Mark Burgin's Contribution to the Foundation of Mathematics

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Abstract

In this paper I attempt to describe Mark Burgin's results in non-Diophantine mathematics which are important for foundation of mathematics and its applications in quantum field theory. In particular, the elimination of divergences in Quantum Electrodynamics is described.

Keywords: non-Diophantine arithmetic; foundation of mathematics; divergences

1 Problems in foundation of classical mathematics

The title of the famous Wigner's paper [1] is: "The unreasonable effectiveness of mathematics in the natural sciences", and the paper is concluded as follows:

"*The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.*"

Wigner is known mainly as a famous physicist, and it is seen from those words that he treated mathematics mainly as a powerful tool for applications (e.g., in physics, information theory, chemistry, biology etc.). However, when I discussed the problem of foundation of mathematics with mathematicians, I was surprised that many of them treat mathematics only as an abstract science, and for them it is not important whether or not there are problems in applications of mathematics.

In principle, such an approach also has the right to exist, and history shows that many mathematical results, which at one time were considered purely abstract, eventually found their application in physics and other sciences. But even if some results are not of practical use, they may have a purely aesthetic value. For we do not demand that poetry or music have any applications for the description of nature. In poetry and music, the main thing is beauty, which cannot be expressed in words. In mathematics, as Dirac said, the main thing is the beauty of formulas. But there are some criteria here. Under the influence of my professors of mathematics, I thought that the rigor of mathematical proofs is sacred for mathematicians, and they will never sacrifice this. But is it?

In a possible approach to foundation of mathematics, which we call ApproachA, it is not posed a question whether mathematics should correctly describe nature. The goal of the approach is to find a complete and consistent set of axioms which will make it possible to conclude whether any mathematical statement is true or false. This problem is also formulated as the Entscheidungsproblem which asks for algorithms that consider statements and answers "Yes" or "No" according to whether the statements are universally valid, i.e., valid in every structure satisfying the axioms.

One of the most famous mathematicians who supported ApproachA was Hilbert. For example, he said: *"No one shall expel us from the paradise that Cantor has created for us"*. Hilbert believed that the problem of foundation of mathematics would be solved mainly within the framework of classical mathematics. One of the definitions of this mathematics in the literature is that this mathematics is based on classical logic and ZFC set theory. This is the mainstream approach to mathematics which is used in applications. In simpler words, one can say that classical mathematics involves all integers, all real numbers, continuity, infinitesimals and infinitely large numbers. Alternatives to classical mathematics in ApproachA are constructive mathematics and predicative mathematics, but they are almost never used in applications.

The problem of foundation of classical mathematics is very difficult. The Gödel's incompleteness theorems state that mathematics involving standard arithmetic of natural numbers is incomplete and cannot demonstrate its own consistency. The problem widely discussed in the literature is whether the problems posed by the theorems can be circumvented by nonstandard approaches to natural numbers, e.g., by treating them in the framework of Robinson arithmetic, finitistic arithmetic, transfinite numbers etc. However, the results obtained by Tarski, Turing and others show that, in ApproachA, the problem of foundation of mathematics remains, and this problem has not been resolved yet.

Gödel's works on the incompleteness theorems, saying that any mathematics involving the set of all natural numbers has foundational problems, are written in highly technical terms of mathematical logics. However, this fact is obvious from the philosophy of verificationism. In the 20s of the 20th century the Viennese circle of philosophers under the leadership of Schlick developed an approach called logical positivism which contains verification principle: *A proposition is only cognitively meaningful if it can be definitively and conclusively determined to be either true or false* (see e.g., [2, 3, 4]). However, this principle does not work in classical mathematics. For example, from the point of view of verificationism, it cannot be determined whether the statement that $a + b = b + a$ for all natural numbers a and b is true or false.

However, in scientific community, there are strong opponents of verificationism. For example, as noted by Grayling [5], *"The general laws of science are not, even in principle, verifiable, if verifying means furnishing conclusive proof of their truth. They can be strongly supported by repeated experiments and accumulated evidence but they cannot be verified completely"*. So, from the point of view of standard mathematics and standard physics, verification principle is too strong.

Also, Popper proposed the concept of falsificationism [6]: *If no cases where a claim is false can be found, then the hypothesis is accepted as provisionally true.* In particular, the statement that $a+b=b+a$ for all natural numbers a and b can be treated as provisionally true until one has found some numbers a and b for which $a+b \neq b+a$.

However, according to the philosophy of quantum theory, there should be no statements accepted without proof and based on belief in their correctness (i.e., axioms). The theory should contain only those statements that can be verified, where by "verified" physicists mean an experiment involving only a finite number of steps. So, the philosophy of quantum theory is similar to verificationism, not falsificationism. Note that Popper was a strong opponent of quantum theory and supported Einstein in his dispute with Bohr.

In particular, quantum theory should not be based on mathematics that has foundational problems, but, according to Gödel's incompleteness theorems, classical mathematics does have such problems. Quantum theory has made great progress in fulfilling its program. For example, in this theory, physical quantities are not abstract concepts, but only those that are described by well-defined operators. However, existing quantum theory still is based on classical mathematics because it involves space-time coordinates for which there are no well defined operators in relativistic quantum theory. Therefore, the current version of most general quantum theory does not yet satisfy all the principles of this theory and, as a consequence, in this theory, some physical quantities are described by divergent integrals (see below).

From the point of view of verificationism and the philosophy of quantum theory, standard classical mathematics is not well defined not only because it contains an infinite number of numbers. For example, let us pose a problem whether 10+20 equals 30. Then we should describe an experiment which should solve this problem. Any such experiment must use some kind of computing device. Therefore, the answer to the question posed should be given not from any abstract considerations, but from how this computing device works.

Any computing device can operate only with a finite amount of resources and cannot work with numbers greater than some number L. If a and b are natural numbers then we assume that our computing device gives for $a+b$ and $a \cdot b$ the same numbers as in standard mathematics if those numbers are less than L but at no circumstances it can give a number greater than L. Consider two possibilities:

- a) If those numbers are greater than L then the result equals L. Say $L=40$, then the experiment will confirm that 10+20=30 while if $L=25$ then we will get that 10+20=25.

- b) If those numbers are $\geq L$ then the result equals standard result but modulo L. Say $L=40$, then the experiment will confirm that 10+20=30 while if $L=25$ then we will get that 10+20=5.

So the statements that 10+20=30 and even that $2+2=4$ *are ambiguous because they* *do not contain information on how they should be verified.*

In contrast to ApproachA, we define ApproachB as an approach to mathematics where mathematics not only correctly describes experimental data, but also does not contain foundational problems. As is clear from Mark Burgin's approach to foundation of mathematics described in the subsequent sections, he was a proponent of ApproachB. Even in the spirit of the last example, the title of Mark Burgin's paper [7] written in 1997 reads: "Non-Diophantine Arithmetics or is it Possible that 2+2 is not Equal to 4?" Meanwhile, for now, in this section, we will still describe problems in ApproachA.

We believe the following observation is very important: although classical mathematics (including its constructive version) is a part of our everyday life, people typically do not realize that *classical mathematics is implicitly based on the assumption that one can have any desired amount of resources.* Classical mathematics is based on the implicit assumption that we can consider an idealized case when a computing device can operate with an infinite amount of resources. In other words, from the point of view of verificationism, standard operations with natural numbers are implicitly treated as limits of operations with a finite natural L when $L \to \infty$. As a rule, every limit in mathematics is thoroughly investigated but in the case of standard operations with natural numbers it is not even mentioned that those operations are formal limits of operations with a finite L when $L \to \infty$. In real life such limits even might not exist if, for example, the universe contains a finite number of elementary particles.

One of the key concepts in classical mathematics is the concept of infinitesimals proposed by Newton and Leibniz more than 300 years ago. Since that time, a titanic work has been done on foundation of classical mathematics. As noted above, this problem has not been solved till the present time, but, for many mathematicians, the most important thing is not whether a rigorous foundation exists but that standard mathematics is a powerful tool for solving many problems.

The idea of infinitesimals was in the spirit of existed belief that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts, and, in the times of Newton and Leibniz, people did not know about the existence of atoms and elementary particles. But now we know that when we reach the level of atoms and elementary particles then standard division loses its usual meaning and in nature there are no arbitrarily small parts and no continuity.

For example, typical energies of electrons in modern accelerators are millions of times greater than the electron rest energy, and such electrons experience many collisions with different particles. If it were possible to break the electron into parts, then it would have been noticed long ago.

Another example is that if we draw a line on a sheet of paper and look at this line with a microscope then we see that the line is strongly discontinuous because it consists of atoms. That is why standard geometry (the concepts of continuous lines and surfaces) can describe nature only in the approximation when sizes of atoms are neglected, standard macroscopic theory can work well only in this approximation etc. For example, differential geometry (DG) is used in General Relativity, which is a purely classical (i.e., non-quantum) theory that uses standard continuum mathematics and does not take into account that matter consists of atoms and elementary particles. DG is also used in quantum field theories involving a curved space-time background. Those theories contain not only mathematical foundational problems related to Gödel's incompleteness theorems, but also to the problem that physical quantities are described by divergent integrals.

Of course, when we consider water in the ocean and describe it by differential equations of hydrodynamics, this works well but this is only an approximation since water consists of atoms. However, it seems unnatural that even quantum theory is based on continuous mathematics. Even the name "quantum theory" reflects a belief that nature is quantized, i.e., discrete, and this name has arisen because in quantum theory some quantities have discrete spectrum (i.e., the spectrum of the angular momentum operator, the energy spectrum of the hydrogen atom etc.). But this discrete spectrum has appeared in the framework of classical mathematics, i.e., mathematics which involves infinitesimals and has foundational problems.

I asked mathematicians whether, in their opinion, the indivisibility of the electron shows that in nature there are no infinitesimals and standard division does not work always. Some of them say that sooner or later the electron will be divided but, as a rule, mathematicians agree that the electron is indivisible and in nature there are no infinitesimals. They say that, for example in practice, dx/dt should be understood as $\Delta x/\Delta t$ where Δx and Δt are small but not infinitesimal. I ask them: but you work with dx/dt, not $\Delta x/\Delta t$. They reply that since mathematics with derivatives works well then there is no need to philosophize and develop something else.

One of the key problems of modern quantum theory is the problem of divergences: the theory gives divergent expressions for the S-matrix in perturbation theory. In renormalized theories, the divergencies are eliminated by the renormalization procedure where finite observable quantities are formally expressed as products of singularities. Although this procedure is not well substantiated mathematically, in some cases it results in excellent agreement with experiment. At the same time, in nonrenormalized theories, infinities cannot be eliminated by the renormalization procedure, and this a great obstacle in several fundamental problems, e.g., for constructing quantum gravity based on quantum field theory. As the famous physicist and the Nobel Prize laureate Steven Weinberg writes in his book [8]: "*Disappointingly this problem appeared with even greater severity in the early days of quantum theory, and although greatly ameliorated by subsequent improvements in the theory, it remains with us to the present day*". The title of Weinberg's paper [9] is "Living with infinities".

So, classical mathematics has foundational problems which so far have not been solved in spite of efforts of such great mathematicians as Cantor, Fraenkel, Gödel, Hilbert, Kronecker, Russell, Zermelo and others, and, as noted above, classical mathematics is problematic from the point of view of verificationism and the philosophy of quantum theory. The philosophy of those great mathematicians was implicitly based on macroscopic experience in which the concepts of infinitely small/large, continuity and standard division are natural. However, as noted above, those concepts contradict the existence of elementary particles and are not natural in quantum theory. The illusion of continuity arises when one neglects the discrete structure of matter.

The above discussion gives reason to think that the problem of foundation of mathematics can only be solved if significant changes are made to existing mathematics. However, this does not mean that existing mathematics will be canceled. The history of science shows that new fundamental theories do not cancel existing theories that have proven themselves in many problems. New theories usually only show that existing theories are not universal because there are conditions under which they do not work. Probably, the most famous example: the theory of relativity does not cancel classical mechanics, but shows that it only works when all speeds are much less than the speed of light, while when they are already comparable to the speed of light, then it is necessary to apply the theory of relativity. The next section describes what changes Mark Burgin proposed to make to mathematics to solve the problem of its foundation.

2 Mark Burgin's approach to the problem of foundation of mathematics

Mark Burgin studied at the Faculty of Mechanics and Mathematics of Moscow University. In those years, this department was considered the Mecca of mathematics throughout the Soviet Union. Many students and professors of the faculty believed that other sciences are inferior with respect to mathematics. Therefore, they believed that the problem of foundation of mathematics should be considered only from the point of view of ApproachA described in Sec. 1. However, Mark believed that applications of mathematics were also very important, and this is clear even from the title of his book with Czachor [10].

Apparently, Mark's first paper on foundation of mathematics was [11] written in 1977. It begins with the words: "Even at the very beginning of the emergence of mathematics as a science in ancient Greece, doubts arouse about how true many basic mathematical logical abstractions were. In this case, the most important, apparently, are the concept of infinity and the construction of natural numbers... These problems were formulated in detail by P.K. Rashevsky [12], who pointed out on the need to construct a natural series that differs significantly in its properties from the classic natural series."

In his paper [12], Rashevsky writes that "... the natural series is still the only mathematical idealization of real counting processes. This monopoly position dawns its aura of a certain truth in the ultimate instance, absolute, the only possible, recourse to which is inevitable in all cases when a mathematician works with recalculation their objects. Moreover, since the physicist uses only the apparatus that mathematics offers him, then the absolute power of the natural series extends and on physics and — through the number line — predetermines to a large extent possibilities of physical theories."

In [11], Mark defined concepts that later became basic in his approach to non-Diophantine Arithmetic. Then in [7, 10, 13, 14, 15] he builds a non-Diophantine arithmetic **A** of integer numbers using weak projectivity with the Diophantine arithmetic **Z** of all integer numbers where the definition of weak projectivity is as follows.

Let us take two abstract arithmetics $\mathbf{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ and $\mathbf{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ and consider two mappings g: $A_1 \to A_2$ and h: $A_2 \to A_1$.

Definition. *a*) An abstract arithmetic $\mathbf{A}_1 = (A_1; +_1, \circ_1, \leq_1)$ is called weakly projective with respect to an abstract arithmetic $\mathbf{A}_2 = (A_2; +_2, \circ_2, \leq_2)$ if there are following relations between orders and operations in $\mathbf{A}_1$ and in $\mathbf{A}_2$:

$$a +_1 b = h(g(a) +_2 g(b))$$

$$a \circ_1 b = h(g(a) \circ_2 g(b))$$

$$a \leq_1 b \text{ only if } g(a) \leq_2 g(b))$$

b) The mapping g is called the *projector* and the mapping h is called the *coprojector* for the pair $(\mathbf{A}_1, \mathbf{A}_2)$.

The functions g and h determine a *weak projectivity* between the arithmetic $\mathbf{A}_1$ and the arithmetic $\mathbf{A}_2$.

Informally, it means that to perform an operation, e.g., addition or multiplication, in $\mathbf{A}_1$ with two numbers a and b, we map these numbers into $\mathbf{A}_2$, perform this operation there, and map the result back to $\mathbf{A}_1$.

For instance, let us take $\mathbf{A}_2 = \mathbf{Z}$, $g(x) = x + 1$ and $h(x) = x - 1$. Taking $a = 2$ and $b = 3$, we have

$$2 +_1 3 = h(g(a) + g(b)) = h(g(2) + g(3)) = h(3 + 4) = h(7) = 6$$

In such a way, these two functions g and h define the non-Diophantine arithmetic $\mathbf{A}_1$ of integer numbers. Note that $\mathbf{A}_1$ contains the same integer numbers as the conventional arithmetic **Z** but operations with them are defined in a different way.

In his papers and joint book with Czachor [10], Mark considers various choices of functions g and h for various problems in non-Diophantine arithmetic. However, from the point of view of foundation of mathematics, the set of all possible pairs (g, h) must be significantly narrowed. As discussed in Sec. 1, from the point of view of verificationism and philosophy of quantum theory, only those versions of mathematics can be substantiated which do not contain the concept of infinity. Those versions necessarily should contain a parameter L such that the theory does not contain numbers greater than L.

As noted in Sec. 1, from the point of view of verificationism and the philosophy of quantum theory, classical mathematics is not well defined because it does not contain information on how all operations with numbers should be verified. They can be verified only by using computing devices which can operate only with a finite amount of resources and cannot work with numbers greater than some number L. Nevertheless, the way of thinking of most mathematicians and physicists is such that fundamental mathematics and fundamental physics should not contain such a number L.

However, the laws of how fundamental mathematics correctly describes physics are determined by the universe in which we live, and this universe can be considered a computer that determines these laws. For example, if there is only a finite number of elementary particles in the universe, then the presence of L is mandatory in these laws. This number is determined by the state of the universe. Since this state is changing, the number L will be different at different stages of the evolution of the universe.

Based on these considerations, Mark proposed in [16] the following option for the functions g and h. Let us take a natural number L as the boundary parameter of the non-Diophantine arithmetic $\mathbf{A}_L$. We build the non-Diophantine arithmetic $\mathbf{A}_L$ taking the following functions g and h for establishing a weak projectivity between $\mathbf{A}_L$ and $\mathbf{Z}$:

$$g(x) = \begin{cases} x & \text{if } -L \leq x \leq L \\ L & \text{if } x > L \\ -L & \text{if } x < -L \end{cases}$$

and

$$h(x) = \begin{cases} x & \text{if } -L \leq x \leq L \\ L & \text{if } x > L \\ -L & \text{if } x < -L \end{cases}$$

Then the operations, that is, addition, subtraction, and multiplication, in $\mathbf{A}_L$ are defined in the following way:

$$a +_L b = h(g(a) + g(b))$$

$$a \times_L b = h(g(a) \times g(b))$$

$$a -_L b = h(g(a) - g(b))$$

The number L is called the *upper boundary number* of the arithmetic $\mathbf{A}_L$. Note that formally the non-Diophantine arithmetic $\mathbf{A}_L$ contains all integer numbers but with the above choice of the functions g and h, only numbers greater than $-L$ and less than L are accessible. All other integer numbers do not impact operations with accessible numbers. As a result, arithmetic $\mathbf{A}_L$ exactly models computer arithmetic with integer numbers [17, 18]. It is also possible to suggest that arithmetic $\mathbf{A}_L$ will be useful for building finite physics based on sound and adequate mathematical structures.

As shown in [16], the arithmetic $\mathbf{A}_L$ becomes the standard arithmetic $\mathbf{Z}$ in the formal limit $L \to \infty$. However, as noted above, from the point of view of verificationism, the value of L should be finite. For illustration, Mark considers examples of operations in the arithmetic $\mathbf{A}_L$ where $L = 10^{100}$. If $\oplus$, $\ominus$ and $\otimes$ are used to denote addition, subtraction and multiplication in this arithmetic, respectively, then:

$$1000 \oplus 1000 = 2000$$

$$10^{90} \oplus 10^{90} = 2 \times 10^{90}$$

$$10^{200} \oplus 10^{10} = 10^{100}$$

$$1000 \otimes 1000 = 1000000$$

$$10^{90} \otimes 10^{90} = 10^{100}$$

$$10^{200} \otimes 10^{10} = 10^{100}$$

$$10^{90} \ominus 10^{80} = (10^{10}–1)10^{80}$$

$$10^{200} \ominus 10^{10} = 10^{100}$$

$$(10^{200} + 1000) \ominus 10^{200} = 10^{100}$$

Direct application of the definition of operations in the arithmetic $\mathbf{A}_L$ gives the following result:

Proposition. For any natural numbers L and n, we have the following identities in the arithmetic $\mathbf{A}_L$:

$$L +_L n = L$$

$$L \times_L n = L$$

$$-L +_L (-n) = -L -_L n = -L$$

$$-L \times_L n = -L$$

$$-n +_L n = n -_L n = 0$$

$$L \times_L (-n) = -L$$

$$n -_L L = 0 \ \text{ if } n > L$$

$$0 \times_L n = 0$$

$$0 +_L n = n \text{ if } -L \leq n \leq L$$

$$0 +_L n = L \text{ if } n > L$$

$$0 +_L (-n) = -L \ \text{ if } n > L$$

and one can prove [16] the following

Theorem For any natural number L, we have:

a) addition and multiplication are commutative in the arithmetic $\mathbf{A}_L$;

b) addition in the arithmetic $\mathbf{A}_L$ is not always associative;

c) multiplication in the arithmetic $\mathbf{A}_L$ is always associative;

d) multiplication in the arithmetic $\mathbf{A}_L$ is not always distributive with respect to addition;

e) The results of addition, subtraction, and multiplication in the arithmetic $\mathbf{A}_L$ cannot be greater than L and less than $-L$.

The next section describes typical divergent integrals which appear in quantum field theory (QFT) as a consequence of the fact that standard approach to QFT is not well-defined. Then it is explained, how with Mark Burgin's approach to non-Diophantine arithmetic described above, divergent integrals in QFT can be considered without divergences in the framework of a consistent mathematical theory.

3 Elimination of divergences in quantum electrodynamics

As noted in Sec. 1, one of the key problems of modern quantum theory is the problem of divergences. Typical divergences in QFT are similar to the divergences in quantum electrodynamics (QED). As shown in [16], after the integration over hyperspherical angular variables, the integrals for the Feynman diagrams describing the electron self-energy, the photon self-energy and the electron-photon vertex are:

$$J_1 = \int_0^\infty \frac{p^3 dp}{(p^2+l)^2}; \quad J_2 = \int_0^\infty \frac{p^5 dp}{(p^2+l)^2}; \quad J_3 = \int_0^\infty \frac{p^5 dp}{(p^2+l)^3} \tag{1}$$

In standard theory, those integrals are divergent, and in the literature this is sometimes illustrated as follows. Let $J_i(p_{max})$ $(i = 1, 2, 3)$ be the integrals in Eq. (1) where the upper limit is not ∞ but p_{max}. Then a simple integration gives that if p_{max} is very large then

$$J_1(p_{max}) = \frac{1}{2}(ln\frac{p_{max}^2}{l} - 1); \quad J_2(p_{max}) = \frac{1}{2}(p_{max}^2 - 2p_{max}ln\frac{p_{max}^2}{l} + l);$$
$$J_3(p_{max}) = \frac{1}{2}(ln\frac{p_{max}^2}{l} - \frac{3}{2}) \tag{2}$$

From the point of view of formal construction of QED, one should take the limits of those expressions when $p_{max} \to \infty$ because standard QED is based on standard mathematics where there is no maximum for the momentum. However, these limits do not exist. This is an indication that mathematically QED is not well-defined. Then a question arises why, nevertheless, QED describes experimental data with a high accuracy.

The answer is as follows. Perturbation theory in QED starts from the bare electron mass m_0 and bare electron electric charge e_0. However, the description of experiment should involve not those quantities but real electron mass m and real electron charge e. It has been proved that, in each order of perturbation theory, all singularities of unknown quantities m_0 and e_0 and all singularities of QED perturbation theory are fully absorbed by m and e such that the resulting formulas expressed in terms of m and e do not contain singularities anymore.

This property of QED is characterized such that QED is a renormalizable theory. A very impressive property of QED is that it describes the electron and muon magnetic moments with the accuracy eight decimal digits. This result has been achieved in the third order of perturbation theory, and so far no comparisons of theory and experiment in higher orders is possible. It has been also proved that electroweak theory and quantum chromodynamics also are renormalizable theories. At the same time, QED and those theories cannot answer the question whether the perturbation series in them are convergent or asymptotic. Also, the existing versions of quantum gravity are not renormalizable.

Despite successes of renormalizable theories in describing experimental data, the above discussion shows that those theories are not well-defined mathematically. One of the reasons, indicated in known textbooks (see e.g., [19]) is that they contain products of quantized fields at the same points. This is not a correct mathematical operation because quantized fields are distributions.

We now consider how the integrals in Eq. (1) should be treated in non-Diophantine mathematics (NDM). We will use the version of NDM where the functions g and h are described in the preceding section and it has been noted that those functions have been proposed by Mark Burgin in [16].

A detailed description of NDM has been given in [10] and the very basic facts of NDM have been described in Sec. 2. Let $S(x)$ be a set of integer, rational or real numbers x. Then, as follows from **Theorem** in Sec. 2, in NDM there always exists a number L with the following properties: if $x_1, x_2 \in S$, then the results of addition, subtraction and multiplication of x_1 and x_2 will be the same as in standard mathematics if $|x_1|$ and $|x_2|$ are much less than L but can essentially differ from the results in standard mathematics is x_1 and/or x_2 are comparable to L. Therefore, we can say that Mark Burgin's approach is applicable not only to foundation of arithmetic, but also, in the general case, to foundation of mathematics.

Consider, for example, the integral J_1 in Eq. (1). The Riemann sums for this integral are defined as follows. We represent the interval $[0, \infty)$ as the union $[0, \infty) = \cup_{i=0}^{\infty}[p_i, p_{i+1})$ where $p_i = i\Delta p$, $(i = 0, 1, ...\infty)$ where $\Delta p > 0$. Then the

Riemann sum for J_1 is

$$S(n) = \sum_{i=1}^{n} \frac{p_i^3}{(p_i^2 + l)^2} \Delta p \tag{3}$$

and J_1 is the limit of $S(n)$ when $\Delta p \to 0$ and $n \to \infty$.

Let us note that p and l are the dimensional quantities, and their dimensions depends on systems of units. For example, in SI the dimension of p is $kg \cdot m/s$ while in the system of units $\hbar = c = 1$, which is often used in particle theory, the dimension is $1/length$. To obtain the corresponding descriptions in NDM, it is necessary to use non-Grassmannian linear spaces [10] where integer, rational and real numbers are dimensionless. For this reason one can define $p = ax$ where a is a constant having the dimension of momentum, and $x \in [0, \infty)$ is the dimensionless variable. Then

$$S(n) = \sum_{i=1}^{n} \frac{x_i^3}{(x_i^2 + b)^2} \Delta x \tag{4}$$

where $\Delta x = \Delta p/a$, $p_i = ax_i$ and $b = l/a^2$.

Since all the terms in the sum (4) are positive, in standard theory, J_1 diverges and J_1 is a limit of the sums $S(n)$ when $\Delta x \to 0$ and $n \to \infty$, then in standard theory, $\forall L > 0\ \exists \delta > 0$ and $\exists n_0$ such that $S(n) > L\ \forall \Delta x < \delta$ and $\forall n > n_0$.

To eliminate the unwelcome divergence of the considered integrals, one can use the non-Diophantine mathematics R_L obtained from A_L by replacing Z by the set of real numbers R [10]. Then operations with numbers and functions cannot go beyond the boundary number L in the positive direction and the boundary number $-L$ in the negative direction, as well as in A_L. At the same time, results of contemporary physics in general, and quantum theory in particular, which do not involve infinity in the form of divergence, are preserved in this new setting if we take L sufficiently large because for all numbers from R_L from the interval $(-L^{1/2}/2, L^{1/2}/2)$, all basic arithmetical operations are the same as in the conventional Diophantine mathematics R of real numbers. Note that if L is very large, the interval $(-L^{1/2}/2, L^{1/2}/2)$ is sufficiently large.

Contemporary quantum physics is based on the Diophantine mathematics because in it, all operations with numbers and functions are performed according to the rules of this mathematics. Using operations from a non-Diophantine mathematics in QFT, we obtain ND quantum physics. When we utilize the non-Diophantine mathematics R_L for building ND quantum physics, the Riemann sum (3) is transformed to the non-Diophantine Riemann sum

$$S_L(n) = \sum_{i=1}^{\oplus n} [(p_i^{3\otimes}) \oslash (p_i^{2\otimes} \oplus b)^{2\otimes}] \otimes \Delta p \tag{5}$$

Here $\oplus$ denotes addition, $\ominus$ denotes subtraction, $\oslash$ denotes division, $\sum^{\oplus}$ denotes multiple addition, and $\otimes$ denotes multiplication in the non-Diophantine mathematics

R_L. Taking the limit of $S_L(n)$ when $\Delta p \to 0$ and $n \to \infty$, we obtain a non-Newtonian integral [10]

$$J_L = \int^{\oplus} [(p^{3\otimes}) \oslash (p^{2\otimes} \oplus b)^{2\otimes}] \otimes dp \tag{6}$$

In ND quantum physics based on the non-Diophantine mathematics R_L, it is the counterpart of the integral J_1 that describes the electron self-energy. By construction of the non-Diophantine mathematics R_L, the sum (5) cannot be greater than the number L. Consequently, the integral (6) also cannot be greater than the number L.

By the same technique as before, in the non-Diophantine mathematics one can transform the Riemann sum (4) to the non-Diophantine Riemann sum

$$S_L(n) = \sum_{i=1}^{\oplus n} [(x_i^{3\otimes}) \oslash (x_i^{2\otimes} \oplus b)^{2\otimes}] \otimes \Delta x \tag{7}$$

Taking the limit of this sum when $\Delta x \to 0$ and $n \to \infty$, we obtain a non-Diophantine integral [10]

$$J_L = \int^{\oplus} [(x^{3\otimes}) \oslash (x^{2\otimes} \oplus b)^{2\otimes}] \otimes dx \tag{8}$$

In ND quantum physics based on the non-Diophantine mathematics R_L, the sum (7) is also the counterpart of the integral J_1 that describes the electron self energy. By construction, this sum also cannot be larger than the number L. Consequently, the integral (8) also cannot be larger than the number L.

As noted by Mark in [16], in a similar way, it is possible to demonstrate that in the non-Diophantine mathematics R_L, for all integrals — J_{L1} that describes the electron self-energy, J_{L2} that describes the photon self-energy, and J_{L3} that describes the electron-photon vertex, we have the following inequalities

$$J_{L1} \leq L, \quad J_{L2} \leq a^2 L, \quad J_{L3} \leq L \tag{9}$$

This shows that, in contrast to standard mathematics where the values of J_i $(i = 1, 2, 3)$, are infinite, their counterparts in ND quantum physics based on the non-Diophantine mathematics R_L are finite. Therefore, in ND quantum physics, the renormalization procedure can be performed in a fully legitimate mathematical way.

4 Discussion

The consideration in Secs. 1 and 2 shows that only such versions of mathematics can be free from foundational problems that do not involve the concept of infinity. One such possibility is the version of non-Diophantine mathematics where the functions g and h are as suggested by Mark Burgin in [16]. However, in the joint book of Mark Burgin with Marek Czachor [10], another version of non-Diophantine arithmetic without infinity is proposed. This is a version based on modular arithmetic where the ring $\mathbf{Z}$ is replaced by the modular ring $\mathbf{Z}_p$ where p is the characteristic of the ring.

The rules of addition, subtraction and multiplication in $\mathbf{Z}_p$ are inherited from the rules in $\mathbf{Z}$ but all operation are taken modulo p. The authors of [10] give an example that in $\mathbf{Z}_{10}$, 2+2=4 but 5+5=0.

In Sec. 1, when we discussed verificationism and the philosophy of quantum theory, we mentioned possibilities a) and b). The version of non-Diophantine mathematics discussed in Sec. 2 and its applications discussed in Sec. 3 is in the spirit of possibility a). The applications of modular arithmetic are in the spirit of possibility b). Such applications are discussed in [20].

At present, it is unclear which version of mathematics will be more promising. The immediate advantage of Mark Burgin's approach is that (as the discussion in Sec. 3 shows) the results of this approach can be immediately applied to various concrete problems, while applying the approach of [20] requires a considerable preparatory work. In any case, in fundamental mathematics, the foundational problems must be solved, and, as follows from the principle of verificationism, such mathematics should not contain the concept of infinity. As can be seen from the discussion above, Mark Burgin has made considerable contributions in this direction.

Now I will describe a problem that Mark started working on, but, unfortunately, did not finish it.

Let A and B be two processes such that:

- the initial states in A and B are the same;
- all particles in the finite states of A and B have the same momenta and spins but the final state of B contains an additional (soft) photon with a very small energy $E = \hbar\omega$.

Then, as shown in a wide literature, with a high accuracy, the differential cross sections $d\sigma_A$ and $d\sigma_B$ of the processes A and B are related as:

$$d\sigma_B = \alpha d\sigma_A F do \frac{d\omega}{\omega} \tag{10}$$

where $\alpha \approx 1/137$ is the fine structure constant, $\mathbf{k}$ is the momentum of the soft photon, o is the range of solid angles for the unit vector $\mathbf{n} = \mathbf{k}/\omega$ and F is a function of the initial and final momenta in the process A. If we consider only cases where $\omega \in [\omega_1, \omega_2]$ and integrate over o and ω then we get

$$d\sigma_B = \alpha d\sigma_A G ln \frac{\omega_2}{\omega_1} \tag{11}$$

where the function G is of the order of unity and depends only on the initial and final momenta in the process A. As noted in the literature on QED, with the logarithmic accuracy, ω_2 can be replaced by the energy of the particle which emitted the soft photon but the value of ω_1 is not limited from below. Therefore when $\omega_1 \to 0$, $d\sigma_B$ becomes infinite, and this situation is called infrared catastrophe.

As explained in the literature, the reason of the infrared catastrophe is that the result (11) is obtained in the perturbation theory over α while in fact, as follows from Eq. (11), the parameter of the perturbation theory is $\alpha \cdot ln(\omega_2/\omega_1)$ and this quantity is not less than unity when ω_1 is small. It is also explained that the infrared catastrophe can be avoided if a cutoff for ω_1 in the intermediate stage of calculations is introduced, and this cutoff disappears at the final stage of calculations. Although this rule is not well substantiated mathematically, in practice it results in avoiding the infrared catastrophe.

So, the situation is analogous to the divergences in QED discussed in Sec. 3 but now the divergences appear not at very large momenta but *vice versa*, at very small momenta. But Mark believed that well defined theories should not contain divergences at all. His idea was that the problem with the infrared catastrophe can be solved in a rigorous mathematical way in the framework of non-Diophantine mathematics, and he started working on this problem.

I had not met Mark in person and knew nothing about his family. But we communicated by phone and email. I was very impressed that, unlike many mathematicians and physicists (who believe that foundational questions are not very important and that the main thing is that the theory should work well in concrete problems), Mark believed that foundational questions are a very important part of fundamental mathematics and fundamental physics.

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