Complex curvature and complex radius

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Abstract

I define the notions of complex curvature and complex radius and prove that one of these complex numbers is exactly the inverse of the other.

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MSC: 53A04, 15B10, 85-10, 70F10

Introduction

At the beginning of the 19th century, long after Galileo, Kepler and Newton, following precious research by French mathematicians such as Lancret ([3]) and Frenet ([2]), important notions and theorems of differential geometry appeared that also involved the *torsion* of a curve, not just its curvature.

In this context of great revelations the properties of curves, of the general helix and of the circular helix were outlined.

So, the formulas for the curvature of the circular helix and its torsion are well known ([1], [4], [5], [6]). If a is the radius of the helix, and b is the pitch of the helix divided by 2π (barred pitch), then the curvature and torsion of the circular helix have the expressions:

$$\kappa = \frac{a}{a^2 + b^2}$$

and, respectively,

$$\tau = -\frac{b}{a^2 + b^2}.$$

Definitions

In what follows I will introduce two new notions in differential geometry, about which I have not found references in my researches. Thus, I will name ***complex radius*** the complex number formed by the radius of the circular helix and its barred pitch

$$R = a + b\mathbf{i}$$

and I will name ***complex curvature*** the complex number formed with the curvature and torsion of the circular helix

$$\lambda = \kappa + \tau\mathbf{i},$$

where $\mathbf{i}$ is *imaginary unit*, with the property that $\mathbf{i}^2 = -1$.

Complex curvature is the inverse of complex radius

I will now show that the complex curvature is the inverse of the complex radius. Specifically, I will make the product of the complex curvature with the complex radius and I show that it is equal to unity.

We have

$R \cdot \lambda = (a + \mathbf{i}b) \cdot (\kappa + \mathbf{i}\tau) =$

$= a\kappa + a\tau\mathbf{i} + b\kappa\mathbf{i} - b\tau =$

$= a \cdot \frac{a}{a^2 + b^2} - a \cdot \frac{b}{a^2 + b^2}\mathbf{i} + b \cdot \frac{a}{a^2 + b^2}\mathbf{i} + b \cdot \frac{b}{a^2 + b^2} =$

$= \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} + \left(-\frac{ab}{a^2 + b^2} + \frac{ba}{a^2 + b^2}\right)\mathbf{i} =$

$= \frac{a^2 + b^2}{a^2 + b^2} + \frac{-ab + ba}{a^2 + b^2}\mathbf{i} = 1 + 0\mathbf{i} = 1.$

So,

$$\lambda = \frac{1}{R}.$$

Of course, this result does not depend on the sign of the torsion of the circular helix (the sign of b), being valid for both right and left helices.

Conclusions

I consider this result to be of special importance, which allows the generalization of the notion of curvature and radius, known up to now. Also, this result allows new approaches in the theory of curves and once again highlights the importance of the circular helix, especially due to the fact that, based on the fundamental theorem of curves, on sufficiently small portions (therefore locally) any smooth curve can be approximated through a circular helix. With these unifying conclusions regarding the circular helix, we open the door wider to helical physics, which postulates the principle that free bodies move *on circular helices*, not in straight lines.

If you believe, as I do, in the importance of this material, spread it with munificence across the four horizons to get the word out.

Thank you!

References

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