ON WILKER-TYPE INEQUALITIES

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Abstract. In this paper, we present elementary proofs of Wilker-type inequalities involving trigonometric and hyperbolic functions. In addition, we propose some conjectures which extend and generalize the Wilker-type inequalities.

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1. Introduction

The inequalities in the following Theorem 1.1 are referred to as Wilker inequalities ([3]).

Theorem 1.1 (Wilker inequalities).
(a)

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \quad x \in (0, \pi/2). \tag{1.1}$$

(b)

$$2 + \frac{8}{45}x^3 \tan x > \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x, \quad x \in (0, \pi/2). \tag{1.2}$$

Inequality (1.1) was proposed by J. B. Wilker in [3]. The question as to find the largest constant c such that the inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x, \quad x \in (0, \pi/2) \tag{1.3}$$

holds was also proposed ([3]). A refinement of this inequality is established as in (1.2), where the constants $\frac{8}{45}$ and $\left(\frac{2}{\pi}\right)^4$ are best possible. The proof of the inequalities in (1.2) can be given by using the Bernoulli numbers ([2]) or the Taylor series expansion ([6]).

Since Wilker's inequality was proposed in [3], there have been various generalization, refinements, sharpeness and extension of this inequality. We list some of them as follows:

- Let $x \in (0, \pi/2)$ and $0 < \lambda \le \mu$. Then a weighted generalization of Wilker's inequality

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^2 + \frac{\mu}{\mu+\lambda}\frac{\tan x}{x} > 1 \tag{1.4}$$

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holds. Wilker's inequality follows as a special case of the inequality when $\mu = \lambda = 1$ ([5]).

- A refinement of Wilker's inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \quad x \in (0, \pi/2) \tag{1.5}$$

was given in [4].

- The sharp Wilker-type inequalities

$$\frac{8}{45}x^4 + \left(\frac{2}{\pi}\right)^6 x^5 \tan x > \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 > \frac{8}{45}x^4 + \frac{16}{315}x^5 \tan x \tag{1.6}$$

and

$$\frac{8x^4}{45} + \frac{16x^6}{315} + \left(\frac{2}{\pi}\right)^8 x^7 \tan x > \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 > \frac{8x^4}{45} + \frac{16x^6}{315} + \frac{104x^7 \tan x}{4725} \tag{1.7}$$

for $x \in (0, \pi/2)$, were shown to hold by using the Bernoulli numbers ([1]).

- Using monotonicity theorem and Taylor expansion of hyperbolic functions, the two Wilker-type inequalities

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2, \quad x \in (0, \pi/2) \tag{1.8}$$

and

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 + \frac{8}{45}x^3 \tanh x, \quad x \in (0, \pi/2) \tag{1.9}$$

involving hyperbolic functions were established ([8]).

The inequality (1.1) can be proved based on the use of the elementary trigonometric inequality $\tan x > x > \sin x$ for $x \in (0, \pi/2)$ ([7]). In Section 2, we give an alternative proof for (1.1). In addition, we establish two new Wilker-type inequalities for trigonometric and hyperbolic functions, respectively in Theorem 2.4 and Theorem 2.5. Section 3 is devoted to conjectures involving the generalization of the inequalities in Theorem 2.4 and Theorem 2.5.

2. The main theorems

The following proof shows an alternative approach to establishing inequality (1.1).

Proof. We write

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 = \frac{-2x^2 + \sin^2 x + x \tan x}{x^2} \tag{2.1}$$

and let $f(x) = -2x^2 + \sin^2 x + x \tan x$. Direct calculations show that

$$f'(x) = -4x + \sin(2x) + \tan x + x \sec^2 x, \tag{2.2}$$

$$f''(x) = \frac{1}{2} \tan x \sec^2 x g(x), \tag{2.3}$$

where $g(x) = 4x - \sin(4x)$. Since $g(0) = 0$ and $g'(x) = 4(1 - \cos(4x)) = 8\sin^2(2x) > 0$ for $x \in (0, \pi/2)$, we have $g(x) > 0$ for $x \in (0, \pi/2)$ and thus $f''(x) > 0$ for $x \in (0, \pi/2)$. Now $f'(0) = 0$, so $f'(x) > 0$ for $x \in (0, \pi/2)$. Finally, $f'(x) > 0$ for

$x \in (0, \pi/2)$ together with $f(0) = 0$ leads to $f(x) > 0$ for $x \in (0, \pi/2)$ and the proof of Theorem 1.1 is completed. □ □

To establish our Wilker-type inequality, we need Lemma 2.1∼Lemma 2.3.

Lemma 2.1. *Let $x \in (0, \pi/2)$.*

(1) For some $\xi \in (0, x)$,

$$\cos x = \sum_{k=0}^{4}(-1)^k \frac{x^{2k}}{(2k)!} - \frac{x^{10}}{10!}\cos\xi \le 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320}. \tag{2.4}$$

(2) For some $\xi \in (0, x)$,

$$\sin x = \sum_{k=0}^{3}(-1)^k \frac{x^{2k+1}}{(2k+1)!} + \frac{x^9}{9!}\cos\xi \ge x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}. \tag{2.5}$$

(3) For some $\xi \in (0, x)$,

$$\begin{aligned} \cos(2x) &= \sum_{k=0}^{7}(-1)^k \frac{(2x)^{2k}}{(2k)!} + \frac{65536x^{16}}{16!}\cos(2\xi) \\ &= 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \frac{4x^{10}}{14175} + \frac{4x^{12}}{467775} - \frac{8x^{14}}{42567525} \\ &+ \frac{65536x^{16}}{16!}\cos(2\xi) \\ &\ge 1 - 2x^2 + \frac{2x^4}{3} - \frac{4x^6}{45} + \frac{2x^8}{315} - \frac{4x^{10}}{14175} = \sum_{k=0}^{5}(-1)^k \frac{(2x)^{2k}}{(2k)!}. \end{aligned} \tag{2.6}$$

Proof. (2.4) and (2.5) follow immediately from Taylor's Theorem with Lagrange's form of remainder. To prove (2.6), it suffices to show that for $x \in (0, \pi/2)$,

$$\frac{4x^{12}}{467775} - \frac{8x^{14}}{42567525} - \frac{65536x^{16}}{16!} \ge 0 \tag{2.7}$$

or

$$-x^4 - 60x^2 + 2730 \ge 0. \tag{2.8}$$

Since $-\left(\frac{\pi}{2}\right)^4 - 60\left(\frac{\pi}{2}\right)^2 + 2730 \ge 0$, the proof is completed. □ □

Lemma 2.2. *Let $x \in (0, \pi/2)$. Then*

$$-128x^6 + 2655x^4 - 30240x^2 + 75600 > 0. \tag{2.9}$$

Proof. Let $f(y) = -128y^3 + 2655y^2 - 30240y + 75600$. We find $f'(y) = -384y^2 + 5310y - 30240$ and $f''(y) = -768y + 5310 > 0$ on $(0, \pi^2/4)$. This means that $f'(y)$ is increasing on $(0, \pi^2/4)$, and thus $f'(y) \le f'(\pi^2/4) = -30240 + \frac{2655\pi^2}{2} - 24\pi^4 < 0$ on $(0, \pi^2/4)$. Therefore $f(y)$ is decreasing on $(0, \pi^2/4)$ and thus $f(y) \ge f(\pi^2/4) = 75600 - 7560\pi^2 + \frac{2655\pi^4}{16} - 2\pi^6 > 0$ on $(0, \pi^2/4)$. This completes the proof. □ □

Lemma 2.3. *Let $x \in (0, \pi/2)$. Then*

$$1 - x^2 - 2\cos x + \cos^2 x + x\sin x > 0. \tag{2.10}$$

Proof. Using Lemma 2.1 and Lemma 2.2, we have

$$
\begin{aligned}
&1-x^2-2\cos x+\cos^2 x+x\sin x\\
&=1-x^2-2\cos x+\frac{\cos(2x)+1}{2}+x\sin x\\
&\geq 1-x^2-2\sum_{k=0}^{4}(-1)^k\frac{x^{2k}}{(2k)!}+\frac{1}{2}\sum_{k=0}^{5}(-1)^k\frac{(2x)^{2k}}{(2k)!}+\frac{1}{2}+x\sum_{k=0}^{3}(-1)^k\frac{x^{2k+1}}{(2k+1)!}
\end{aligned}
$$

(2.11)
$$
=\frac{x^4\left(-128x^6+2655x^4-30240x^2+75600\right)}{907200}>0
$$

for $x\in(0,\pi/2)$. □ □

Withe the help of Lemma 2.3, we prove the following Wilker-type inequality.

Theorem 2.4 (Wilker-type inequality for trigonometric functions)**.** *If* $x\in(0,\pi/2)$, *then*

$$
\left(\frac{\cos x-1}{x}\right)^2+\frac{\sin x}{x}>1. \tag{2.12}
$$

Proof. Employing Lemma 2.3, we prove for $x\in(0,\pi/2)$,

$$
\left(\frac{\cos x-1}{x}\right)^2+\frac{\sin x}{x}-1=\frac{1-x^2-2\cos x+\cos^2 x+x\sin x}{x^2}>0. \quad \square \tag{2.13}
$$

□

Finally, we prove the following theorem.

Theorem 2.5 (Wilker-type inequality for hyperbolic functions)**.** *Let* $x\in\mathbb{R}\setminus\{0\}$. *Then*

$$
\left(\frac{\cosh x-1}{x}\right)^2+\frac{\sinh x}{x}>1. \tag{2.14}
$$

Proof. Due to symmetry, it suffices to show (2.14) for $x>0$. We write

$$
\left(\frac{\cosh x-1}{x}\right)^2+\frac{\sinh x}{x}-1=\frac{h(x)}{x^2}, \tag{2.15}
$$

where $h(x)=-x^2+x\sinh x+\cosh^2 x-2\cosh x+1$. A straightforward calculation gives

$$
h'(x)=-2x-\sinh x+\sinh(2x)+x\cosh x, \tag{2.16}
$$

$$
h''(x)=\sinh x(x+4\sinh x). \tag{2.17}
$$

Since $h'(0)=0$ and $h''(x)>0$ for $x>0$, we have $h'(x)>0$ for $x>0$. Now $h(0)=0$, so $h(x)>0$ for $x>0$ and the proof of Theorem 1.1 is completed. □

□

3. Concluding Remarks

In this paper, two Wilker-type inequalities are established and are proved by elementary arguments. We propose the conjecture that the following Wilker-type inequalities hold:

- (An extension of Wilker-type inequality) For $x \in (0, \pi/2)$,

$$\left(\frac{x}{\cos x - 1}\right)^2 + \frac{x}{\sin x} > \left(\frac{\cos x - 1}{x}\right)^2 + \frac{\sin x}{x} > 1. \tag{3.1}$$

  The above inequality is an analogue of (1.5) in [4].
- (An weighted Wilker-type inequality for hyperbolic functions)
  Let $0 \leq p \leq \frac{1}{2}$. For $x \in \mathbb{R} \setminus \{0\}$,

$$p\left(\frac{\cosh x - 1}{x}\right)^2 + (1-p)\frac{\sinh x}{x} > \frac{1}{2}. \tag{3.2}$$

  Inequality (2.14) in Theorem 2.5 follows as a special case of the inequality when $p = \frac{1}{2}$.
- (Weighted Wilker-type inequalities for trigonometric functions)
  - Let $0 \leq p \leq \frac{1}{2}$. For $x \in (0, \pi/2)$,

$$p\left(\frac{\cos x - 1}{x}\right)^2 + (1-p)\frac{\sin x}{x} > \frac{1}{2}. \tag{3.3}$$

  - Let $p_0 \leq p \leq 1$. For $x \in (0, \pi/2)$,

$$p\left(\frac{\cos x - 1}{x}\right)^2 + (1-p)\frac{\sin x}{x} < \frac{1}{2}, \tag{3.4}$$

    where $p_0 = \frac{\pi(4-\pi)}{4(\pi-2)} \approx 0.590571$.

    Inequality (2.12) in Theorem 2.4 follows as a special case of the inequality when $p = \frac{1}{2}$.

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