \det \begin{pmatrix} -\lambda & -1 & 1 & 0 \\ 0 & -\frac{\lambda^2+1}{\lambda} & \frac{1}{\lambda} & 1 \\ 0 & 0 & -\frac{\lambda(\lambda^2+2+4a\varepsilon)}{\lambda^2+1} & \frac{4a\varepsilon-\lambda^2}{\lambda^2+1} \\ 0 & 0 & 0 & -\frac{\lambda^4+(4+8a\varepsilon)\lambda^2+16a^2\varepsilon^2}{\lambda(\lambda^2+2+4a\varepsilon)} \end{pmatrix} \\
&= \lambda^4 \;+\; (4+8a\varepsilon)\lambda^2 \;+\; 16a^2\varepsilon^2
\end{aligned}$$

According to the solution formula for quadratic equation this determinant equals zero when

$$\lambda^2 \;=\; -2(1+2a\varepsilon) \;\pm\; 2\sqrt{1+4a\varepsilon}.$$

We should keep in mind that a can be either positive or negative. Anyway, since in the end we are interested in the limit $\varepsilon \to 0$, we can here assume

that $1 + 4a\varepsilon > 0$. It turns out that this square root expression can be simplified, and the previous equation is actually equivalent with

$$\lambda^2 \;=\; -\big(\sqrt{1+4a\varepsilon} \;\mp\; 1\big)^2.$$

It can be difficult to discover this, but once the simplified formula has been seen, it is straightforward to check that it is true. We conclude that the eigenvalues of the matrix encountered in the Hamilton's equations of motion are

$$\pm\frac{i}{2\varepsilon}\big(\sqrt{1+4a\varepsilon} \;\pm\; 1\big).$$

Here the choices in "$\pm$"-signs can be made independently, which gives four different eigenvalues.

How do you solve non-trivial $x, y, p_x, p_y \in \mathbb{R}$ such that

$$\begin{pmatrix} -y + p_x \\ x + p_y \\ -(1+4a\varepsilon)x - p_y \\ -(1+4a\varepsilon)y + p_x \end{pmatrix} \;=\; \pm i\big(\sqrt{1+4a\varepsilon} - 1\big) \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}?$$

One way is that we substitute $x = 1$ and ignore the last coordinate of the equation. Then we have three unknowns and three equations, and we can solve the unknowns with a finite amount of algebraic operations. This way we find eigenvectors

$$\begin{pmatrix} 1 \\ \pm i \\ \pm i\sqrt{1+4a\varepsilon} \\ -\sqrt{1+4a\varepsilon} \end{pmatrix}.$$

When we do the same with the equation

$$\begin{pmatrix} -y + p_x \\ x + p_y \\ -(1+4a\varepsilon)x - p_y \\ -(1+4a\varepsilon)y + p_x \end{pmatrix} \;=\; \pm i\big(\sqrt{1+4a\varepsilon} + 1\big) \begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix},$$

we find eigenvectors

$$\begin{pmatrix} 1 \\ \mp i \\ \pm i\sqrt{1+4a\varepsilon} \\ \sqrt{1+4a\varepsilon} \end{pmatrix}.$$

We are only interested in those linear combinations of the basis solutions where the imaginary parts cancel. When we do this, we get the result that if $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$ are some constants, and if the path $t \mapsto (x(t), y(t), p_x(t),$

$p_y(t)$) is defined according to the formula

$$\begin{aligned}\begin{pmatrix} x(t) \\ y(t) \\ p_x(t) \\ p_y(t) \end{pmatrix} &= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_2\sqrt{1+4a\varepsilon} \\ -\alpha_1\sqrt{1+4a\varepsilon} \end{pmatrix} \cos\Big(\frac{\sqrt{1+4a\varepsilon}-1}{2\varepsilon}t\Big) \\ &+ \begin{pmatrix} \alpha_2 \\ -\alpha_1 \\ -\alpha_1\sqrt{1+4a\varepsilon} \\ -\alpha_2\sqrt{1+4a\varepsilon} \end{pmatrix} \sin\Big(\frac{\sqrt{1+4a\varepsilon}-1}{2\varepsilon}t\Big) \\ &+ \begin{pmatrix} \alpha_3 \\ \alpha_4 \\ -\alpha_4\sqrt{1+4a\varepsilon} \\ \alpha_3\sqrt{1+4a\varepsilon} \end{pmatrix} \cos\Big(\frac{\sqrt{1+4a\varepsilon}+1}{2\varepsilon}t\Big) \\ &+ \begin{pmatrix} -\alpha_4 \\ \alpha_3 \\ -\alpha_3\sqrt{1+4a\varepsilon} \\ -\alpha_4\sqrt{1+4a\varepsilon} \end{pmatrix} \sin\Big(\frac{\sqrt{1+4a\varepsilon}+1}{2\varepsilon}t\Big),\end{aligned}$$

then this path satisfies the time evolution equation. The initial values $x(0), y(0), p_x(0), p_y(0)$ are related to the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ according to a relation

$$\begin{pmatrix} x(0) \\ y(0) \\ p_x(0) \\ p_y(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & \sqrt{1+4a\varepsilon} & 0 & -\sqrt{1+4a\varepsilon} \\ -\sqrt{1+4a\varepsilon} & 0 & \sqrt{1+4a\varepsilon} & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}.$$

The determinant of this matrix is $4(1+4a\varepsilon)$. We have already assumed above that $1+4a\varepsilon > 0$, so since the determinant is non-zero, we can conclude that if arbitrary initial values $x(0), y(0), p_x(0), p_y(0)$ are fixed, it is always possible to find the coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ that determine the solution $t \mapsto (x(t), y(t), p_x(t), p_y(t))$ that has those initial values. This means that we have now found all the solutions to the time evolution equation.

By substituting this initial value formula into the Hamiltonian we get a result that the energies of the solutions depend on the coefficients $\alpha_1, \alpha_2, \alpha_3$, α_4 according to a formula

$$\begin{aligned}H_\varepsilon\big(x(0), y(0), p_x(0), p_y(0)\big) &= \frac{1+4a\varepsilon-\sqrt{1+4a\varepsilon}}{2\varepsilon}(\alpha_1^2+\alpha_2^2) \\ &+ \frac{1+4a\varepsilon+\sqrt{1+4a\varepsilon}}{2\varepsilon}(\alpha_3^2+\alpha_4^2).\end{aligned}$$

We know from the theory of Hamiltonian mechanics that this will also be the constant value of $H_\varepsilon(x(t), y(t), p_x(t), p_y(t))$ for all $t > 0$. Let's have a

look at what happens to these solutions in the limit $\varepsilon \to 0$. By using the Taylor series $\sqrt{1+A} = 1 + \frac{1}{2}A + O(A^2)$ we get an approximation

$$\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \\ p_x(t) \\ p_y(t) \end{pmatrix} &= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_2 + O(\varepsilon) \\ -\alpha_1 + O(\varepsilon) \end{pmatrix} \cos\big(\big(a + O(\varepsilon)\big)t\big) \\
&+ \begin{pmatrix} \alpha_2 \\ -\alpha_1 \\ -\alpha_1 + O(\varepsilon) \\ -\alpha_2 + O(\varepsilon) \end{pmatrix} \sin\big(\big(a + O(\varepsilon)\big)t\big) \\
&+ \begin{pmatrix} \alpha_3 \\ \alpha_4 \\ -\alpha_4 + O(\varepsilon) \\ \alpha_3 + O(\varepsilon) \end{pmatrix} \cos\Big(\Big(\frac{1}{\varepsilon} + O(1)\Big)t\Big) \\
&+ \begin{pmatrix} -\alpha_4 \\ \alpha_3 \\ -\alpha_3 + O(\varepsilon) \\ -\alpha_4 + O(\varepsilon) \end{pmatrix} \sin\Big(\Big(\frac{1}{\varepsilon} + O(1)\Big)t\Big).
\end{aligned}$$

We see that the solutions that use the coefficients α_1 and α_2 converge to the solutions of the original contrained system. These solutions rotate around the origin, and the sign of a determines the direction of the rotation: If $a > 0$, the solutions rotate clockwise, and if $a < 0$, then counterclockwise. The solutions that use the coefficients α_3 and α_4 instead do not converge: They rotate counterclockwise with an angular speed that diverges to infinity. The energies of these solutions are approximately

$$\begin{aligned}
H_\varepsilon\big(x(0), y(0), p_x(0), p_y(0)\big) &= \big(a + O(\varepsilon)\big)(\alpha_1^2 + \alpha_2^2) \\
&+ \Big(\frac{1}{\varepsilon} + O(1)\Big)(\alpha_3^2 + \alpha_4^2).
\end{aligned}$$

We see that the energies of the solutions that use the coefficients α_1 and α_2 converge to the energies of the original constrained system. These energies work so that if $a > 0$, the energy is positive, and if $a < 0$, the energy is negative. The energies of the solutions that use the coefficients α_3 and α_4 instead diverge to infinity.

At this point we are ready to ponder the question that does the system defined by the regularized Lagrangian L_ε converge to the original constrained system defined by the Lagrangian L defined in Equation (1) in the limit $\varepsilon \to 0$. One possible answer is that well the values of Lagrangian L_ε certainly converge pointwisely to the values of Lagrangian L. That is obvious, but do the solutions of the time evolution equation of the regularized system converge to the solutions of the time evolution equation of

the original constrained system? The simple answer is that actually no. This maybe means that the system defined by L_ε does not converge to the system defined by L, but let's think about this more carefully. One way of approaching this issue is that we let $t \mapsto (x(t), y(t))$ be some solution to the original constrained system with some initial values $(x(0), y(0))$. We then let $t \mapsto (\overline{x}(t), \overline{y}(t))$ be a part of some solution to the regularized system with some small $\varepsilon > 0$. Let's assume that these solutions have the same initial values of these two coordinates, meaning that $(x(0), y(0)) = (\overline{x}(0), \overline{y}(0))$. Do these assumptions imply that the solutions are approximately the same, meaning that $(x(t), y(t)) \approx (\overline{x}(t), \overline{y}(t))$ for (at least small) $t > 0$? The answer is no; these solutions can differ significantly. So in this sense the regularized system does not approximate the original constrained system. However, if the solution $t \mapsto (\overline{x}(t), \overline{y}(t))$ is significantly different from $t \mapsto (x(t), y(t))$, this implies that the solution $t \mapsto (\overline{x}(t), \overline{y}(t))$ is an extreme high energy solution. Suppose we are not interested in extreme high energy solutions. Let's fix some upper bound $E_{\max} > 0$ that does not depend on ε, and say that all solutions whose energies exceed this bound will be discarded. If a solution $t \mapsto (\overline{x}(t), \overline{y}(t), \overline{p}_x(t), \overline{p}_y(t))$ survives this condition, then the approximation $(x(t), y(t)) \approx (\overline{x}(t), \overline{y}(t))$ will be true. The approximation will become extremely accurate in the limit $\varepsilon \to 0$. We can conclude that yes the regularized system defined by L_ε does converge to the original constrained system defined by L defined in Equation (1) in the limit $\varepsilon \to 0$, but this convergence works only if we interpret it in such way that the extreme high energy solutions get discarded in the limit process.

The Schrödinger eigenvalue equation of the regularized system is

$$\Big(-\frac{\hbar^2}{4\varepsilon}(\partial_x^2+\partial_y^2) \,+\, \frac{i\hbar}{2\varepsilon}(y\partial_x - x\partial_y) \,+\, \Big(\frac{1}{4\varepsilon}+a\Big)(x^2+y^2)\Big)\psi(x,y) \;=\; E\psi(x,y).$$

Let's switch to polar coordinates defined by the relations

$$x \;=\; r\cos(\theta) \qquad \text{and} \qquad y \;=\; r\sin(\theta).$$

By using the differential operator relations

$$\begin{aligned}\partial_x \;&=\; \cos(\theta)\partial_r \;-\; \frac{1}{r}\sin(\theta)\partial_\theta,\\ \partial_y \;&=\; \sin(\theta)\partial_r \;+\; \frac{1}{r}\cos(\theta)\partial_\theta,\end{aligned}$$

we can transform the Schrödinger eigenvalue equation into a form

$$\Big(-\frac{\hbar^2}{4\varepsilon}\Big(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2\Big) \;-\; \frac{i\hbar}{2\varepsilon}\partial_\theta \;+\; \Big(\frac{1}{4\varepsilon}+a\Big)r^2\Big)\psi(r,\theta) \;=\; E\psi(r,\theta).$$

We get started in solving the Schrödinger eigenvalue equation by using an ansatz

$$\psi(r,\theta) \;=\; e^{-Ar^2}.$$

When we substitute this into the Schrödinger eigenvalue equation, the result is that it works if

$$A \;=\; \frac{\sqrt{1+4a\varepsilon}}{2\hbar} \qquad \text{and} \qquad E \;=\; \frac{\hbar\sqrt{1+4a\varepsilon}}{2\varepsilon}.$$

Having found this solution, we can next try a more elaborate ansatz with infinite series

$$\psi(r,\theta) \;=\; e^{-\frac{\sqrt{1+4a\varepsilon}}{2\hbar}r^2} \sum_{k_1=0}^{\infty} \sum_{k_2=-\infty}^{\infty} b_{k_1,k_2} r^{k_1} e^{ik_2\theta}$$

with some coefficients $b_{k_1,k_2} \in \mathbb{C}$ where $(k_1,k_2) \in \mathbb{N}\times\mathbb{Z}$. When we substitute this into the Schrödinger eigenvalue equation, we get an equation

$$\begin{aligned} e^{-\frac{\sqrt{1+4a\varepsilon}}{2\hbar}r^2} \sum_{k_2=-\infty}^{\infty} \Big(& \frac{\hbar^2}{4\varepsilon} k_2^2 b_{0,k_2} \frac{1}{r^2} \;+\; \frac{\hbar^2}{4\varepsilon}(k_2^2-1) b_{1,k_2} \frac{1}{r} \\ & + \sum_{k_1=0}^{\infty} \Big(\Big(\frac{\hbar}{2\varepsilon}\big(\sqrt{1+4a\varepsilon}(k_1+1)+k_2\big) - E \Big) b_{k_1,k_2} \\ & \qquad + \frac{\hbar^2}{4\varepsilon}\big(k_2^2-(k_1+2)^2\big) b_{k_1+2,k_2} \Big) r^{k_1} \Big) e^{ik_2\theta} \;=\; 0. \end{aligned}$$

If we ignore the convergence question, we can conclude that the ansatz works if

$$\begin{aligned} k_2^2 b_{0,k_2} \;&=\; 0 \qquad \forall\, k_2 \in \mathbb{Z}, \\ (k_2^2-1) b_{1,k_2} \;&=\; 0 \qquad \forall\, k_2 \in \mathbb{Z}, \end{aligned}$$

and

$$\Big(\frac{\hbar}{2\varepsilon}\big(\sqrt{1+4a\varepsilon}(k_1-1)+k_2\big) - E \Big) b_{k_1-2,k_2} \;+\; \frac{\hbar^2}{4\varepsilon}\big(k_2^2-k_1^2\big) b_{k_1,k_2} \;=\; 0$$
$$\forall\, k_1 \in \{2,3,4,\ldots\},\; k_2 \in \mathbb{Z}.$$

The first condition is equivalent with the condition that $b_{0,k_2} = 0$ if $k_2 \neq 0$ (and in the light of this condition $b_{0,0}$ can be arbitrary). The second condition is equivalent with the condition that $b_{1,k_2} = 0$ if $k_2 \neq 1$ and $k_2 \neq -1$ (and in the light of this condition $b_{1,1}$ and $b_{1,-1}$ can be arbitrary). The third condition is equivalent with the condition that if $|k_2| = |k_1|$ and $k_1 \in \{2,3,4,\ldots\}$, then

$$\Big(\frac{\hbar}{2\varepsilon}\big(\sqrt{1+4a\varepsilon}(k_1-1)+k_2\big) - E \Big) b_{k_1-2,k_2} \;=\; 0$$

(that turns out to be a redundant relation,) (and in the light of this condition $b_{k_1,\pm k_1}$ can be arbitrary) and if $|k_2| \neq |k_1|$ and $k_1 \in \{2,3,4,\ldots\}$, then

$$b_{k_1,k_2} \;=\; \frac{\frac{\hbar}{2\varepsilon}\big(\sqrt{1+4a\varepsilon}(k_1-1)+k_2\big) - E}{\frac{\hbar^2}{4\varepsilon}(k_1^2-k_2^2)} b_{k_1-2,k_2}.$$

If one spends some time thinking about these conditions, one can eventually see that they imply the b-matrix to have the form

$$b \;=\; \begin{pmatrix} \cdots & 0 & 0 & 0 & 0 & b_{0,0} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & b_{1,-1} & 0 & b_{1,1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & b_{2,-2} & 0 & b_{2,0} & 0 & b_{2,2} & 0 & 0 & \cdots \\ \cdots & 0 & b_{3,-3} & 0 & b_{3,-1} & 0 & b_{3,1} & 0 & b_{3,3} & 0 & \cdots \\ \cdots & b_{4,-4} & 0 & b_{4,-2} & 0 & b_{4,0} & 0 & b_{4,2} & 0 & b_{4,4} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The elements $b_{0,0}, b_{1,\pm1}, b_{2,\pm2}, \ldots$ can have arbitrary values, and the rest of the non-trivial elements are determined by the recursion relation. It looks like a good idea to study such solutions where only one of the coefficients $b_{0,0}, b_{1,\pm1}, b_{2,\pm2}, \ldots$ is non-zero, so let's fix some $k_2 \in \mathbb{Z}$, and set $b_{|k_2'|,k_2'} = 0$ for $k_2' \neq k_2$. Then for any fixed $b_{|k_2|,k_2}$ the recursion relation determines a sequence $b_{|k_2|,k_2}, b_{|k_2|+2,k_2}, b_{|k_2|+4,k_2}, \ldots$, and all other b-coefficients will be zero. Let's demand that E must have a such value that the sequence $b_{|k_2|,k_2}, b_{|k_2|+2,k_2}, b_{|k_2|+4,k_2}, \ldots$ contains only a finite amount of non-zero elements. This technique usually works in quantum mechanics. This means that

$$E \;=\; \frac{\hbar}{2\varepsilon}\big(\sqrt{1+4a\varepsilon}(k_1-1)+k_2\big)$$

with some $k_1 \in \{|k_2|+2, |k_2|+4, |k_2|+6, \ldots\}$. If we denote $n = \frac{k_1-|k_2|-2}{2}$, then the corresponding values of n are $0, 1, 2, \ldots$. Let's define energies E_{n,k_2} with the formula

$$E_{n,k_2} \;=\; \frac{\hbar}{2\varepsilon}\big(\sqrt{1+4a\varepsilon}(2n+|k_2|+1)+k_2\big) \qquad \forall\, n \in \mathbb{N}.$$

Now $E_{0,k_2}, E_{1,k_2}, E_{2,k_2}, \ldots$ are the energies that make the sequence $b_{|k_2|,k_2}$, $b_{|k_2|+2,k_2}, b_{|k_2|+4,k_2}, \ldots$ contain only a finite amount of non-zero elements.

We should be interested to check that which one the found energies E_{n,k_2}, where $(n,k_2) \in \mathbb{N}\times\mathbb{Z}$, is the lowest energy. It can be seen easily that if the lowest energy exists, it must be one of the energies $E_{0,0}, E_{0,-1}, E_{0,-2}, \ldots$. We can calculate that if $k_2 \leq 0$, then

$$E_{0,k_2-1} \;-\; E_{0,k_2} \;=\; \frac{\hbar}{2\varepsilon}\big(\sqrt{1+4a\varepsilon}-1\big).$$

We see that if $a > 0$, then $E_{0,0}$ is the lowest energy, but if $a < 0$, then the lowest energy doesn't exist because $E_{0,0} > E_{0,-1} > E_{0,-2} > \cdots$. Anyway, the energy $E_{0,0}$ still feels like some kind of "zero-point energy", because it is associated with the simplest found wave function that is also localized around the origin.

We can use the index $(n,k_2) \in \mathbb{N}\times\mathbb{Z}$ to define a wave function $\psi_{n,k_2}(r,\theta)$ as that function that the energy E_{n,k_2} and the recursion relation determine

when we use the initial value $b_{|k_2|,k_2} = 1$. The factor $\sqrt{1+4a\varepsilon}$ appears in these wave functions a lot, so let's make our formulas slightly nicer by using a notation $\Delta := \sqrt{1+4a\varepsilon}$. The solutions to the Schrödinger eigenvalue equation that we have now found look like this:

$$\begin{aligned}
\psi_{0,0}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2} & E_{0,0} &= \tfrac{\hbar\Delta}{2\varepsilon}\\
\psi_{1,0}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2}\big(1-\tfrac{\Delta}{\hbar}r^2\big) & E_{1,0} &= \tfrac{3\hbar\Delta}{2\varepsilon}\\
\psi_{2,0}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2}\big(1-\tfrac{2\Delta}{\hbar}r^2+\tfrac{\Delta^2}{2\hbar^2}r^4\big) & E_{2,0} &= \tfrac{5\hbar\Delta}{2\varepsilon}\\
\psi_{3,0}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2}\big(1-\tfrac{3\Delta}{\hbar}r^2+\tfrac{3\Delta^2}{2\hbar^2}r^4-\tfrac{\Delta^3}{6\hbar^3}r^6\big) & E_{3,0} &= \tfrac{7\hbar\Delta}{2\varepsilon}\\
&\vdots & &\vdots
\end{aligned}$$

$$\begin{aligned}
\psi_{0,1}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+i\theta}r & E_{0,1} &= \tfrac{\hbar(2\Delta+1)}{2\varepsilon}\\
\psi_{1,1}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+i\theta}\big(r-\tfrac{\Delta}{2\hbar}r^3\big) & E_{1,1} &= \tfrac{\hbar(4\Delta+1)}{2\varepsilon}\\
\psi_{2,1}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+i\theta}\big(r-\tfrac{\Delta}{\hbar}r^3+\tfrac{\Delta^2}{6\hbar^2}r^5\big) & E_{2,1} &= \tfrac{\hbar(6\Delta+1)}{2\varepsilon}\\
\psi_{3,1}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+i\theta}\big(r-\tfrac{3\Delta}{2\hbar}r^3+\tfrac{\Delta^2}{2\hbar^2}r^5-\tfrac{\Delta^3}{24\hbar^3}r^7\big) & E_{3,1} &= \tfrac{\hbar(8\Delta+1)}{2\varepsilon}\\
&\vdots & &\vdots
\end{aligned}$$

$$\begin{aligned}
\psi_{0,2}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+2i\theta}r^2 & E_{0,2} &= \tfrac{\hbar(3\Delta+2)}{2\varepsilon}\\
\psi_{1,2}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+2i\theta}\big(r^2-\tfrac{\Delta}{3\hbar}r^4\big) & E_{1,2} &= \tfrac{\hbar(5\Delta+2)}{2\varepsilon}\\
\psi_{2,2}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+2i\theta}\big(r^2-\tfrac{2\Delta}{3\hbar}r^4+\tfrac{\Delta^2}{12\hbar^2}r^6\big) & E_{2,2} &= \tfrac{\hbar(7\Delta+2)}{2\varepsilon}\\
\psi_{3,2}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+2i\theta}\big(r^2-\tfrac{\Delta}{\hbar}r^4+\tfrac{\Delta^2}{4\hbar^2}r^6-\tfrac{\Delta^3}{60\hbar^3}r^8\big) & E_{3,2} &= \tfrac{\hbar(9\Delta+2)}{2\varepsilon}\\
&\vdots & &\vdots
\end{aligned}$$

$$\begin{aligned}
\psi_{0,3}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+3i\theta}r^3 & E_{0,3} &= \tfrac{\hbar(4\Delta+3)}{2\varepsilon}\\
\psi_{1,3}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+3i\theta}\big(r^3-\tfrac{\Delta}{4\hbar}r^5\big) & E_{1,3} &= \tfrac{\hbar(6\Delta+3)}{2\varepsilon}\\
\psi_{2,3}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+3i\theta}\big(r^3-\tfrac{\Delta}{2\hbar}r^5+\tfrac{\Delta^2}{20\hbar^2}r^7\big) & E_{2,3} &= \tfrac{\hbar(8\Delta+3)}{2\varepsilon}\\
\psi_{3,3}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2+3i\theta}\big(r^3-\tfrac{3\Delta}{4\hbar}r^5+\tfrac{3\Delta^2}{20\hbar^2}r^7-\tfrac{\Delta^3}{120\hbar^3}r^9\big) & E_{3,3} &= \tfrac{\hbar(10\Delta+3)}{2\varepsilon}\\
&\vdots & &\vdots
\end{aligned}$$

$\dots,$

$$\begin{aligned}
\psi_{0,-1}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2-i\theta}r & E_{0,-1} &= \tfrac{\hbar(2\Delta-1)}{2\varepsilon}\\
\psi_{1,-1}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2-i\theta}\big(r-\tfrac{\Delta}{2\hbar}r^3\big) & E_{1,-1} &= \tfrac{\hbar(4\Delta-1)}{2\varepsilon}\\
\psi_{2,-1}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2-i\theta}\big(r-\tfrac{\Delta}{\hbar}r^3+\tfrac{\Delta^2}{6\hbar^2}r^5\big) & E_{2,-1} &= \tfrac{\hbar(6\Delta-1)}{2\varepsilon}\\
\psi_{3,-1}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2-i\theta}\big(r-\tfrac{3\Delta}{2\hbar}r^3+\tfrac{\Delta^2}{2\hbar^2}r^5-\tfrac{\Delta^3}{24\hbar^3}r^7\big) & E_{3,-1} &= \tfrac{\hbar(8\Delta-1)}{2\varepsilon}\\
&\vdots & &\vdots
\end{aligned}$$

$$\begin{aligned}
\psi_{0,-2}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2-2i\theta}r^2 & E_{0,-2} &= \tfrac{\hbar(3\Delta-2)}{2\varepsilon}\\
\psi_{1,-2}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2-2i\theta}\big(r^2-\tfrac{\Delta}{3\hbar}r^4\big) & E_{1,-2} &= \tfrac{\hbar(5\Delta-2)}{2\varepsilon}\\
\psi_{2,-2}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2-2i\theta}\big(r^2-\tfrac{2\Delta}{3\hbar}r^4+\tfrac{\Delta^2}{12\hbar^2}r^6\big) & E_{2,-2} &= \tfrac{\hbar(7\Delta-2)}{2\varepsilon}\\
\psi_{3,-2}(r,\theta) &= e^{-\frac{\Delta}{2\hbar}r^2-2i\theta}\big(r^2-\tfrac{\Delta}{\hbar}r^4+\tfrac{\Delta^2}{4\hbar^2}r^6-\tfrac{\Delta^3}{60\hbar^3}r^8\big) & E_{3,-2} &= \tfrac{\hbar(9\Delta-2)}{2\varepsilon}\\
&\vdots & &\vdots
\end{aligned}$$

....

So we have now found a large set of solutions to the Schrödinger eigenvalue equation. The question that we should be interested at this point is that are these all the solutions, or could there still be more to be found. This is maybe the same question as that do these wave functions span the Hilbert space $L^2(\mathbb{R}^2)$. At the time of writing this, I don't know how to rigorously prove that these functions would span the Hilbert space $L^2(\mathbb{R}^2)$, or that these would be all the solutions, but we can make few remarks that suggest that this probably is the case. We can assume it to be known that the finite complex span of the polynomials $x^n y^m$, where $n, m \in \mathbb{N}$, is dense in many relevant function spaces where the functions' support, that is some subset of $\mathbb{R}^2$, is bounded. Let's denote $z = x + iy$ and $z^* = x - iy$. Then $x = \frac{1}{2}(z + z^*)$ and $y = -\frac{i}{2}(z - z^*)$. We see that any finite linear combination of $x^n y^m$ can also be written as a finite linear combination of $z^n (z^*)^m$, where $n, m \in \mathbb{N}$. Using this result, it is maybe possible to next prove that the finite complex span of the functions $e^{-|z|^2} z^n (z^*)^m$ also is dense in many relevant function spaces where we assume the functions to converge to zero sufficiently fast as $|z| \to \infty$. Let's denote $z = re^{i\theta}$. Then $z^n (z^*)^m = r^{n+m} e^{i(n-m)\theta}$. Notice that if $n + m$ is even, also $n - m$ is even, and if $n + m$ is odd, also $n - m$ is odd. Also $n + m \geq |n - m|$ is true for all $n, m \in \mathbb{N}$. So any finite linear combination of the functions $e^{-|z|^2} z^n (z^*)^m$ can be written as a finite linear combination the functions $e^{-r^2} r^{k_1} e^{ik_2\theta}$, where $(k_1, k_2) \in \mathbb{N} \times \mathbb{Z}$, and where in each term k_1 and k_2 have the same parity, and $k_1 \geq |k_2|$. If we ignore the $\frac{\Delta}{2\hbar}$ factor in the exponent, then if $k_1 \in \mathbb{N}$ and $k_2 \in \mathbb{Z}$ have the same parity and $k_1 \geq |k_2|$, $e^{-r^2} r^{k_1} e^{ik_2\theta}$ can be written as a linear combination of $\psi_{0,k_2}, \psi_{1,k_2}, \ldots, \psi_{\frac{k_1 - |k_2|}{2}, k_2}$. So a finite linear combination of $e^{-|z|^2} z^n (z^*)^m$, where $n, m \in \mathbb{N}$, can be written as a finite linear combination of $\psi_{n,k}$, where $(n, k) \in \mathbb{N} \times \mathbb{Z}$. In the light of these remarks it looks like that the finite complex span of the functions $\psi_{n,k}$ is dense in many relevant function spaces, and it looks like extremely likely that these are all the solutions to the Schrödinger eigenvalue equation. We will next continue under the assumption that we know the solutions $(\psi_{n,k}, E_{n,k})$ to be all the solutions, although strictly speaking we didn't prove this rigorously.

In the limit $\varepsilon \to 0$ the energies are approximately

$$E_{n,k_2} \;=\; \frac{\hbar(2n + |k_2| + k_2 + 1)}{2\varepsilon} \;+\; \hbar a(2n + |k_2| + 1) \;+\; O(\varepsilon).$$

We see that all these energies diverge to positive infinity in the limit $\varepsilon \to 0$. The key to making progress in understanding the behavior of this quantized system in this limit is to start studying the energy differences $E_{n,k_2} - E_{0,0}$. They are

$$E_{n,k_2} - E_{0,0} \;=\; \frac{\hbar(2n + |k_2| + k_2)}{2\varepsilon} \;+\; \hbar a(2n + |k_2|) \;+\; O(\varepsilon).$$

We see that if $n > 0$, then $E_{n,k_2} - E_{0,0} \to \infty$ as $\varepsilon \to 0$, and if $k_2 > 0$, also then $E_{0,k_2} - E_{0,0} \to \infty$ as $\varepsilon \to 0$, but if $k_2 \leq 0$, then $E_{0,k_2} - E_{0,0} \to \hbar a|k_2|$ as $\varepsilon \to 0$. So sometimes $E_{n,k_2} - E_{0,0}$ diverges to infinity, and sometimes it converges to a finite value. We interpret this to mean that if $E_{n,k_2} - E_{0,0}$ diverges to infinity, then the state described by the index (n, k_2) vanishes in this limit. This means that only the states $(n, k_2) = (0, 0), (0, -1), (0, -2), \ldots$ remain. We conclude that the eigenstates of the limit system are described by the wave functions and the energies

$$\psi_k(r, \theta) \;=\; \frac{1}{\sqrt{\pi \hbar^{k+1} k!}} e^{-\frac{1}{2\hbar} r^2 - ik\theta} r^k \qquad \text{and} \qquad E_k \;=\; E_{\text{zero}} + \hbar k a$$

for all $k \in \mathbb{N}$. The normalization has been done using the integral formula

$$\int_0^\infty e^{-Ax^2} x^{2k+1} dx \;=\; \frac{k!}{2A^{k+1}},$$

that works for $k \in \mathbb{N}$. We have now successfully found the energy eigenstates of the constrained system defined in Equation (1) with the Lagrangian $L = \dot{x}y - \dot{y}x - a(x^2 + y^2)$, which is a major achievement. There's a lot to wonder about this quantization. One of the features of these energy eigenstates is that they do not span the full Hilbert space $L^2(\mathbb{R}^2)$, but instead some non-trivial subspace of it.

If $a > 0$, then $E_0 < E_1 < E_2 < \cdots$, and if $a < 0$, then $E_0 > E_1 > E_2 > \cdots$, which is somewhat consistent with the classical energies of this system. The value of E_{zero} should not have a physical significance, so we can maybe ignore it, but if somebody insists that some value has to be assigned to it, the answer is $E_{\text{zero}} = \infty$.

Suppose we define some initial value as

$$\psi(0, r, \theta) \;=\; \sum_{k=0}^{\infty} c_k \psi_k(r, \theta)$$

with some coefficients $c_0, c_1, c_2, \ldots \in \mathbb{C}$. The time evolution of this wave function is then

$$\begin{aligned}
\psi(t, r, \theta) \;&=\; \sum_{k=0}^{\infty} c_k e^{-\frac{i}{\hbar} E_k t} \psi_k(r, \theta) \\
&=\; e^{-\frac{i}{\hbar} E_{\text{zero}} t} \sum_{k=0}^{\infty} \frac{c_k}{\sqrt{\pi \hbar^{k+1} k!}} e^{-\frac{1}{2\hbar} r^2 - ik(\theta + a)} r^k \\
&=\; e^{-\frac{i}{\hbar} E_{\text{zero}} t} \sum_{k=0}^{\infty} c_k \psi_k(r, \theta + a) \\
&=\; e^{-\frac{i}{\hbar} E_{\text{zero}} t} \psi(0, r, \theta + a).
\end{aligned}$$

We see that the time evolution is simple rotation around the origin. If $a > 0$, then the graph of the wave function rotates clockwise, and if $a < 0$, then counterclockwise. This implies the correct classical time evolution that we saw in the beginning. There is also a complex phase factor that rotates with an infinite angular speed, but it can be ignored.

Let's have a look at the operators $M_x, M_y, -i\hbar\partial_x$ and $-i\hbar\partial_y$ in the energy eigenbasis. Here M_x and M_y mean the multiplication operators defined by the formulas $(M_x f)(x,y) = xf(x,y)$ and $(M_y f)(x,y) = yf(x,y)$. The matrix elements are

$$\begin{aligned}
&\langle\psi_k|M_x|\psi_j\rangle \\
&= \frac{1}{\pi\hbar}\frac{1}{\sqrt{\hbar^{k+j}k!j!}}\int_0^\infty dr\int_0^{2\pi} d\theta\; re^{-\frac{1}{2\hbar}r^2+ik\theta}r^k\cdot r\cos(\theta)\cdot e^{-\frac{1}{2\hbar}r^2-ij\theta}r^j \\
&= \frac{1}{\pi\hbar}\frac{1}{\sqrt{\hbar^{k+j}k!j!}}\Big(\int_0^\infty e^{-\frac{1}{\hbar}r^2}r^{k+j+2}dr\Big)\Big(\underbrace{\int_0^{2\pi}\frac{1}{2}\Big(e^{i(k-j+1)\theta}+e^{i(k-j-1)\theta}\Big)d\theta}_{=\pi(\delta_{k+1,j}+\delta_{k-1,j})}\Big) \\
&= \frac{\sqrt{\hbar}}{2}\big(\sqrt{j+1}\delta_{k-1,j}+\sqrt{j}\delta_{k+1,j}\big),
\end{aligned}$$

$$\begin{aligned}
&\langle\psi_k|M_y|\psi_j\rangle \\
&= \frac{1}{\pi\hbar}\frac{1}{\sqrt{\hbar^{k+j}k!j!}}\int_0^\infty dr\int_0^{2\pi} d\theta\; re^{-\frac{1}{2\hbar}r^2+ik\theta}r^k\cdot r\sin(\theta)\cdot e^{-\frac{1}{2\hbar}r^2-ij\theta}r^j \\
&= -\frac{i}{\pi\hbar}\frac{1}{\sqrt{\hbar^{k+j}k!j!}}\Big(\int_0^\infty e^{-\frac{1}{\hbar}r^2}r^{k+j+2}dr\Big)\Big(\underbrace{\int_0^{2\pi}\frac{1}{2}\Big(e^{i(k-j+1)\theta}-e^{i(k-j-1)\theta}\Big)d\theta}_{=\pi(\delta_{k+1,j}-\delta_{k-1,j})}\Big) \\
&= \frac{i\sqrt{\hbar}}{2}\big(\sqrt{j+1}\delta_{k-1,j}-\sqrt{j}\delta_{k+1,j}\big),
\end{aligned}$$

$$
\begin{aligned}
&\langle\psi_k| - i\hbar\partial_x|\psi_j\rangle \\
&= \frac{1}{\pi\hbar}\frac{1}{\sqrt{\hbar^{k+j}k!j!}}\int_0^\infty dr\int_0^{2\pi} d\theta\; re^{-\frac{1}{2\hbar}r^2+ik\theta}r^k \\
&\quad (-i\hbar)\Big(\cos(\theta)\partial_r - \frac{1}{r}\sin(\theta)\partial_\theta\Big)e^{-\frac{1}{2\hbar}r^2-ij\theta}r^j \\
&= -\frac{i}{\pi}\frac{1}{\sqrt{\hbar^{k+j}k!j!}}\int_0^\infty e^{-\frac{1}{\hbar}r^2}\Big(\Big(jr^{k+j} - \frac{1}{2\hbar}r^{k+j+2}\Big)\underbrace{\int_0^{2\pi} e^{i(k-j+1)\theta}d\theta}_{=2\pi\delta_{k+1,j}} \\
&\qquad\qquad - \frac{1}{2\hbar}r^{k+j+2}\underbrace{\int_0^{2\pi} e^{i(k-j-1)\theta}d\theta}_{=2\pi\delta_{k-1,j}}\Big)dr \\
&= \frac{i\sqrt{\hbar}}{2}\big(\sqrt{j+1}\delta_{k-1,j} - \sqrt{j}\delta_{k+1,j}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle\psi_k| - i\hbar\partial_y|\psi_j\rangle \\
&= \frac{1}{\pi\hbar}\frac{1}{\sqrt{\hbar^{k+j}k!j!}}\int_0^\infty dr\int_0^{2\pi} d\theta\; re^{-\frac{1}{2\hbar}r^2+ik\theta}r^k \\
&\quad (-i\hbar)\Big(\sin(\theta)\partial_r + \frac{1}{r}\cos(\theta)\partial_\theta\Big)e^{-\frac{1}{2\hbar}r^2-ij\theta}r^j \\
&= \frac{1}{\pi}\frac{1}{\sqrt{\hbar^{k+j}k!j!}}\int_0^\infty e^{-\frac{1}{\hbar}r^2}\Big(\Big(\frac{1}{2\hbar}r^{k+j+2} - jr^{k+j}\Big)\underbrace{\int_0^{2\pi} e^{i(k-j+1)\theta}d\theta}_{=2\pi\delta_{k+1,j}} \\
&\qquad\qquad - \frac{1}{2\hbar}r^{k+j+2}\underbrace{\int_0^{2\pi} e^{i(k-j-1)\theta}d\theta}_{=2\pi\delta_{k-1,j}}\Big)dr \\
&= -\frac{\sqrt{\hbar}}{2}\big(\sqrt{j+1}\delta_{k-1,j} + \sqrt{j}\delta_{k+1,j}\big).
\end{aligned}
$$

So if we first define matrices B^+ and B^- as

$$B^{\pm} = \begin{pmatrix} 0 & \pm 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & \pm\sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \pm\sqrt{3} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \pm\sqrt{4} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \pm\sqrt{5} & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

we can then use these to define the matrices

$$X = \frac{\sqrt{\hbar}}{2}B^+, \quad Y = \frac{i\sqrt{\hbar}}{2}B^-, \quad P_x = \frac{i\sqrt{\hbar}}{2}B^- \quad \text{and} \quad P_y = -\frac{\sqrt{\hbar}}{2}B^+,$$

and these are the operators corresponding to the quantities x, y, p_x and p_y in the energy eigenbasis. If we first calculate

$$B^{\pm}B^{\mp} = \begin{pmatrix} \pm 1 & 0 & -\sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & \pm 1 & 0 & -\sqrt{2\cdot 3} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & \pm 1 & 0 & -\sqrt{3\cdot 4} & 0 & \cdots \\ 0 & \sqrt{2\cdot 3} & 0 & \pm 1 & 0 & -\sqrt{4\cdot 5} & \cdots \\ 0 & 0 & \sqrt{3\cdot 4} & 0 & \pm 1 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4\cdot 5} & 0 & \pm 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

we can then use this formula to easily calculate the commutation relations

$$[X, Y] = \frac{i\hbar}{2}, \qquad [X, P_x] = \frac{i\hbar}{2}, \qquad [X, P_y] = 0,$$
$$[Y, P_x] = 0, \qquad [Y, P_y] = \frac{i\hbar}{2} \qquad \text{and} \qquad [P_x, P_y] = \frac{i\hbar}{2}.$$

We see that the commutation structure of the matrices X, Y, P_x and P_y is the same as the Dirac-Poisson bracket structure shown in Equation (2). It is unlikely that we got a result like this by chance, so this supports the hypothesis that there probably isn't a major mistake in the above calculations. Notice, strictly speaking the matrices X, Y, P_x and P_y are not representations of the operators $M_x, M_y, -i\hbar\partial_x$ and $-i\hbar\partial_y$; instead they are representations of the operators $Q \circ M_x$, $Q \circ M_y$, $Q \circ (-i\hbar\partial_x)$ and $Q \circ (-i\hbar\partial_y)$, where Q is the orthogonal projection to the non-trivial space spanned by the wave functions $\psi_0, \psi_1, \psi_2, \ldots$. This is why it is possible that the matrices X, Y, P_x and P_y have a different commutation structure than the operators $M_x, M_y, -i\hbar\partial_x$ and $-i\hbar\partial_y$.

These matrices satisfy the relations $P_x = Y$ and $P_y = -X$, so apparently the classical constraints apply to the operators too.

If we first calculate

$$(B^{\pm})^2 \;=\; \begin{pmatrix} \pm 1 & 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & \pm 3 & 0 & \sqrt{2\cdot 3} & 0 & 0 & \cdots \\ \sqrt{2} & 0 & \pm 5 & 0 & \sqrt{3\cdot 4} & 0 & \cdots \\ 0 & \sqrt{2\cdot 3} & 0 & \pm 7 & 0 & \sqrt{4\cdot 5} & \cdots \\ 0 & 0 & \sqrt{3\cdot 4} & 0 & \pm 9 & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{4\cdot 5} & 0 & \pm 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

we can then use this formula to easily calculate that

$$a(X^2+Y^2) \;=\; \frac{\hbar a}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 5 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 7 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 9 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We see that $a(X^2+Y^2)$ functions as a Hamiltonian operator too. The only difference between $a(X^2+Y^2)$ and the original Hamiltonian operator implicitly found above is that the zero-point energies are different.

At the time of publication of this article many people believe that it is an important axiom or feature of quantum mechanics that the energy eigenstates of a system always span the full Hilbert space. Above we just found a quantized system that has the feature that the energy eigenstates span only some non-trivial subspace of the full Hilbert space. For example, this means that if an initial value $\psi(0,r,\theta)$ is some arbitrary element of $L^2(\mathbb{R}^2)$, then the time evolution $\psi(t,r,\theta)$ doesn't necessarily exist at all. The time evolution exists only if the initial value $\psi(0,r,\theta)$ is in the right subspace spanned by the energy eigenstates. Some people might feel that this is a sign of something having gone wrong in the quantization. If you think about this more, you can eventually see that most likely nothing has gone wrong in this quantization, but instead this is the way quantized constrained systems are. For example, let's think about what kind of wave packets can be constructed as superpositions of some energy eigenstates. Suppose somebody has fixed a vector $(x,y) \in \mathbb{R}^2$. Suppose you want to construct a wave packet $|\psi\rangle$ as a superposition of some energy eigenstates in a such way that $\langle\psi|(M_x,M_y)|\psi\rangle = (x,y)$. This can be done. Suppose the available energy eigenstates span the full Hilbert space $L^2(\mathbb{R}^2)$. This means that there are a lot of eigenstates available, and it is possible to construct many different kinds of wave packets $|\psi\rangle$ that all have the same property $\langle\psi|(M_x,M_y)|\psi\rangle = (x,y)$. Then suppose somebody fixes a vector

$(p_x, p_y) \in \mathbb{R}^2$, and that we want the equation $\langle\psi| - i\hbar(\partial_x, \partial_y)|\psi\rangle = (p_x, p_y)$ to be true too. Because it is possible to construct many different kinds of wave packets that keep the condition $\langle\psi|(M_x, M_y)|\psi\rangle = (x, y)$ true, it is possible adjust the wave packet in a such way that also the condition $\langle\psi| - i\hbar(\partial_x, \partial_y)|\psi\rangle = (p_x, p_y)$ becomes true. So these two conditions can be true simultaneously. Then suppose instead that we are allowed to only use the energy eigenstates $\psi_0, \psi_1, \psi_2, \ldots$ found above that only span some non-trivial subspace of the full Hilbert space $L^2(\mathbb{R}^2)$. Also in this case we can construct a wave packet $|\psi\rangle$ in a such way that $\langle\psi|(M_x, M_y)|\psi\rangle = (x, y)$. However, because the set of eigenstates is now more limited, it is not possible to construct many different kinds of wave packets with this property in the same way as with the full Hilbert space $L^2(\mathbb{R}^2)$. Then suppose somebody fixes a vector $(p_x, p_y) \in \mathbb{R}^2$. Now, because the set of eigenstates is so limited, it turns out that it will not necessarily be possible to adjust the wave packet in a such way that $\langle\psi| - i\hbar(\partial_x, \partial_y)|\psi\rangle = (p_x, p_y)$ would become true. It can be seen from the relations between the matrices X, Y, P_x and P_y that if $\langle\psi|(M_x, M_y)|\psi\rangle$ has been fixed, at that point also $\langle\psi| - i\hbar(\partial_x, \partial_y)|\psi\rangle$ has been fixed. So the fact that the energy eigenstates span only some non-trivial subspace of the full Hilbert space implies a constraint between the quantities $\langle\psi|(M_x, M_y)|\psi\rangle$ and $\langle\psi| - i\hbar(\partial_x, \partial_y)|\psi\rangle$. This constraint then becomes the classical constraint in the classical limit.

How strange was it that all energies diverged to infinity in the limit $\varepsilon \to 0$? It is perhaps enlightening to compare this phenomenon to some simpler example with a similar limit. Suppose we are interested in a Newtonian point particle in two dimensions, meaning that we have a Hamiltonian $H = \frac{p_x^2 + p_y^2}{2m} + U(x, y)$. Let's define the potential $U(x, y)$ to be 0 when (x, y) is in a rectangle $[0, \ell_1] \times [0, \ell_2]$ with some positive constants $\ell_1, \ell_2 \in \mathbb{R}$, and to be ∞ when (x, y) is outside this rectangle. What are the solutions $t \mapsto (x(t), y(t))$ that describe the motion of the point particle in this rectangle? The answer is that the particle moves in straight lines inside the rectangle and bounces off the walls elastically. What happens to the system and the solutions, if we take the limit $\ell_2 \to 0$? The answer is that the system becomes a one-dimensional system. Then the particle moves right and left on the one-dimensional interval $[0, \ell_1] \times \{0\}$, and bounces off the end points elastically. Suppose we are interested in the quantum mechanical model of this two-dimensional system. We can solve the Schrödinger eigenvalue equation. The wave functions are $\psi_{n_1,n_2}(x, y) = \sin\left(\frac{n_1 \pi x}{\ell_1}\right) \sin\left(\frac{n_2 \pi y}{\ell_2}\right)$, where $n_1, n_2 \in \{1, 2, 3, \ldots\}$, and the energies are $E_{n_1,n_2} = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_1^2}{\ell_1^2} + \frac{n_2^2}{\ell_2^2}\right)$. What happens to the quantized system in the limit $\ell_2 \to 0$? Does it become a one-dimensional quantized system? As can be seen from the formula for the energies, actually $E_{n_1,n_2} \to \infty$ for all $n_1, n_2 \in \{1, 2, 3, \ldots\}$ in the limit $\ell_2 \to 0$. So if you have a problem with the energies diverging to infinity,

then you don't get the one-dimensional quantized system. Let's have a look at the energy differences $E_{n_1,n_2} - E_{1,1}$. If $n_2 > 1$, then $E_{n_1,n_2} - E_{1,1} \to \infty$ in the limit $\ell_2 \to 0$, but $E_{n_1,1} - E_{1,1} = \frac{\hbar^2\pi^2}{2m}\frac{n_1^2-1}{\ell_1^2}$ remain constant (and converge trivially) for all $n_1 \in \{1, 2, 3, \ldots\}$. If we interpret this to mean that the energy states where $n_2 > 1$ vanish in this limit, then we are left with the correct one-dimensional quantized system. The energies of the one-dimensional system become $E_{n_1} = E_{\text{zero}} + \frac{\hbar^2\pi^2}{2m}\frac{n_1^2-1}{\ell_1^2}$, where $E_{\text{zero}} = \infty$, which is not precisely the same as what we get with the ordinary one-dimensional quantization, but which is essentially the same. This example suggests that it is probably a common phenomenon that if a limit process reduces the dimension of some system, then all energies diverge to infinity in that limit. It doesn't mean that the limit would be nonsense; we just have to use the differences of the energies to the zero-point energy. In the light of what we saw in this article above it seems that the same is also true in limit processes that produce constraints between spatial coordinates and canonical momenta.