On the sum of reciprocals of primes

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Abstract

Suppose that $y > 0$, $0 \leq \alpha < 2\pi$ and $0 < K < 1$. Let P^+ be the set of primes p such that $\cos(y \ln p + \alpha) > K$ and P^- the set of primes p such that $\cos(y \ln p + \alpha) < -K$. In this paper we prove $\sum_{p \in P^+} \frac{1}{p} = \infty$ and $\sum_{p \in P^-} \frac{1}{p} = \infty$.

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1 Introduction

Let P be the set of primes and $\mathbb{N}$ be the set of natural numbers. In 1737, Euler[2] proved the sum of reciprocals of primes is divergent.

$$\sum_{p \in P} \frac{1}{p} = \infty$$

Definition 1.1. Suppose that $y > 0$, $0 \leq \alpha < 2\pi$ and $0 < K < 1$. Let

$$P^+(y, \alpha, K) = \{p \in P \mid \cos(y \ln p + \alpha) > K\}$$

and

$$P^-(y, \alpha, K) = \{p \in P \mid \cos(y \ln p + \alpha) < -K\}.$$

We write P^+ and P^- for the sake of simplicity.

Throughout this paper we always assume that $y > 0$. In this paper we prove

Theorem 1.2.

$$\sum_{p \in P^+} \frac{1}{p} = \infty \quad \textit{and} \quad \sum_{p \in P^-} \frac{1}{p} = \infty.$$

2 Proof of Theorem 1.2

We will use the prime number theorem in the proof of Theorem 1.2.

Prime Number Theorem ([1, 3])**.** *Let $\pi(x)$ be the number of primes less than or equal to x. Then*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\ln x} = 1.$$

Lemma 2.1. *Recall that $y > 0$. Let $0 \le \gamma < 2\pi$. There are at most two primes p such that*

$$y \ln p = 2n\pi + \gamma$$

for some $n \in \mathbb{N} \cup \{0\}$.

Proof. Suppose that there exist three distinct primes $p_1 < p_2 < p_3$ and $\ell, m, n \in \mathbb{N} \cup \{0\}$ such that

$$y \ln p_1 = 2\ell\pi + \gamma, \quad y \ln p_2 = 2m\pi + \gamma, \quad y \ln p_3 = 2n\pi + \gamma. \tag{1}$$

We will get a contradiction. From eq. (1), we have

$$y(\ln p_2 - \ln p_1) = 2(m - \ell)\pi, \qquad y(\ln p_3 - \ln p_1) = 2(n - \ell)\pi. \tag{2}$$

Notice that $\ell < m < n$. Let $m - \ell = h$ and $n - \ell = k$. From eq. (2), we have

$$\frac{\ln p_3 - \ln p_1}{\ln p_2 - \ln p_1} = \frac{k}{h}.$$

Therefore

$$h(\ln p_3 - \ln p_1) = k(\ln p_2 - \ln p_1)$$

and hence

$$\left(\frac{p_3}{p_1}\right)^h = \left(\frac{p_2}{p_1}\right)^k.$$

Thus

$$p_1^k p_3^h = p_1^h p_2^k.$$

This contradicts to the uniqueness of prime factorization. □

Definition 2.2. Recall that $y > 0$ and $0 < K < 1$. Let β be the number such that

$$\cos\beta = K, \quad 0 < \beta < \frac{\pi}{2}.$$

For each $n \in \mathbb{N} \cup \{0\}$, let

$$\begin{aligned} A_n &= \{p \in P \mid 2n\pi - \beta < y \ln p + \alpha \le 2n\pi + \beta\}, \\ B_n &= \{p \in P \mid (2n+1)\pi - \beta < y \ln p + \alpha \le (2n+1)\pi + \beta\} \end{aligned}$$

and

$$A = \bigcup_{n=0}^{\infty} A_n, \qquad B = \bigcup_{n=0}^{\infty} B_n.$$

Proof of Theorem 1.2

Notice that $P^+ \subset A$ and $P^- \subset B$. From Lemma 2.1, we know that $A - P^+$ has at most two elements and $B - P^-$ also has at most two elements. Therefore it is enough to show that

$$\sum_{p\in A} \frac{1}{p} = \infty \quad \textit{and} \quad \sum_{p\in B} \frac{1}{p} = \infty.$$

Recall that $y > 0$. By the prime number theorem, there exists $M > 0$ such that if $x > M$ then

$$e^{-\frac{\beta}{2y}} \frac{x}{\ln x} \leq \pi(x) \leq e^{\frac{\beta}{2y}} \frac{x}{\ln x}. \tag{3}$$

From Definition 2.2, we have

$$A_n = \left\{ p \in P \mid e^{\frac{2n\pi}{y} - \frac{\beta+\alpha}{y}} < p \leq e^{\frac{2n\pi}{y} + \frac{\beta-\alpha}{y}} \right\}$$

and

$$B_n = \left\{ p \in P \mid e^{\frac{(2n+1)\pi}{y} - \frac{\beta+\alpha}{y}} < p \leq e^{\frac{(2n+1)\pi}{y} + \frac{\beta-\alpha}{y}} \right\}.$$

Notice that $A_1, B_1, A_2, B_2, \cdots$ are mutually disjoint. There exists $N \in \mathbb{N}$ such that if $n > N$ then

$$e^{\frac{2n\pi}{y} - \frac{\beta+\alpha}{y}} > M.$$

From now on, we assume that $n > N$. By eq. (3), we can find the lower bounds of the number of elements of A_n and B_n. We have

$$\begin{aligned} |A_n| \quad \geq \quad & e^{-\frac{\beta}{2y}} \frac{e^{\frac{2n\pi}{y} + \frac{\beta-\alpha}{y}}}{\frac{2n\pi}{y} + \frac{\beta-\alpha}{y}} - e^{\frac{\beta}{2y}} \frac{e^{\frac{2n\pi}{y} - \frac{\beta+\alpha}{y}}}{\frac{2n\pi}{y} - \frac{\beta+\alpha}{y}} \\ & = \frac{ye^{\frac{2n\pi}{y} + \frac{\beta-2\alpha}{2y}}}{2n\pi + \beta - \alpha} - \frac{ye^{\frac{2n\pi}{y} - \frac{\beta+2\alpha}{2y}}}{2n\pi - \beta - \alpha} \end{aligned} \tag{4}$$

and

$$\begin{aligned} |B_n| \quad \geq \quad & e^{-\frac{\beta}{2y}} \frac{e^{\frac{(2n+1)\pi}{y} + \frac{\beta-\alpha}{y}}}{\frac{(2n+1)\pi}{y} + \frac{\beta-\alpha}{y}} - e^{\frac{\beta}{2y}} \frac{e^{\frac{(2n+1)\pi}{y} - \frac{\beta+\alpha}{y}}}{\frac{(2n+1)\pi}{y} - \frac{\beta+\alpha}{y}} \\ & = \frac{ye^{\frac{(2n+1)\pi}{y} + \frac{\beta-2\alpha}{2y}}}{(2n+1)\pi + \beta - \alpha} - \frac{ye^{\frac{(2n+1)\pi}{y} - \frac{\beta+2\alpha}{2y}}}{(2n+1)\pi - \beta - \alpha}. \end{aligned} \tag{5}$$

Notice that if $p \in A_n$ then

$$\frac{1}{p} \geq e^{-\frac{2n\pi}{y} - \frac{\beta-\alpha}{y}} \tag{6}$$

and if $p \in B_n$ then

$$\frac{1}{p} \geq e^{-\frac{(2n+1)\pi}{y} - \frac{\beta-\alpha}{y}}. \tag{7}$$

From eq. (4) and (6), we have

$$\begin{aligned}
\sum_{p\in A_n}\frac{1}{p} &\geq \left(\frac{ye^{\frac{2n\pi}{y}+\frac{\beta-2\alpha}{2y}}}{2n\pi+\beta-\alpha}-\frac{ye^{\frac{2n\pi}{y}-\frac{\beta+2\alpha}{2y}}}{2n\pi-\beta-\alpha}\right)e^{-\frac{2n\pi}{y}-\frac{\beta-\alpha}{y}}\\
&= \frac{ye^{-\frac{\beta}{2y}}}{2n\pi+\beta-\alpha}-\frac{ye^{-\frac{3\beta}{2y}}}{2n\pi-\beta-\alpha}\\
&= y\frac{(2n\pi-\beta-\alpha)e^{-\frac{\beta}{2y}}-(2n\pi+\beta-\alpha)e^{-\frac{3\beta}{2y}}}{(2n\pi-\alpha)^2-\beta^2}\\
&= y\frac{(2n\pi-\alpha)\left(e^{-\frac{\beta}{2y}}-e^{-\frac{3\beta}{2y}}\right)-\beta\left(e^{-\frac{\beta}{2y}}+e^{-\frac{3\beta}{2y}}\right)}{(2n\pi-\alpha)^2-\beta^2}\\
&= \frac{2cn-d}{(2n\pi-\alpha)^2-\beta^2}.
\end{aligned}$$

where

$$c=y\pi\left(e^{-\frac{\beta}{2y}}-e^{-\frac{3\beta}{2y}}\right)>0 \tag{8}$$

and

$$d=y\alpha\left(e^{-\frac{\beta}{2y}}-e^{-\frac{3\beta}{2y}}\right)+y\beta\left(e^{-\frac{\beta}{2y}}+e^{-\frac{3\beta}{2y}}\right).$$

Similarly from eq. (5) and (7), we have

$$\begin{aligned}
\sum_{p\in B_n}\frac{1}{p} &\geq \left(\frac{ye^{\frac{(2n+1)\pi}{y}+\frac{\beta-2\alpha}{2y}}}{(2n+1)\pi+\beta-\alpha}-\frac{ye^{\frac{(2n+1)\pi}{y}-\frac{\beta+2\alpha}{2y}}}{(2n+1)\pi-\beta-\alpha}\right)e^{-\frac{(2n+1)\pi}{y}-\frac{\beta-\alpha}{y}}\\
&= \frac{ye^{-\frac{\beta}{2y}}}{(2n+1)\pi+\beta-\alpha}-\frac{ye^{-\frac{3\beta}{2y}}}{(2n+1)\pi-\beta-\alpha}\\
&= y\frac{((2n+1)\pi-\beta-\alpha)e^{-\frac{\beta}{2y}}-((2n+1)\pi+\beta-\alpha)e^{-\frac{3\beta}{2y}}}{(2n\pi+\pi-\alpha)^2-\beta^2}\\
&= y\frac{((2n+1)\pi-\alpha)\left(e^{-\frac{\beta}{2y}}-e^{-\frac{3\beta}{2y}}\right)-\beta\left(e^{-\frac{\beta}{2y}}+e^{-\frac{3\beta}{2y}}\right)}{(2n\pi+\pi-\alpha)^2-\beta^2}\\
&= \frac{c(2n+1)-d}{(2n\pi+\pi-\alpha)^2-\beta^2}.
\end{aligned}$$

Recall eq. (8). Since $c>0$, we have

$$\sum_{p\in A}\frac{1}{p}\geq\sum_{n=N+1}^{\infty}\sum_{p\in A_n}\frac{1}{p}\geq\sum_{n=N+1}^{\infty}\frac{2cn-d}{(2n\pi-\alpha)^2-\beta^2}=\infty$$

and

$$\sum_{p\in B}\frac{1}{p}\geq\sum_{n=N+1}^{\infty}\sum_{p\in B_n}\frac{1}{p}\geq\sum_{n=N+1}^{\infty}\frac{c(2n+1)-d}{(2n\pi+\pi-\alpha)^2-\beta^2}=\infty.$$

Thus

$$\sum_{p\in A}\frac{1}{p}=\infty \quad and \quad \sum_{p\in B}\frac{1}{p}=\infty.$$

□

References

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