$\pi - e$, $\pi + e$, πe and $\frac{\pi}{e}$ all are irrational numbers

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Abstract

It is proved that $\sqrt{3} - \sqrt{2}$ and $\sqrt{3} + \sqrt{2}$, e and π , $\pi - e$, $\pi + e$, πe and $\frac{\pi}{e}$, all are irrational numbers . It is an argument by contradiction.

Notation and reminder

π : *known as Archimedes constant , is the ratio of a circle's circumference to its diameter and* $3 < \pi < 4$*.*

$e = \sum_{m=0}^{+\infty} \frac{1}{m!}$: *known as Euler's number and* $2 < e < 3$*.*

$\mathbb{N}^* := \{1,2,3,4, \dots\}$ *the natural numbers .*

$\mathbb{Z} := \{\dots, -4, -3, -2, -1, 0,1,2,3,4, \dots\}$ *the integers and* $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$ *.*

$\mathbb{Q} := \{ \frac{p}{q} : (p, q) \in \mathbb{Z}\text{x}\mathbb{Z}^* \text{and } p \wedge q = 1\}$ *the set of rational numbers*.

$\mathbb{R}$: *the set of real numbers*.

$\mathbb{R} \setminus \mathbb{Q} := \{x \in \mathbb{R} \text{ and } x \notin \mathbb{Q} : \mathbb{Q} \subset \mathbb{R}\}$ *the set of irrational numbers*.

$p \wedge q := \max\{d \in \mathbb{N}^* : d/p \text{ and } d/q\}$ *the greatest common divisor of* p *and* q*.*

$\forall$: *the universal quantifier and* $\exists$: *the existential quantifier.*

Introduction

Irrational numbers are the type of real numbers that cannot be expressed in the rational form $\frac{p}{q}$, where p , q are integers and $q \neq 0$. In simple words, all the real numbers that are not rational numbers are irrational. In this paper we show that $\sqrt{3} - \sqrt{2}$ and $\sqrt{3} + \sqrt{2}$, e and π , $\pi - e$, $\pi + e$, πe and $\frac{\pi}{e}$, all are irrational numbers. It is an argument by contradiction.

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Theorem 1. $\sqrt{6} \in \mathbb{R} \setminus \mathbb{Q}$. *In other words,* $\sqrt{6}$ *is an irrational number.*

Proof. An argument by contradiction. Suppose that $\sqrt{6} \in \mathbb{Q}$, and as $\sqrt{6} > 0$ then $\exists\, p , q \in \mathbb{N}^*$ such that $\sqrt{6} = \frac{p}{q}$ and $p \wedge q = 1$, then $\left(\sqrt{6}\right)^2 = \left(\frac{p}{q}\right)^2$, then $6 = \frac{p^2}{q^2}$ and $6q^2 = p^2 \Rightarrow p^2$ is even and $p \in \mathbb{N}^* \Rightarrow p$ is even or $p = 2k$: $k \in \mathbb{N}^*$ $\Rightarrow 6q^2 = (2k)^2 = 4k^2 \Rightarrow 3q^2 = 2k^2$and $3 \wedge 2 = 1 \Rightarrow 2$ divides q^2 and 2 is prime $\Rightarrow$ 2 divides q and $q \in \mathbb{N}^* \Rightarrow q$ is even or $q = 2k'$: $k' \in \mathbb{N}^*$, hence $p \wedge q \geq 2$, and we get a contradiction because $p \wedge q = 1$.

Main Theorem 1. $\sqrt{3} - \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ *and* $\sqrt{3} + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$.
In other words, $\sqrt{3} - \sqrt{2}$ *and* $\sqrt{3} + \sqrt{2}$ *both are irrational numbers .*

Proof. An argument by contradiction. Suppose that $\sqrt{3} - \sqrt{2} \in \mathbb{Q}$, then $\exists\, r \in \mathbb{Q}$ such that $\sqrt{3} - \sqrt{2} = r$ implies that $\left(\sqrt{3} - \sqrt{2}\right)^2 = r^2 \in \mathbb{Q}$ $\Rightarrow 5 - 2\sqrt{6} = r^2 \in \mathbb{Q} \Rightarrow \sqrt{6} = \frac{5 - r^2}{2} \in \mathbb{Q}$, and we get a contradiction .
On the other hand, suppose that $\sqrt{3} + \sqrt{2} \in \mathbb{Q}$, then $\exists\, r \in \mathbb{Q}$ such that $\sqrt{3} + \sqrt{2} = r$ implies that $(\sqrt{3} + \sqrt{2})^2 = r^2 \in \mathbb{Q} \Rightarrow 5 + 2\sqrt{6} = r^2 \in \mathbb{Q}$ $\Rightarrow \sqrt{6} = \frac{r^2 - 5}{2} \in \mathbb{Q}$, and we get a contradiction .

Lemma 2. *We have* $\lim_{n \to +\infty} \sum_{m=n+1}^{+\infty} \frac{n!}{m!} = 0$ *and* $\lim_{n \to +\infty} n . \sum_{m=n+1}^{+\infty} \frac{n!}{m!} = 1$.

Proof. $\forall\, n \in \mathbb{N}^*$, $\sum_{m=n+1}^{+\infty} \frac{n!}{m!} = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots$

$$< \frac{1}{n+1} + \frac{1}{(n+1)(n+1)} + \frac{1}{(n+1)(n+1)(n+1)} + \cdots$$

$$= \sum_{k=1}^{+\infty} \frac{1}{(n+1)^k} = \frac{1}{n} \ ,$$

then $0 < \sum_{m=n+1}^{+\infty} \frac{n!}{m!} < \frac{1}{n}$ and $\lim_{n \to +\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n \to +\infty} \sum_{m=n+1}^{+\infty} \frac{n!}{m!} = 0$.

On the other hand , we have $\frac{1}{n+1} < \sum_{m=n+1}^{+\infty} \frac{n!}{m!} < \frac{1}{n} \Rightarrow \frac{n}{n+1} < n . \sum_{m=n+1}^{+\infty} \frac{n!}{m!} < 1$

, and $\lim_{n \to +\infty} \frac{n}{n+1} = 1 \Rightarrow \lim_{n \to +\infty} n . \sum_{m=n+1}^{+\infty} \frac{n!}{m!} = 1$.

For more details about irrational numbers , we refer the reader and our students to [**1**] and to [**2**].

Theorem 2. *We have* $\lim_{n\to+\infty} n.\sin(2\pi n!\,e) = 2\pi$.

Proof. Indeed , $\lim_{n\to+\infty} n.\sin(2\pi n!\,e) = \lim_{n\to+\infty} n.\sin(2\pi n! \sum_{m=0}^{+\infty} \frac{1}{m!})$

$$= \lim_{n\to+\infty} n.\sin(2\pi n!\,(\sum_{m=0}^{n} \frac{1}{m!} + \sum_{m=n+1}^{+\infty} \frac{1}{m!}))$$

$$= \lim_{n\to+\infty} n.\sin(2\pi \sum_{m=0}^{n} \frac{n!}{m!} + 2\pi \sum_{m=n+1}^{+\infty} \frac{n!}{m!}) ,$$

we put $a_n = \sum_{m=0}^{n} \frac{n!}{m!} \in \mathbb{N}^*$ and $b_n = \sum_{m=n+1}^{+\infty} \frac{n!}{m!} \to 0$ and $nb_n \to 1$ when $n \to +\infty$,

then $\lim_{n\to+\infty} n.\sin(2\pi n!\,e) = \lim_{n\to+\infty} n.\sin(2\pi a_n + 2\pi b_n)$

$$= \lim_{n\to+\infty} n.\sin(2\pi b_n) = \lim_{n\to+\infty} n.\,2\pi b_n.\frac{\sin(2\pi b_n)}{2\pi b_n}$$

$$= \lim_{n\to+\infty} 2\pi.\,nb_n.\frac{\sin(2\pi b_n)}{2\pi b_n} = 2\pi.\,1.1 = 2\pi.$$

Main Theorem 2. $e \in \mathbb{R} \setminus \mathbb{Q}$ *and* $\pi \in \mathbb{R} \setminus \mathbb{Q}$.
In other words, e and π both are irrational numbers .

Proof. An argument by contradiction . First , we prove that e is irrational . Suppose that $e \in \mathbb{Q}$, and as $e > 0$, then $\exists\, p , q \in \mathbb{N}^*$ such that $e = \frac{p}{q}$ and $p \wedge q = 1$. Then, $\lim_{n\to+\infty} n.\sin(2\pi n!\,e) = \lim_{n\to+\infty} n.\sin(2\pi n! \frac{p}{q})$,
we put $a_n = n! \frac{p}{q} : n \in \mathbb{N}^*$, and it is clear that a_n is strictly increasing and $\{a_n : n \geq q\} \subset \mathbb{N}^*$, then $n.\sin(2\pi a_n) = 0 \; : n \geq q$,
this implies that $\lim_{n\to+\infty} n.\sin(2\pi a_n) = 0$, and we get a contradiction according to [**Theorem 2**] . Thus , e is an irrational number. Another proof presented by Dimitris Koukoulopoulos and was found by Fourier in 1815 is available at [**3**, **Théorème15.2**]. Second , we prove that π is irrational . A simple proof that π is irrational made by Ivan Niven in 1947 is available at [**4**] and Lambert's proof of the irrationality of π in 1760 is available at [**5**].

The sine function (or $\sin(x)$) is defined , continuous , odd and 2π−periodic on $\mathbb{R}$ and $\forall\, \theta \in \mathbb{R}$ we have $\sin(\theta) = 0 \Leftrightarrow \theta \in \{k\pi : k \in \mathbb{Z}\}$. For more details about sine function and its properties , we refer the reader and our students to [**6**, **page 101**].

$\pi - e$, $\pi + e$, πe and $\frac{\pi}{e}$ all are irrational numbers

Theorem 3. *We have* $\begin{cases} \lim\limits_{n\to+\infty} \sin(n!\,(\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!}) \;=\; 0 \\ \lim\limits_{n\to+\infty} \sin(n!\,(\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!}) \;=\; 0 \\ \lim\limits_{n\to+\infty} \sin\left(n!\,\pi e - \pi.\sum_{m=0}^{n} \frac{n!}{m!}\right) \;=\; 0 \\ \lim\limits_{n\to+\infty} \sin\left(n!\,pe - p.\sum_{m=0}^{n} \frac{n!}{m!}\right) \;=\; 0 \end{cases}$.

Proof. First,

$$\lim_{n\to+\infty} \sin(n!\,(\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!}) \;=\; \lim_{n\to+\infty} \sin(n!\,\pi - \; n!\,e + \sum_{m=0}^{n} \frac{n!}{m!})$$

$$= \lim_{n\to+\infty} \sin(n!\,\pi - \; \sum_{m=0}^{+\infty} \frac{n!}{m!} + \sum_{m=0}^{n} \frac{n!}{m!})$$

$$= \lim_{n\to+\infty} \sin(n!\,\pi - \; \sum_{m=n+1}^{+\infty} \frac{n!}{m!})$$

$$= \lim_{n\to+\infty} - \sin(\sum_{m=n+1}^{+\infty} \frac{n!}{m!}) \;=\; - \sin(0) \;= 0\,.$$

Second,

$$\lim_{n\to+\infty} \sin(n!\,(\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!}) \;=\; \lim_{n\to+\infty} \sin(n!\,\pi + \; n!\,e - \sum_{m=0}^{n} \frac{n!}{m!})$$

$$= \lim_{n\to+\infty} \sin(n!\,\pi + \; \sum_{m=0}^{+\infty} \frac{n!}{m!} - \sum_{m=0}^{n} \frac{n!}{m!})$$

$$= \lim_{n\to+\infty} \sin(n!\,\pi + \; \sum_{m=n+1}^{+\infty} \frac{n!}{m!})$$

$$= \lim_{n\to+\infty} \sin(\sum_{m=n+1}^{+\infty} \frac{n!}{m!}) \;=\; \sin(0) \;= 0\,.$$

Third,

$$\lim_{n\to+\infty} \sin\left(n!\,\pi e - \pi.\sum_{m=0}^{n} \frac{n!}{m!}\right) \;=\; \lim_{n\to+\infty} \sin\left(\pi.\sum_{m=0}^{+\infty} \frac{n!}{m!} - \pi.\sum_{m=0}^{n} \frac{n!}{m!}\right)$$
$$= \lim_{n\to+\infty} \sin\left(\pi.\sum_{m=n+1}^{+\infty} \frac{n!}{m!}\right) = \;\sin(0) \;= 0\,.$$

Fourth, let $p \in \mathbb{N}^*$ we have

$$\lim_{n\to+\infty} \sin\left(n!\,pe - p.\sum_{m=0}^{n} \frac{n!}{m!}\right) \;=\; \lim_{n\to+\infty} \sin\left(p.\sum_{m=0}^{+\infty} \frac{n!}{m!} - p.\sum_{m=0}^{n} \frac{n!}{m!}\right)$$
$$= \lim_{n\to+\infty} \sin\left(p.\sum_{m=n+1}^{+\infty} \frac{n!}{m!}\right) = \;\sin(0) \;= 0\,.$$

Main Theorem 3. *$\pi - e \in \mathbb{R} \setminus \mathbb{Q}$ and $\pi + e \in \mathbb{R} \setminus \mathbb{Q}$ and $\pi e \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{\pi}{e} \in \mathbb{R} \setminus \mathbb{Q}$. In other words, $\pi - e$, $\pi + e$, πe and $\frac{\pi}{e}$ all are irrational numbers.*

Proof. An argument by contradiction . First, suppose that $\pi - e \in \mathbb{Q}$, and as $\pi - e > 0$, then $\exists\, p, q \in \mathbb{N}^*$ such that $\pi - e = \frac{p}{q}$ and $p \wedge q = 1$.

We recall that, $\forall\, n \in \mathbb{N}^* : n!\,(\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!} > 0$.

Then, $\lim\limits_{n\to+\infty} \sin\left(n!\,(\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim\limits_{n\to+\infty} \sin\left(n!\frac{p}{q} + \sum_{m=0}^{n} \frac{n!}{m!}\right)$, we put $a_n = n!\frac{p}{q} + \sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^*$, and it is clear that a_n is strictly increasing and $\{a_n : n \geq q\} \subset \mathbb{N}^*$, then $\lim\limits_{n\to+\infty} a_n \notin \{k\pi : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\}$,

this implies that $\lim\limits_{n\to+\infty} \sin(a_n) \neq 0$, and we get a contradiction according to [**Theorem 3**].

Second, suppose that $\pi + e \in \mathbb{Q}$, and as $\pi + e > 0$, then $\exists\, p, q \in \mathbb{N}^*$ such that $\pi + e = \frac{p}{q}$ and $p \wedge q = 1$.

We recall that, $\forall\, n \in \mathbb{N}^* : n!\,(\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!} > 0$.

Then, $\lim\limits_{n\to+\infty} \sin\left(n!\,(\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim\limits_{n\to+\infty} \sin\left(n!\frac{p}{q} - \sum_{m=0}^{n} \frac{n!}{m!}\right)$, we put $a_n = n!\frac{p}{q} - \sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^*$, and it is clear that a_n is strictly increasing and $\{a_n : n \geq q\} \subset \mathbb{N}^*$, then $\lim\limits_{n\to+\infty} a_n \notin \{k\pi : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\}$,

this implies that $\lim\limits_{n\to+\infty} \sin(a_n) \neq 0$, and we get a contradiction according to [**Theorem 3**].

Third, suppose that $\pi e \in \mathbb{Q}$, and as $\pi e > 0$, then $\exists\, p, q \in \mathbb{N}^*$ such that $\pi e = \frac{p}{q}$ and $p \wedge q = 1$.

Then,
$$\lim_{n\to+\infty} \sin\left(n!\,\pi e - \pi.\sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim_{n\to+\infty} \sin\left(n!\frac{p}{q} - \pi.\sum_{m=0}^{n} \frac{n!}{m!}\right) = \lim_{n\to+\infty} (-1)^{n+1}.\sin\left(n!\frac{p}{q}\right),$$

we put $a_n = n!\frac{p}{q} : n \in \mathbb{N}^*$, and it is clear that a_n is strictly increasing and $\{a_n : n \geq q\} \subset \mathbb{N}^*$, then $\lim\limits_{n\to+\infty} a_n \notin \{k\pi : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\}$,

this implies that $\lim\limits_{n\to+\infty} \sin(a_n) \neq 0$ and $\lim\limits_{n\to+\infty} (-1)^{n+1}.\sin(a_n) \neq 0$, and we get a contradiction according to [**Theorem 3**].

$$\pi - e\,,\pi + e\,,\pi e \text{ and } \frac{\pi}{e} \text{ all are irrational numbers}$$

Fourth, suppose that $\frac{\pi}{e} \in \mathbb{Q}$, and as $\frac{\pi}{e} > 0$, then $\exists\, p\,,q \in \mathbb{N}^*$ such that $\frac{\pi}{e} = \frac{p}{q}$ and $p \wedge q = 1$, then $q\pi = pe$ and $\forall\, n \in \mathbb{N}^* : n!q\pi = n!\,pe$. Then, $\lim_{n\to+\infty} \sin\left(n!\,pe - p.\sum_{m=0}^{n}\frac{n!}{m!}\right) = \lim_{n\to+\infty} \sin\left(n!\,q\pi - p.\sum_{m=0}^{n}\frac{n!}{m!}\right)$

$$= \lim_{n\to+\infty} -\sin\left(p.\sum_{m=0}^{n}\frac{n!}{m!}\right) ,$$

we put $a_n = p.\sum_{m=0}^{n}\frac{n!}{m!} : n \in \mathbb{N}^*$, and it is clear that a_n is strictly increasing and $\{a_n : n \in \mathbb{N}^*\} \subset \mathbb{N}^*$, then $\lim_{n\to+\infty} a_n \notin \{k\pi : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\}$, this implies that $\lim_{n\to+\infty} \sin(a_n) \neq 0$ and $\lim_{n\to+\infty} -\sin(a_n) \neq 0$, and we get a contradiction according to [**Theorem 3**].

Thus , we conclude that $\pi - e\,,\pi + e\,,\pi e$ and $\frac{\pi}{e}$, all are irrational numbers.

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