CALCULATING THE CURVATURE OF SPACE-TIME

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Abstracts

We are working on concrete applications of the relativistic Schwarzschild metric to the cosmos. In this report we calculate the Gaussian curvature of space-time. The relativistic Schwarzschild metric solves Einstein's equations exactly assuming a point gravitational mass and empty space in its vicinity. This metric leads to a static and symmetric solution 2D of the mathematical equation of space-time that allows to calculate the Gaussian curvature at each point. We have calculated some curvature values and found a simple equation to calculate them which allows us to extend the results to a wider range of distances. Finally using this equation and the Birkhoff–Jebsen theorem we have studied the curvature in a homogeneous and isotropic universe with a constant energy density and obtain a value very close to zero for the curvature at any inner point of that universe.

Keywords: general relativity, curvature of space-time., Schwarzschild metric.

1 -The problem of the curvature of space-time

First, we are concerned with the problem of calculating the curvature of space-time caused by a spherical and static black hole at a point located at a distance "r" from the center of the black hole. This point will always be further away from the event horizon or Schwarzschild radius, “Rs”. Schwarzschild solves the equations of the general theory of relativity [1] for an assumption of a point gravitational mass and a surrounding empty space, establishing a metric and an equation for space-time that turns out to be stationary in time and with spherical symmetry, resulting in a 2D surface, (the Flamm paraboloid), which is represented in Fig. 1.

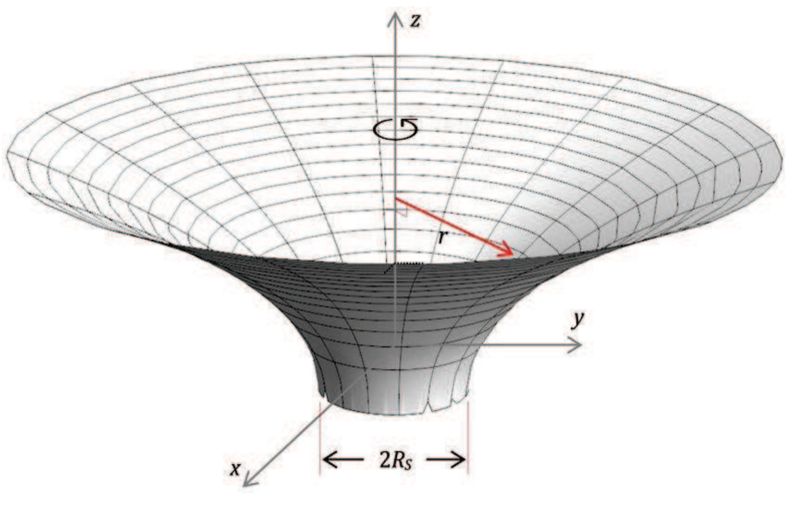


Fig. 1 Space-time in the Schwarzschild metric. Flamm paraboloid

2.- Resolution of the mathematical problem

Flamm's paraboloid, mathematical solution to the Schwarzschild metric, is a 2D surface inserted in a space R^3. Its geometry allows us to parameterize the paraboloid as a function of the observer's distance from the point mass "r" and the azimuth angle "φ". The problem admits a mathematical treatment of differential geometry of surfaces [2], and with it we are going to calculate values of Gaussian Curvature.

Surface parameters (r, φ)

$0 \leq r < \infty, \quad 0 \leq \varphi < 2\pi$

which has this parametric equation:

$x = r \cos\varphi$

$y = r\ \mathrm{sen}\varphi$

$z = 2(Rs(r- Rs))^{1/2}$

and by vector equation:

$f(x,y,z) = (r \cos\varphi,\ r\ \mathrm{sen}\varphi,\ 2(Rs(r- Rs))^{1/2})$

Determination of velocity, acceleration, and normal vectors to the surface

$ðf/ð\varphi = (-r\ \mathrm{sen}\varphi, r\cos\varphi, 0)$ $\qquad ð^2f/ð\varphi^2 = (-r \cos\varphi,\ -r\ \mathrm{sen}\varphi,\ 0)$

$ðf/ðr = (\cos\varphi, \mathrm{sen}\varphi, (r/Rs\ -1)^{-1/2})$ $\qquad ð^2f/ðr^2 = (0,\ 0,\ (-1/(2R_s)).\ (r/Rs\ -1)^{-3/2})$

$ðf/ð\varphi ðr = (-\mathrm{sen}\varphi,\ \cos\varphi,\ 0)$

$n = (ðf/ð\varphi \times ðf/ðr) = (r\cos\varphi/(r/Rs\ -1)^{1/2},\ r\mathrm{sen}\varphi/(r/Rs\ -1)^{1/2},\ -r)$

$[n] = r\,((1/(r/Rs\ -1)) +1)^{1/2}$

$\mathbf{n} = n/[n]$

Curvature and curvature parameters

Gauss curvature $\quad$ $\mathbf{K = LN\text{-}M^2/EG\text{-}F^2}$

$L = ð^2f/ð\varphi^2.\ \mathbf{n}$ $\qquad E = ðf/ð\varphi.\ ðf/ð\varphi$

$N = ð^2f/ðr^2.\ \mathbf{n}$ $\qquad G = ðf/ðr.\ ðf/ðr$

$M = (ðf/ð\varphi ðr).\ \mathbf{n}$ $\qquad F = ðf/ð\varphi.\ ðf/ð\varphi$

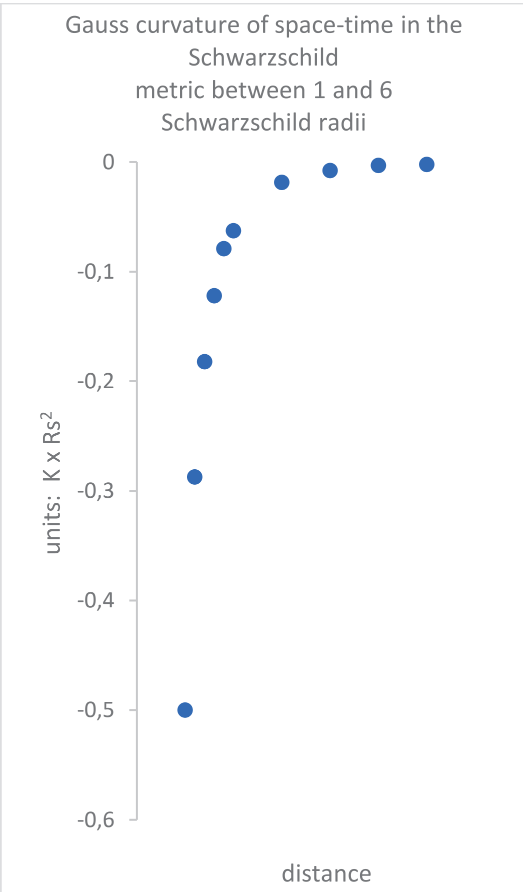
Completing a previous work of ours, [3], we have particularized the equations in 20 points between 1 and 1400 Schwarzschild radii, Rs, calculating the corresponding curvatures as shown in the results table 1. Fig. 2

Thus, although in the metric there is a singularity at the point 1Rs, the value of the Gaussian curvature for the singularity is resolved mathematically calculating a limit. We have calculated that limit for Gauss curvature.

3.- Results of curvature values

Table 1. Gaussian curvature values according to the Schwarzschild metric

Distance to the point mass	**Value of Gauss Curvature k**	Distance to the point mass	**Value of Gauss Curvature k**
1Rs	**-0,5000 x Rs^{-2}**	60Rs	**-2,325.10^{-6} x Rs^{-2}**
1,2Rs	**-0,2873 x Rs^{-2}**	80Rs	**-9,596.10^{-7} x Rs^{-2}**
1,4Rs	**-0,1821 x Rs^{-2}**	100Rs	**-4,925.10^{-7} x Rs^{-2}**
1,6Rs	**-0,1220 x Rs^{-2}**	200Rs	**-5,963.10^{-8} x Rs^{-2}**
1,8 Rs	**-0,0790 x Rs^{-2}**	400Rs	**-4,800.10^{-9} x Rs^{-2}**
2Rs	**-0,0625 x Rs^{-2}**	600Rs	**-2,376.10^{-9} x Rs^{-2}**
3Rs	**-0,0186 x Rs^{-2}**	800Rs	**-9,710.10^{-10} x Rs^{-2}**
4Rs	**-0,0078 x Rs^{-2}**	1000Rs	**-5,059.10^{-10} x Rs^{-2}**
5Rs	**-0,0030 x Rs^{-2}**	1200Rs	**-2,883.10^{-10}x Rs^{-2}**
6Rs	**-0,0023 x Rs^{-2}**	1400Rs	**-1,810.10^{-10} x Rs^{-2}**



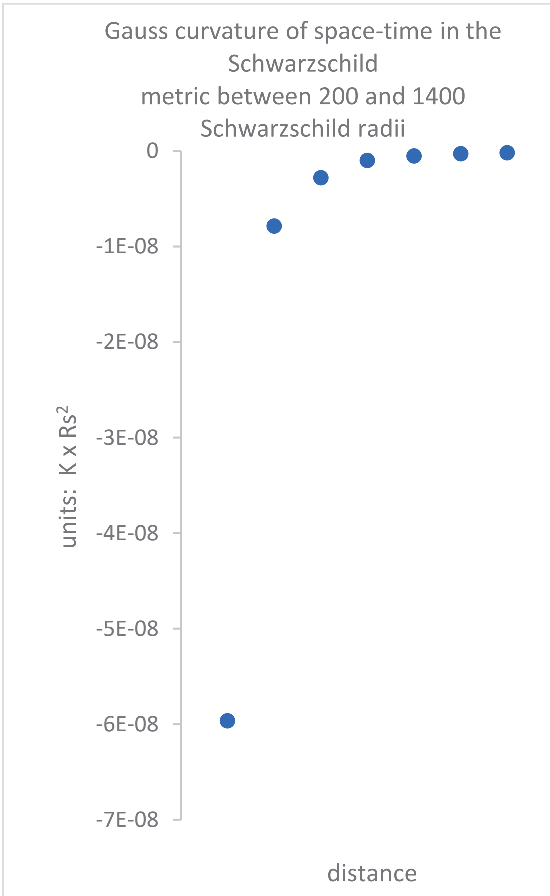


Fig 2 Gauss curvature of space-time in the Schwarzschild metric

4.- An equation to calculate the curvature of space-time according to the Schwarzschild metric

An adjustment equation has been obtained using an Excel program by regression methods throughout this wide range of distances. The degree of quality of the fit obtained by calculating the parameter R^2 is very high, 0.9999. For this reason, it is to be expected that this equation allows interpolate the calculation of Gaussian curvature values, in this wide range of distances, with high accuracy without the need to carry out the laborious calculations that would otherwise have to be done.

Fit equation between 1 and 1400 Schwarzschild radii

Gaussian curvature: $k = -0{,}5268\ (r/Rs)^{-3{,}054} \times Rs^{-2}$

Fit quality $R^2 = 0{,}9999$

Rounding decimals and according to definition of Schwarzschild radium, Rs

$Rs = 2GM/c^2$

where G is the universal gravitation constant, and M is the mass of the black hole, we can express the adjustment equation we have found as the following approximate equation:

$$\mathbf{k = -GM/c^2r^3} \qquad (1)$$

where k is the Gaussian curvature of space-time according to the Schwarzschild metric.

5.- Calculating the curvature of space-time in a homogeneous and isotropic universe

We're going to calculate the dimensionless curvature for a homogeneous, isotropic universe with an energy density ρ and we're going to do it at a generic point that lies within it. This universe will have a constant curvature, let's calculate it. This model of the universe is the same as the cosmological standard model ΛCDM.

To do this, we are going to use the results obtained in this work regarding our curvature formula (1), the Birkhoff–Jebsen theorem.

5.1- Birkhoff–Jebsen theorem

We make a brief comment on this theorem of mathematics applied to the theory of generalized relativity. First, we summarize Professor Fulvio Melia in reference [5] to explain it.

"If we have a spherical universe of mass-energy density ρ and radius r and within it a concentric sphere of radius r_s smaller than r, it is true that the acceleration due to gravity at any point on the surface of the sphere of relative radius r_s to an observer at its origin, depends solely on the mass-energy relation contained within this sphere".

Thus, according to this, to calculate the curvature of the gravitational field of a point located at a distance "r_s" from the geometric center that we are considering in our continuous universe, it is only necessary to consider its interaction with the points that are at a radius smaller than "r_s", therefore, the mass "m" to be considered will only be that contained in the sphere of radius "r_s".

In general relativity, Birkhoff's theorem [6] states that any spherically symmetric solution of the vacuum field equations must be statically and asymptotically flat. This means that the outer solution (that is, the spacetime outside a gravitational, non-rotating, spherical body) must be given by the Schwarzschild metric.

5.2- Curvature calculations

Let us consider applying our formula to our universe, for this we consider a sphere of radius r inside, the Birkhoff-Jebsen theorem assures us that if we want to calculate the space-time curvature on the surface of the sphere, we could consider only the interaction with the gravitational mass found inside. Furthermore, as the Birkhoff-Jebsen theorem assures us that the solution is given by the Schwarzschild metric. The curvature formula that we have obtained can be applicable in this case, taking into account that the interaction with the interior points of the sphere is, that is, the gravitational field on the surface of the sphere is reduced to an interaction with a point mass of equal magnitude in the center of the sphere and in this case the equation to calculate curvatures of the Schwarzschild model is applicable. that we have found.

Since the energy density "ρ" in this universe is constant, it will be constant in every sphere that we are considering and thus we can write:

$$M = \rho.(4\pi r^3/3) \quad (2)$$

According to our curvature adjustment equation, we have:

$$K = -GM/c^2r^3 \quad (3)$$

substituting (2) in (3) we get:

$$K = -4\pi G\, \rho/3c^2$$

Equation found, which relates the curvature of space-time to energy density, valid at any point in our model of the universe.

$$\mathbf{K/\rho = -4\pi G/3c^2 = -0{,}3104.10^{-26}\ m/Kg}$$

k is the Gaussian curvature (m^{-2}) and ρ is the energy density (Kg/m^{-3})

Thus, this equation assigns us the same curvature value at each point of the universe we are studying, which is characterized by a constant energy density and presents the properties of isotropy and uniformity throughout it. That curvature value, which depends only on the energy density and is the same at every point in our model of our universe, we will call it the "curvature of space-time" and now we will calculate it:

How the energy density of the universe is close to the critical density:

$$\rho = 0{,}9.10^{-26} Kg/m^3,$$

$$K = -4\pi G\rho/3c^2 = (-0{,}3104.10^{-26}).\rho$$

the curvature of space-time results in

Curvature of space-time
K = 0,3.10^{-52} m^{-2} ≅ $ZERO$

6.- Conclusions

We have calculated the curvature value of space-time according to a model of an isotropic, homogeneous universe with a constant energy density at all its points. We have obtained that the curvature only depends on the value of the energy density and a constant value calculated based on the universal gravitation constant and the value of the speed of light. This leads us to the same curvature value in each of its points if these are inner points in the topological sense. To obtain this result we have relied on the Schwarzschild metric applied in a very special way to our universe model and the Birkhoff–Jebsen theorem. The result obtained from a curvature value very close to zero agrees with the experimental values obtained by the Planck Mission [6] and with the opinion of many scientists currently.

7.- References

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