Hidden nonlinearity in Newton's second law of gravitation in (2+1)-dimensional space-time

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By assuming that the Ricci curvature tensor consists of a subset (scalar) field, we propose that Newton's second law of gravitation in (2+1)-dimensional space-time, a linear equation, could have hidden nonlinearity. This subset field satisfies a non-linear subset field theory where in the case of an empty space-time or the weak field, it reduces to Newton's linear theory of gravitation.

Keywords: *hidden nonlinearity, subset field, Newton's second law of gravitation, weak field, (2+1)-dimensional empty space-time.*

I. INTRODUCTION

It is commonly believed that there exists no nonlinearity in Newton's theory of gravitation (Newtonian field equation, Newton's second law of gravitation)[1–6]. It is because explicitly Newton's theory of gravitation is written in a linear form. Related to the general theory of relativity, Newton's theory of gravitation is the weak-field limit of Einstein's non-linear theory of gravitation. So, how could nonlinearity exist in Newton's theory of gravitation?

We assume that the curvature tensor (the set of the solutions of Einstein field equations) in an empty space-time consists of a subset field, a scalar field. An empty space-time here means that there is no matter present and there is no physical fields exist except the weak gravitational field. The weak gravitational field does not disturb the emptiness. But other fields disturb the emptiness[7].

A subset field is locally equal to the curvature tensor i.e. the curvature tensor can be obtained by patching together subset fields (except in a zero-measure set) but globally different. The difference between the subset fields and the curvature tensor in an empty space-time is global instead of local since the subset fields obey the topological quantum condition but the curvature tensor does not.

Newton's theory of gravitation expressed using the curvature tensor satisfies a linear field equation only, but a subset field satisfies linear and non-linear field equations. Both, the curvature tensor and a subset field, satisfy a linear field equation in the case of the weak field of gravitation. It means that, in the case of the weak field, a non-linear subset field theory reduces to Newton's linear theory of gravitation.

Inspired by the works of Ranada[8,9], we assume that the Riemann curvature tensor could consist of the subset fields and we propose that Newton's second law of gravitation in (2+1)-dimensional empty space-time, a linear equation, could have hidden nonlinearity. This nonlinearity could exist because Newton's theory of gravitation in an empty space-time is the weak-field limit of a non-linear subset field theory. To the best of our knowledge[1–6], the formulation of hidden nonlinearity in Newton's theory of gravitation has not been done yet.

II. THE NEWTONIAN LIMIT

In the general theory of relativity, the motion of test bodies in (3+1)-dimensional curved space-time is governed by the geodesic equation which can be written as[1,2]

$$\frac{d^2x^\alpha}{d\tau^2} + \sum_{\mu,\nu} \Gamma^\alpha_{\ \mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \tag{1}$$

where $x^\alpha(\tau)$ is the world line of the particle in global inertial coordinates, and $\alpha, \mu, \nu = 0, 1, 2, 3$.

In the Newtonian limit, we treat that the motion of a body is much slower than the speed of light. It has the consequence that the proper time, τ, may be approximated by the coordinate time, t. So, for the time-time components, $\mu, \nu = t$, we may approximate $dx^\mu/d\tau$, $dx^\nu/d\tau$ in the second term of eq.(1) as $(1, 0, 0, 0)$. It means that the space-space components are vanish. Thus, eq.(1) becomes[2]

$$\frac{d^2x^\alpha}{dt^2} = -\Gamma^\alpha_{\ tt} \tag{2}$$

We have for the space components, $\alpha = 1, 2, 3$[1,2]

$$\Gamma^\alpha_{\ tt} = \frac{\partial\phi}{\partial x^\alpha} \tag{3}$$

i.e. the Christoffel symbol, $\Gamma^\alpha_{\ tt}$, is related to the gradient of the gravitational (scalar) potential, ϕ. Here, again, due to the motion of a body being much slower than the speed of light, the time derivatives of ϕ have been neglected.

In the case of (1+1)-dimensional space-time, by substituting eq.(3) into (2), the motion of a body is governed by the equation[2]

$$\vec{a} = -\vec{\nabla}\phi \tag{4}$$

where $\vec{\nabla}$ is the gradient operator with respect to 1-dimensional space and

$$\vec{a} = \frac{d^2\vec{x}}{dt^2} \tag{5}$$

is the acceleration of a body relative to global inertial coordinates of flat metric[2].

We see from eq.(4), in Newton's point of view, test bodies are in motion with acceleration or gravitational field. It means that there exists the gravitational forces act upon test bodies. The gravitational forces cause test bodies to orbit on "a curved line" in a flat space-time. On the other side, roughly speaking, we could say that eq.(1) shows the trajectories of test bodies following the geodesic "straight line" in the curved space-time. These points of view are the important difference between Einstein's general theory of relativity (1) and its Newtonian limit (4).

By using eq.(4), we can write Newton's second law of gravitation in (1+1)-dimensional space-time as

$$\vec{F} = m\ \vec{a} = -m\vec{\nabla}\phi \tag{6}$$

where $\vec{F}$ is the gravitational force, m is mass, and $\vec{a}$ is the gravitational field (gravitational acceleration) and ϕ is the gravitational potential. We see from eq.(6) that the difference in the gravitational potential shows the existence of acceleration or gravitational field. The existence of the gravitational force affects the test bodies to move with the acceleration. In analogy to the relation between the electromagnetic potential and the electromagnetic fields as shown e.g. in the Aharonov-Bohm effect[10], probably we could generalize or interpret from eq.(6) or (4) that the gravitational potential is more fundamental than the gravitational field.

III. WEAK-FIELD LIMIT OF GRAVITATION

In the limit of weak gravitational fields, low velocities or static[3] (of gravitational sources), and small pressure, the general theory of relativity reduces to Newton's theory of gravitation[1].

In the case of the weak field, the metric tensor in (3+1)-dimensional space-time can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, $h_{\mu\nu}$ is small perturbation, $|h_{\mu\nu}| << 1$. Small perturbation have values[4]

$$h_{tt} = -2\phi,\ h_{t\mu} = h_{\mu t} = 0,\ h_{\mu\nu} = -2\ \delta_{\mu\nu}\ \phi \tag{8}$$

so the related metric can be written as

$$ds^2 = (1-2\phi)\ dx^2 - (1+2\phi)\ dt^2 \tag{9}$$

Linearization (we ignore the non-linear terms of connection[3]) of the Ricci curvature tensor, due to the weak field, yields[1]

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} \tag{10}$$

This equation is identical to Abelian field strength in electrodynamics where the curvature (the Ricci tensor), $R_{\mu\nu}$, is identical to the field strength tensor, $F_{\mu\nu}$, and the connection (Christoffel symbol), $\Gamma^\alpha_{\mu\alpha}$, is identical to the gauge potential, A_μ.

By considering the importance of eq.(3), so we could understand how the gravitational potential affects the geometry of space-time, the time-time components of Ricci curvature tensor (10) can be written as[1]

$$R_{tt} = \partial_\alpha \Gamma^\alpha_{tt} - \partial_t \Gamma^\alpha_{t\alpha} \tag{11}$$

where the second term in the right-hand side of (11) is assumed zero, $\partial_t \Gamma^\alpha{}_{t\alpha} = 0$ (due to the test body of the gravitational source moving very slowly or static). Eq.(11) becomes

$$R_{tt} = \partial_\alpha \Gamma^\alpha_{tt} \tag{12}$$

By substituting (3) into (12) we obtain Newton's theory of gravitation written below[1,5,6]

$$R_{tt} = \nabla^2\phi \tag{13}$$

where $\nabla^2 = \vec{\nabla}\cdot\vec{\nabla}$ (div of grad) denotes the Laplace operator or Laplacian of (3-dimensional) space[2], and

$$\nabla^2\phi = 4\pi\rho \tag{14}$$

is Poisson's equation[1,2], ρ is the mass density.

By substituting eq.(14) into eq.(13) we obtain Newton's theory of gravitation expressed as Newtonian field equation[1]

$$R_{tt} = 4\pi\rho \tag{15}$$

We see that Newton's theory of gravitation (13) and Newtonian field equation (15) are obtained by assuming that the gravitational source is moving very slowly or static and the gravitational field is weak. Eq.(15) shows us that by choosing time-time components of the Ricci tensor, we can recover Poisson's equation (14).

IV. SUBSET FIELDS PROPERTY AND MAPS $S^3 \to S^2$

Let us consider maps of subset fields (consisting of complex scalar fields) from a finite radius r to an infinite radius implying from the stronger field to the weak field. A scalar field has properties that, by definition, its value for a finite r depends on the magnitude and the direction of the position vector, $\vec{r}$, but for an infinite r it is well-defined[9] (it depends on the magnitude only). In other words, for an infinite r, a scalar field is isotropic. Throughout this article, we will work with the classical scalar field.

The property of such scalar fields can be interpreted as maps $S^3 \to S^2$[8] where S^3 and S^2 are 3-dimensional and 2-dimensional spheres respectively i.e. after identifying via stereographic projection, 3-dimensional physical space, $R^3 \cup \{\infty\}$, with the sphere S^3 and the complete complex plane, $C \cup \{\infty\}$, with the sphere S^2.

Let us discuss the maps above more formally. Assume that we have a scalar field as a function of the position vector, $e^a(\vec{r})$, with a property that can be interpreted using the non-trivial Hopf map written below[8,9]

$$e^a(\vec{r}) : S^3 \to S^2 \quad (16)$$

These maps $S^3 \to S^2$ can be classified in homotopy classes labeled by the value of the corresponding Hopf indexes, integer numbers, and the topological invariants[8,9]. The other names of the topological invariants are the topological charge, and the winding number (the degree of a continuous mapping)[11]. The topological charge which is independent of the metric tensor could be interpreted as energy[12].

We see there exists (one) dimensional reduction in such maps. We consider this dimensional reduction as a consequence of the isotropic (well-defined) property of a scalar field for an infinite r. The property of a scalar field as a function of space seems likely in harmony with the property of space-time itself. Space-time could be locally anisotropic but globally isotropic (the distribution of matter-energy in the universe is assumed to be homogeneous).

V. NON-LINEAR AND LINEARIZED RICCI THEORIES

We assume that a subset field, a component of the curvature, e^a, as a map of the gravitational theory in (3+1) to (2+1)-dimensional space-time written below

$$e^a(\vec{r}, t) : M^{3+1} \to M^{2+1} \quad (17)$$

where M denotes manifold. This map (17) differs from a time-independent map in eq.(16). This problem could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[13].

By considering that the field strength tensor is identical to the curvature[10], we could write[8,9] the Ricci curvature tensor which its components satisfy the map (17) as follows

$$R^a_{\mu\nu} \approx \frac{\partial_\mu e^{a^*} \partial_\nu e^a - \partial_\nu e^{a^*} \partial_\mu e^a}{(1 + e^{a^*} e^a)^2} \quad (18)$$

where e^a is a subset of Ricci curvature tensor, and e^{a^*} is the complex conjugate of e^a. Eq.(18) is the non-linear equation where the nonlinearity is shown by the $e^{a^*} e^a$ term in the denominator. The superscript index a in e^a represents a set of indices that label the components of the subset field.

In the case of the weak field, the subset field is very small, $|e^{a^*} e^a| << 1$, so eq.(18) reduces to a linear field equation as written below

$$R^a_{\mu\nu} \approx \partial_\mu e^{a^*} \partial_\nu e^a - \partial_\nu e^{a^*} \partial_\mu e^a \quad (19)$$

We assume that this equation is identical to the electromagnetic field strength tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. It means that the linearized Ricci theory (10) could be interpreted as the same as the Ricci theory in the case of the weak field (19).

VI. POTENTIAL AND CLEBSCH VARIABLES

Small perturbations of metric or linearized metric perturbations in eq.(7) take a role as "potentials" in the weak field or the linearized gravitation[4]. Roughly speaking, it (more precisely the Christoffel symbol, a connection) is identical to the potential in electromagnetism which consists of electric (scalar) and magnetic (vector) potentials[4]. In the language of a wave, the small perturbation of metric can be written as[4]

$$h_{ab} = \rho_{ab}\ e^{i\vec{k}\cdot\vec{r}} \quad (20)$$

where ρ_{ab} is amplitude and $\vec{k}$ is wave vector. In empty space, the space of a weak field, the amplitude is constant.

In analogy to eq.(20), we consider the subset field, $e^a(\vec{r}, t)$, as the scalar "sub-potential" and we propose that the subset field could be written as[14]

$$e^a(\vec{r}, t) = \rho^a(\vec{r}, t)\ e^{iq(\vec{r}, t)} \quad (21)$$

where $\rho^a(\vec{r}, t)$ is the amplitude, $q(\vec{r}, t)$ is the phase. As (20), we could interpret the subset field, $e^a(\vec{r}, t)$, as the perturbation or disturbance where the physical disturbance is the real part of $e^a(\vec{r}, t)$[15].

The related "potential" (identical to the Christoffel symbol, a connection) is

$$e_{\nu a} = f_a\ \partial_\nu q \quad (22)$$

where the function of amplitude could be written as

$$f_a = -1/\left\{2\pi[1 + (\rho^a)^2]\right\} \quad (23)$$

We call the functions, $f_a(\vec{r}, t)$ and $q(\vec{r}, t)$ as the Clebsch variables[13] or Gaussian potentials[16,17]. These Clebsch variables are related to any divergenceless vector field[8]. An example of a divergenceless vector field is vorticity, $\vec{\omega}$, in hydrodynamics[17] or the magnetic field, $\vec{B}$, where $\vec{\nabla} \cdot \vec{B} = 0$. The Clebsch variables are not uniquely defined (many different choices are possible for them)[8]. We see from eq.(23) that $e_{\nu a}$ could be viewed as vector potential. Here, $e_{\nu a}$ is not a total derivative, otherwise it would be a pure gauge[13]. The subscript index ν in $e_{\nu a}$ represents space-time coordinates.

By using eq.(22), Ricci tensor (19) could be written as[13]

$$R^a_{\mu\nu} \approx \partial_\mu (f_a\ \partial_\nu q) - \partial_\nu (f_a\ \partial_\mu q) \quad (24)$$

This is the Ricci tensor written in terms of the Clebsch variables. Equation (23) is equivalent to eq.(10).

In analogy to R_{tt} of the weak-field and Newtonian limit (11), (12), we consider (24) as

$$R^a_{tt} \approx \partial_\alpha (f_a\ \partial_\alpha q) - \partial_t (f_a\ \partial_t q) \quad (25)$$

where the index α denotes the space component (space coordinate). The second term on the right-hand side of (25) is equal to zero because, in the Newtonian limit, it is assumed that the speed of the body as the gravitational source is very slow compared to the speed of light. So eq.(25) becomes

$$R^a_{tt} \approx \partial_\alpha (f_a \; \partial_\alpha q) \tag{26}$$

The first ∂_α in eq.(26) means divergence and the second ∂_α gradient.

VII. HIDDEN NONLINEARITY IN NEWTON'S SECOND LAW OF GRAVITATION

By substituting eq.(3) into (2) we have

$$\frac{d^2 x^\alpha}{dt^2} = -\frac{\partial \phi}{\partial x^\alpha} = -\partial_\alpha \phi \tag{27}$$

where ∂_α denotes gradient. In the case of 1-dimensional space, eq.(27) can be written as

$$\frac{d^2 \vec{x}}{dt^2} = -\frac{d\phi}{d\vec{x}} \tag{28}$$

This equation (28) is the same as eq.(4). In analogy to (5), (27), let us define

$$\frac{d^2 x^\alpha}{dt^2} \equiv a_\alpha \tag{29}$$

By substituting eq.(29) into (27) we obtain

$$a_\alpha = -\partial_\alpha \phi \tag{30}$$

Eq.(13) can be written as

$$R_{tt} = \partial^\alpha \partial_\alpha \phi \tag{31}$$

where $\alpha = 1, 2$, denotes 2-dimensional space and $\partial^\alpha = \partial / \partial x_\alpha$ denotes divergence. We see from eq.(31) that the divergence of the gradient of a scalar function is a scalar, so R_{tt} is a scalar. By substituting eq.(30) into eq.(31) we obtain

$$R_{tt} = \partial^\alpha (-a_\alpha) = -\partial^\alpha a_\alpha \tag{32}$$

By integrating both sides of eq.(32) with respect to x_α, we find that

$$a_\alpha = -\int R_{tt} \; dx_\alpha \tag{33}$$

By substituting eq.(33) into (6) and replace $\vec{a}$ by a_α, $\vec{F}$ by F_α, we obtain

$$F_\alpha = -m \int R_{tt} \; dx_\alpha \tag{34}$$

where F_α is the gravitational force defined in (2+1)-dimensional space-time and R_{tt} is given by eq.(11).

By substituting eq.(26) into (34) i.e. by replacement R_{tt} by R^a_{tt}, we obtain

$$F_\alpha \approx -m \int \partial_\alpha (f_a \; \partial_\alpha q) \; dx_\alpha \tag{35}$$

Eq.(35) is the equation of Newton's second law of gravitation in (2+1)-dimensional space-time written using the Clebsch variables.

VIII. DISCUSSION AND CONCLUSION

Roughly speaking, the general theory of relativity is Einstein's (non-linear) theory of gravitation, space, and time[18]. It describes the interplay between the local distribution of matter-energy and the curvature of space-time[19]. In the limit of weak gravitational fields, low velocities of the test body or the gravitational sources, and small pressure, the general theory of relativity reduces to Newton's linear theory of gravitation[1].

Eq.(4) is the basic equation of Newton's theory of gravitation and the general theory of relativity does indeed reduce to Newton's theory of gravitation in the appropriate limit. Note that although the predictions of the general theory of relativity agree with those of Newton's theory of gravitation, the underlying point of view is radically difference[2].

In the general theory of relativity point of view, the mass-energy of the sky object e.g. the Sun produces a curvature of the space-time. The Earth is in free motion without acceleration (there is no forces act upon the Earth). The Earth travels on a geodesic "straight line" of the curved space-time to orbit the Sun. It is shown by the geodesic equation (1). In Newton's point of view, the Sun creates the a gravitational field (gravitational acceleration) as shown in eq.(4) that exerts a gravitational force upon the Earth. This gravitational force causes the Earth to orbit (on curved line) the Sun rather than move in a straight line[2] in a flat space-time.

Newton's theory of gravitation, i.e. the curvature, R_{tt}, expressed using the Christoffel symbol (12), satisfies a linear field equation only, but a subset field (expressed using the Clebsch variables) satisfies linear and non-linear field equations. Both satisfy a linear field equation in the case of the weak field of gravitation. It means that, in the case of the weak field, a non-linear subset field theory reduces to Newton's linear theory of gravitation i.e. the non-linear Ricci curvature tensor (18) reduces to the linearized Ricci curvature tensor (19).

The linearized Ricci curvature tensor (19) is locally equivalent to eq.(10), but globally different. Eq.(10) is no longer valid globally. The difference between the subset fields and Ricci curvature tensor in empty space is global instead of local since the subset fields obey the topological quantum condition but Ricci curvature tensor does not.

In analogy to Hopf map (16), we assume that a subset (scalar) field or a component of Ricci curvature tensor as a map of gravitational theory in (3+1) to (2+1)-dimensional space-time (17). This map (17) differs from a time-independent Hopf map (16). This problem could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[13].

It implies there exists (one) dimensional reduction in such a map (17). We consider this dimensional reduction as a consequence of the isotropic (well-defined) property of a subset field for an infinite r (infinite distance from the source) where the gravitational field is weak. It implies also that the linearized Ricci curvature tensor and the derived Newton's theory of gravitation can be formulated in (2+1)-dimensional space-time.

The related "potential" (22) which is identical to the Christoffel symbol, a connection, can be written using the Clebsch variables[8,13] or Gaussian potentials[16,17]. These Clebsch variables are related to any divergenceless vector field[8]. An example of a divergenceless vector field is vorticity, $\vec{\omega}$, in hydrodynamics[17] or the magnetic field, $\vec{B}$, where $\vec{\nabla} \cdot \vec{B} = 0$. The Clebsch variables are not uniquely defined (many different choices are possible for them)[8]. The related potential (22) could be viewed as vector potential. This vector potential is not a total derivative, otherwise, it would be a pure gauge[13].

By using the related potential (22), Ricci curvature tensor (24) and its time-time components in the case of the weak-field and Newtonian limit (25), (26), can be formulated using the Clebsch variables. In turn, the time-time components of Ricci tensor (26) are useful when we construct Newton's second law of gravitation (35).

We could say that Newton's second law of gravitation (35) contains the hidden nonlinearity. The hidden nonlinearity is contained in the Clebsch variables. The hidden nonlinearity in Newton's second law of gravitation or, in general in Newton's theory of gravitation, has deep consequences as it could be related to the existence of the topological object (gravitational knot)[20].

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