Collatz Conjecture: A countably infinite sequence

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1 Introduction

The Collatz conjecture, an unsolved problem in mathematics, has garnered attention over the years with various proven results. In this work, I present a new result that suggests the sequence can potentially have a countably infinite number of terms.

2 Discussion on result and proof

Consider n be a natural number on which we operate the Collatz Algorithm. We can observe that if n is even, the number would become $\frac{n}{2}$ and eventually reach to 1 if we get an even number, i.e., any number of the form 2^k, where k is an integer would reach 1 eventually.

So the thing that deviates n from reaching 1 is if n is an odd number but that is trivial to notice. Consider n to be an odd number, then we have:

$$n \to 3n+1 \to \frac{3n+1}{2} \to s \tag{1}$$

Here, s can be an odd number or an even number depending on the number produced by the division $\frac{3n+1}{2}$. But here is something to observe, the number s is $1.5n + 0.5$ which is greater than n. Let's assume that s is even. In that case writing the sequence again

$$n \to 3n+1 \to \frac{3n+1}{2} \to \frac{3n+1}{4} \tag{2}$$

We can observe that, $\frac{3n+1}{4}$ which is $0.75n + 0.25$ is less than n.

So, I conclude that if the transformation $n \to 3n+1$ produces a number of the form $2^{2+k} \cdot a$ where k is a whole number and a is an odd number, then we get to a number less than n.

Building on these observations(without proof), consider $n = 8 \cdot 2^k - 1$ for any whole number k which is an odd number. Applying the Collatz Algorithm on such an n, we get:

$$
\begin{aligned}
&(8 \cdot 2^k - 1) \\
&\rightarrow 3 \cdot (8 \cdot 2^k - 1) + 1 \\
&\rightarrow \frac{3 \cdot (8 \cdot 2^k - 1) + 1}{2} \\
&= \frac{3 \cdot 8 \cdot 2^k - 3 + 1}{2} \\
&= (8 \cdot 3 \cdot 2^{k-1} - 1) \\
&\rightarrow 3 \cdot (8 \cdot 3 \cdot 2^{k-1} - 1) + 1 \\
&\rightarrow \frac{3 \cdot (8 \cdot 3 \cdot 2^{k-1} - 1) + 1}{2} \\
&\rightarrow \frac{8 \cdot 3^2 \cdot 2^{k-1} - 3 + 1}{2} \\
&= (8 \cdot 3^2 \cdot 2^{k-2} - 1) \\
&... \\
&\rightarrow (3^{k+3} - 1)
\end{aligned} \tag{3}
$$

We can henceforth observe from the above that the Collatz sequence for any $n = 8 \cdot 2^k - 1$ can have a partial sequence of the form given above. Carrying forward the Collatz Algorithm on $(3^{k+3} - 1)$ would give a smaller or bigger number but it's at least possible to get the lower bound of the Collatz sequence length which depends on chosen k.

3 Conclusion

In this paper, I have explored the Collatz conjecture and presented a new result regarding the behavior of the sequence. The proof demonstrated that for natural numbers n subjected to the Collatz Algorithm, the sequence can potentially have a countably infinite number of terms.