Charles Hutton’s Formula

Edgar Valdebenito

January 3 , 2024

ABSTRACT: Some remarks about a formula of Charles Hutton.

Keywords: Machin-like formulas, series, number Pi

1. Introduction

Charles Hutton (1737-1823) was an English mathematician. He wrote several mathematical texts. In 1774, he was elected a fellow of the Royal Society of London. He suggested Machin’s stratagem in the form:

$$\pi = 8\tan^{-1}\left(\frac{1}{3}\right)+4\tan^{-1}\left(\frac{1}{7}\right)\ ,\ (\sim 1776)$$

Remark 1:

$$\pi = 8\tan^{-1}\left(\frac{1}{3}\right)+4\tan^{-1}\left(\frac{1}{7}\right) = 8\tan^{-1}\left(\frac{1}{3}\right)+8\tan^{-1}\left(5\sqrt{2}-7\right) = 8\sum_{n=0}^{\infty}(-1)^n\left(\frac{15\sqrt{2}-20}{3}\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}\left(\frac{3+3\sqrt{2}}{50}\right)^k$$

Remark 2:

$$\pi = 8\tan^{-1}\left(\frac{1}{3}\right)+4\tan^{-1}\left(\frac{1}{7}\right) = 4\tan^{-1}\left(\frac{3}{4}\right)+4\tan^{-1}\left(\frac{1}{7}\right) = 4\sum_{n=0}^{\infty}(-1)^n\left(\frac{25}{28}\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}\left(\frac{84}{625}\right)^k$$

Remark 3:

$$\pi = 8\tan^{-1}\left(\frac{1}{3}\right)+4\tan^{-1}\left(\frac{1}{7}\right) = 16\tan^{-1}\left(\sqrt{10}-3\right)+16\tan^{-1}\left(\sqrt{100+70\sqrt{2}}-5\sqrt{2}-7\right) =$$

$$16\sum_{n=0}^{\infty}(-1)^n\left(\sqrt{100+70\sqrt{2}}+\sqrt{10}-5\sqrt{2}-10\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}\left(\frac{\sqrt{2}+2\sqrt{2-\sqrt{2}}}{20}\right)^k$$

In this note we give some formulas for π .

2. Formulas for Pi

Eq.(1):

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n\left(\frac{1+x^2}{1+x}\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}\left(\frac{x\left(1-x^2\right)}{\left(1+x^2\right)^2}\right)^k\ ,\ 0<x<1$$

Eq.(2):

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n\left(\frac{5}{6}\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}\left(\frac{6}{25}\right)^k$$

Eq.(3):

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n\left(\frac{3\sqrt{5}-5}{2}\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}5^{-k}$$

Eq.(4):

$$\pi = \frac{24}{5}\sum_{n=0}^{\infty}(-1)^n\left(1+\sqrt{2}-\sqrt{3}\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}\left(\frac{-5+4\sqrt{2}+3\sqrt{3}-2\sqrt{6}}{4}\right)^k$$

Eq.(5):

$$\pi = 4\sum_{n=0}^{\infty}(-1)^n\left(2-\frac{2}{\sqrt{3}}\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}\left(\frac{\sqrt{3}}{8}\right)^k$$

Eq.(6):

$$\pi = \frac{24}{7}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{\sqrt{3}}+\sqrt{2}-1\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}\left(\frac{\sqrt{3\left(49-20\sqrt{6}\right)}+4\sqrt{6}-9}{4}\right)^k$$

Eq.(7):

$$\pi = 4\left(15-10\sqrt{2}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{10\sqrt{2}-13}{2}\right)^{2n}\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}\left(\frac{34+50\sqrt{2}}{961}\right)^k$$

Eq.(8):

$$\pi = 8\left(3-3\sqrt{2}+\sqrt{3}\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{1}{\sqrt{3}}+\sqrt{2}-1\right)^{2n}\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}\left(\frac{\sqrt{3\left(49-20\sqrt{6}\right)}+4\sqrt{6}-9}{4}\right)^k$$

Eq.(9):

$$\pi = 3\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}\left(3\sqrt{3}-5\right)^k$$

Eq.(10):

$$\pi = \frac{24}{5}\sum_{n=0}^{\infty}\binom{2n}{n}2^{-2n}\left(\frac{-\sqrt{2}+\sqrt{6}+2\sqrt{2-\sqrt{2}}}{4}\right)^{2n+1}\sum_{k=0}^{n}\frac{(-1)^k}{2n-k+1}\binom{2n-k+1}{k}\left(\frac{2\sqrt{4-2\sqrt{2}}\left(\sqrt{3}-1\right)}{\left(-\sqrt{2}+\sqrt{6}+2\sqrt{2-\sqrt{2}}\right)^2}\right)^k$$

Eq.(11):

$$\theta+\phi = i\sum_{n=1}^{\infty}\left(2-e^{i\theta}-e^{i\phi}\right)^n\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k}{n-k}\binom{n-k}{k}\left(\frac{\left(e^{i\theta}-1\right)\left(e^{i\phi}-1\right)}{\left(2-e^{i\theta}-e^{i\phi}\right)^2}\right)^k, \quad 0<\theta,\phi<\pi/3,\ i=\sqrt{-1}$$

Eq.(12):

$$\pi = \frac{24i}{7}\sum_{n=1}^{\infty}s^n\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k}{n-k}\binom{n-k}{k}r^k, \quad i=\sqrt{-1}$$

$$s = 2-\frac{\sqrt{3}+\sqrt{2+\sqrt{2}}}{2}-i\left(\frac{1+\sqrt{2-\sqrt{2}}}{2}\right)$$

$$r = \frac{\left(-2+i+\sqrt{3}\right)\left(-2+i\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}\right)}{\left(-4+i+\sqrt{3}+i\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}\right)^2}$$

Eq.(13):

$$\pi = \frac{10}{3}\sum_{n=0}^{\infty}\frac{6^{-n}}{2n+1}\sum_{k=0}^{n}(-1)^k\left(\frac{25}{6}\right)^k\left(\binom{n+k+1}{2k+1}+\binom{n+k}{2k+1}\right)$$

Eq.(14):

$$\pi = \frac{24}{3+3\sqrt{2}+4\sqrt{3}}\left(2+\sum_{n=1}^{\infty}\frac{(-1)^n n}{2n+1}\left(\frac{10-6\sqrt{2}}{3}\right)^n\sum_{k=0}^{\lfloor n/2\rfloor}\frac{(-1)^k}{n-k}\binom{n-k}{k}\left(\frac{27+12\sqrt{2}}{196}\right)^k\right)$$

Eq.(15):

$$\pi = \frac{8}{3}\left(15\sqrt{2}-20\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{5\sqrt{2}-7}{3}\right)^n\sum_{k=0}^{n}\binom{n+k}{2k}\left(\frac{50\sqrt{2}-62}{3}\right)^k$$

3. References

[1] J. U. Arndt, Matters Computational, Springer-Verlag, 2011.

[2] P. Beckmann, A History of Pi, Golem Press, 1971.

[3] B. Berndt, Ramanujans Notebooks, I (1985), Springer.n

[4] B. Berndt, Ramanujans Notebooks, II (1991), Springer.

[5] B. Berndt, Ramanujans Notebooks, III (1991), Springer.

[6] B. Berndt, Ramanujans Notebooks, IV (1994), Springer.

[7] H. Chien-Lih, More Machin-type identities, Math. Gaz. 81, 1997.

[8] C. Hutton, Philosophical Transactions of the Royal Soc. of London, v66, 1776, 476-492.