The infinite series $\sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n}$ on the Lviv Scottish book is bounded

Amine Oufaska

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"In 2017, I managed to solve a problem from the "Lviv Scottish book . The problem had a prize of "butelka miodu pitnego" (a bottle of honey mead). Today, while I was in Warsaw, some representatives from Lviv, Ukraine came (by train, as the Ukraine airspace is obviously closed) I was very touched and honored to unexpectedly receive the prize in person."

Terence Tao

Abstract

In this article we prove that the infinite series $\sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n}$ on the Lviv Scottish book is bounded , consequently it is convergent.

Notation and reminder

$\mathbb{N}^* := \{1,2,3,4,\dots\}$ *the natural numbers .*

$\mathbb{Z} := \{\dots, -4, -3, -2, -1, 0,1,2,3,4,\dots\}$ *the integers .*

$\mathbb{R}$: *the set of real numbers and* $\mathbb{R} \setminus \mathbb{Q}$: *the set of irrational numbers*.

$]0,1[:= \{0 < x < 1 : x \in \mathbb{R}\}$ *the open interval with endpoints 0 and 1.*

$|x| := \max\{-x, x : x \in \mathbb{R}\}$ *the absolute value of* x.

$\forall$: *the universal quantifier and* $\exists$: *the existential quantifier.*

For more details about the infinite series , we refer the reader and our students to [**4**] and to [**5**].

Introduction

Is the infinite series $\sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n}$ is convergent ? The problem was posed on 22.06.2017 by PhD students of H.Steinhaus Center of Wroclaw Polytechnica. The promised prize for solution is a bottle of drinking honey, see [**1**] of the Lviv Scottish book. This problem was solved by Terence Tao on 29.09.2017 [**2**] who is honored on 09.08.2023 [**3**]. In this article we show that this infinite series is bounded , consequently it is convergent.

Lemma. $\forall\, n \in \mathbb{N}^*$ *we have* $0 < |\sin(n)| < 1$.

Proof. $\forall\, n \in \mathbb{N}^*$we have $0 \leq |\sin(n)| \leq 1,$ and $n \notin \{\frac{k\pi}{2} : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\}$, thus $0 < |\sin(n)| < 1$.

Main Theorem. *The infinite series* $\sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n}$ *is bounded.*

Proof. Indeed , $\sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n} = \sum_{n=1}^{+\infty} \left|\frac{\sin(n)}{\sqrt[n]{n}}\right|^n$, and $\forall\, n \in \mathbb{N}^*$ we have $0 < |\sin(n)| < 1$ and $\sqrt[n]{n} \geq 1$, this implies that $0 < \left|\frac{\sin(n)}{\sqrt[n]{n}}\right| < 1$, then $\exists\ \alpha , \beta \in]0,1[$ such that $\alpha = \min\{\left|\frac{\sin(n)}{\sqrt[n]{n}}\right| : n \in \mathbb{N}^*\}$ and $\beta = \max\{\left|\frac{\sin(n)}{\sqrt[n]{n}}\right| : n \in \mathbb{N}^*\}$, then $\sum_{n=1}^{+\infty} \alpha^n < \sum_{n=1}^{+\infty} \left|\frac{\sin(n)}{\sqrt[n]{n}}\right|^n < \sum_{n=1}^{+\infty} \beta^n$, thus $\frac{\alpha}{1-\alpha} < \sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n} < \frac{\beta}{1-\beta}$.
Consequently we have $\sum_{n=1}^{+\infty} \frac{|\sin(n)|^n}{n} < +\infty$.

References

[**1**] math.lviv.ua/szkocka/viewpage.php?vol=1&page=37.

[**2**] mathoverflow.net/questions/282259.

[**3**] youtube.com/watch?v=Gs9ZQ9fYMFQ.

[**4**] Konrad Knopp . Theory And Application Of Infinite Series.

[**5**] Tim Smits . Integration and Infinite Series.

E-mail address : **ao.oufaska@gmail.com**