NODAL LINES OF EIGENFUNCTIONS OF LAPLACIAN IN PLANE

Soo Sun Ha

December 9, 2023

Abstract

We prove Payne's nodal line conjecture for any bounded simply connected, possibly non-convex, C^∞ boundary domain Ω in plane $\mathbb{R}^2$; any second Dirichlet eigenfunction of Δ_e in Ω can not have a closed nodal line.

a Sketch of Proof. According to Z. Liqun [10], multiplicity of the second eigenvalue of Ω is at most two. It is also known that the nodal line of the second eigenfunction in Ω divides Ω precisely into two nodal domains. Supposing the most severe case, we assume that an element $\phi_2(0)$ of the second eigenspace in Ω whose dimension is two has a closed nodal line which meets $\partial\Omega$ at P.

According to [5], given a second eigenfunction in Ω and any smooth deformation $J : \Omega \times [0, 1] \to \mathbb{R}^2$, a linear sum of two eigenfunctions in the deformation $J(\Omega, t)$ converges to the given second eigenfunction as $t \to 0$. But this linear sum is not necessarily an eigenfunction in $J(\Omega, t)$, since two eigenfunctions may not belong to the same eigenspace.

Let $\{\phi_2(0), \phi_3(0)\}$ be an orthonormal basis of the second eigenspace in Ω, and let $\lambda_i\big(J(\Omega, t)\big)$ denote the i-th, $i = 2, 3$, eigenvalue of $J(\Omega, t)$ associated with the i-th normalized eigenfunction $\phi_i(t) := \phi_i\big(J(\Omega, t)\big)$ of Δ_e in $J(\Omega, t)$. Let us denote $J(p, t)$ simply by $J_t(p)$ and denote the pulled-back function $J_t^*\phi_i(t)$ to Ω by $\phi^*_{i,0}(t)$.

Proposition 2.12 states that if $\lambda_2\big(J(\Omega, t)\big)$ is double at $t = 0$ and simple on a deleted neighborhood of $t = 0$ with the condition such that $\lim_{t_0\to 0} \frac{d}{dt}_{t=t_0}\lambda_i\big(J(\Omega, t)\big)$, $i = 2, 3$, exist and $\lim_{t_0\to 0} \frac{d}{dt}_{t=t_0}\lambda_2\big(J(\Omega, t)\big) \neq \lim_{t_0\to 0} \frac{d}{dt}_{t=t_0}\lambda_3\big(J(\Omega, t)\big)$, then $\phi^*_{2,0}(t)$ turns out to converge to a second eigenfunction in Ω as $t \to 0$. From this fact we can infer that $\frac{d}{dt}\lambda^*_i(t)$ and $\frac{d}{dt}\phi^*_{i,0}(t)$ exist and are continuous on a neighborhood of $t = 0$.

We find a condition on deformations such that if a deformation J satisfies this condition, then $\phi^*_{i,0}(t) \to \phi_i(0)$, $i = 2, 3$, as $t \to 0$ with $\frac{d}{dt}_{t=0}\lambda_2\big(J(\Omega, t)\big) \neq \frac{d}{dt}_{t=0}\lambda_3\big(J(\Omega, t)\big)$. This condition is represented by

$$\begin{cases} \frac{d}{dt}_{t=0}\Delta_{J_t^*e}\phi_2(0) \\ = -\rho_2\phi_2(0) - \big(\Delta_e + \lambda_2(0)\big)g_2(0), \text{ and} & \quad .....(51) \\ \frac{d}{dt}_{t=0}\Delta_{J_t^*e}\phi_3(0) \\ = -\rho_3\phi_3(0) - \big(\Delta_e + \lambda_2(0)\big)g_3(0), & \quad .....(52) \end{cases}$$

where $\rho_2 \neq \rho_3$, $\rho_i \in \mathbb{R}$, and $g_i(0) \in C_0^\infty(\Omega)$, $i = 2, 3$. Note that the right hand sides of (51) and (52) have no $\langle \phi_3(0) \rangle$-component and no $\langle \phi_2(0) \rangle$-component, respectively. If (51) holds, then by Proposition 2.10 (52) also holds for a number ρ_3. Provided either (51) or (52) holds and $\rho_2 \neq \rho_3$, one can conclude the followings; for $i = 2, 3$,

$$\begin{cases} \phi^*_{i,0}(t) \to \phi_i(0) \ \text{(in } L_2\text{-norm)}, \\ g_i(0) = \text{component of } \frac{d}{dt}_{t=0}\phi^*_{i,0}(t) \ \text{orthogonal to } \langle \phi_2(0), \phi_3(0) \rangle, \\ \rho_i = \frac{d}{dt}_{t=0}\lambda_i\big(J(\Omega, t)\big). \end{cases}$$

We find a concrete deformation J which satisfies the condition mentioned above (Proposition 2.25). Given $\phi_2(0)$, ρ_2 and ρ_3, $\rho_2 \neq \rho_3$, we will show the existence of deformation J such that J not only satisfies condition (51) and (52) but also makes the following value (82) with subindex k removed be positive, and makes $\phi_3(0)$-component of $\frac{d}{dt}_{t=0}\phi^*_{2,0}(t)$ vanish;

$$-\int_\Omega \frac{\partial}{\partial \nu}_{(\zeta,\tau)=P} K_{\lambda_2(\Omega)}(x, y; \zeta, \tau) \frac{d}{dt}_{t=0} \Delta_{J_t^{k*}e}\phi_2(0)(x, y)dxdy, \quad .....(82)$$

where $K_{\lambda_2(\Omega)}$ stands for Green's function of $\Delta_e + \lambda_2(\Omega)$ in Ω and $\frac{\partial}{\partial \nu}$ denotes the outer normal derivative. If we assume that $\phi_2(0)$ is positive in the inner nodal domain, then the above positive requirement makes nodal line of $\phi^*_{2,0}(t)$ be closed and separate from $\partial\Omega$ over a deleted neighborhood of $t = 0$. Note that from (51) and (52) ρ_i is represented by an integral $-\int_\Omega \{\frac{d}{dt}_{t=0}\Delta_{J_t^*e}\phi_i(0)\}\phi_i(0)dxdy$.

According to Hopf's boundary point lemma, one can show by factorization into linear sums of outer normal derivatives of eigenfunctions that the following function on $\partial\Omega$ has at most four zeros (Proposition 2.23);

$$\sum_{2\le i,j\le 3} \alpha_{i,j} \frac{\partial \phi_i(0)}{\partial \nu} \frac{\partial \phi_j(0)}{\partial \nu}, \quad \alpha_{i,j} \in \mathbb{R}. \quad .....(80)$$

But we can not confirm in this way the existence of segments of $\partial\Omega$ on which the following function does not vanish;

$$\alpha_{2,4} \frac{\partial \phi_2(0)}{\partial \nu} \frac{\partial}{\partial \nu}\left(\frac{\partial}{\partial \nu}_P K_{\lambda_2(\Omega)}\right) + \sum_{2\le i,j\le 3} \alpha_{i,j} \frac{\partial \phi_i(0)}{\partial \nu} \frac{\partial \phi_j(0)}{\partial \nu}, \quad \alpha_{2,4},\ \alpha_{i,j} \in \mathbb{R}. \quad .....(81)$$

In Proposition 2.25 to show the validity of construction of J satisfying the conditions mentioned above we solve a problem related to zeros of function (81) in an elementary and complicated way.

We will construct a piecewise smooth deformation which deforms Ω to Ξ in order that each second eigenfunction of a piecewise differentiable path consisting of the second eigenfunctions in this deformation may have a closed nodal line, where through a dilation and a translation of Ω we set

$$\Xi := \big\{(x, y) : 0 < x < 2,\ 0 < y < 1\big\}, \quad \lambda_2(\Xi) = \lambda_2(\Omega), \quad \Xi \cap \Omega \neq \emptyset.$$

The piecewise smooth deformation is composed alternately and repeatedly of two distinct smooth deformations

$$\mathcal{J}^m : \Omega_m \times [0,1] \to \mathbb{R}^2, \quad \text{and} \quad \mathcal{F}^m : \mathcal{J}^m(\Omega_m, \mathfrak{q}_m) \times [0,1] \to \mathbb{R}^2,$$

where

$$\Omega_{m+1} := \mathcal{F}_1^m \circ \mathcal{J}_{\mathfrak{q}_m}^m(\Omega_m), \quad \Omega_1 := \Omega, \;\; m = 1, 2, 3, \dots .$$

The supports of $\mathcal{J}^m$ and $\mathcal{F}^m$ lie in the closures of $\Omega_m \setminus \Xi$ and $\Omega_m \cap \Xi$, respectively. Deformation $\mathcal{J}^m$ satisfies (51), (52) and makes (82) with J_t^{k*} replaced by $\mathcal{J}_t^{m*}$ be positive, and $\mathcal{J}^m$ is also a deformation with $\phi_3(\Omega_m)$-component eliminated described in Proposition 2.25 which means $\frac{d}{dt}_{t=0}\mathcal{J}_t^* \phi_2\big(\mathcal{J}_t^m(\Omega_m)\big)$ has no $\phi_3(\Omega_m)$-component. If we set $0 < \rho_2 \lneq \rho_3$, then $\lambda_2\big(\mathcal{J}_t^m(\Omega_m)\big)$ and $\lambda_3\big(\mathcal{J}_t^m(\Omega_m)\big) - \lambda_2\big(\mathcal{J}_t^m(\Omega_m)\big)$ strictly increase as t grows from $t = 0$ to $t = \mathfrak{q}_m \le 1$. $\mathcal{F}^m$ fills up a portion of $\Xi \setminus \Omega_m$, and then $\lambda_2\big(\mathcal{F}_s^m \circ \mathcal{J}_{\mathfrak{q}_m}^m(\Omega_m)\big)$ decreases to $\lambda_2(\Omega_m)$, and $\lambda_3\big(\mathcal{F}_s^m \circ \mathcal{J}_{\mathfrak{q}_m}^m(\Omega_m)\big) - \lambda_2\big(\mathcal{F}_s^m \circ \mathcal{J}_{\mathfrak{q}_m}^m(\Omega_m)\big)$ may decrease toward zero as s grows to one.

Let us denote $\mathfrak{F}_m := \Xi \cap \Omega_m$. Deformations stated in the above paragraph will be constructed in order that $\phi_2(\Omega_m)|\mathfrak{F}_m$, the restriction of $\phi_2(\Omega_m)$ to $\mathfrak{F}_m$, may approximate to $\phi_2(\Xi)$ in L_2-norm as m becomes large. Then, a closed nodal line of $\phi_2(\Omega_m)$ compared with the nodal line of $\phi_2(\Xi)$, the nodal line of $\phi_2(\Omega_m)|\mathfrak{F}_m$ may be considered to be sufficiently close to the segments $\{(x,y)\,|\, x = 0,1,\ 0 < y < 1\}$, and $\{(x,y)\,|\, 0 < x < 1,\ y = 0,1\}$. Then, the outer nodal domain Ω_m^- of $\phi_2(\Omega_m)$ contains a sufficiently narrow and long simply connected band

$$W = \Omega_m^- \cap \big\{(x,y) : 0 < x < 1/2,\ 1/4 < y < 3/4\big\}.$$

We will follow methods of David Jerison [7]. From exponential decay theorem (Lemma 3.1) applied to W one can show that for a $\zeta \in \partial W \cap \partial\Omega_m^+$, Ω_m^+ the inner nodal domain of $\phi_2(\Omega_m)$, the magnitude of gradient

$$\big|\nabla\phi_2(\Omega_m)(\zeta)\big|$$

also decays exponentially as ζ moves along $\partial W \cap \partial\Omega_m^+$ and as m becomes large. According to [7], we can show

$$\min_{B(z,R/2)} |\phi_2(\Omega_m)|$$

is bounded to the magnitude $R\big|\nabla\phi_2(\Omega_m)(\zeta)\big|$ for a disk $B(z,R) \subset \Omega_m^+$ such that $\zeta \in \partial B(z,R) \cap \partial\Omega_m^+$. It implies that $\phi_2(\Omega_m)|\mathfrak{F}_m$ can not approximate to $\phi_2(\Xi)$ in Ω_m^+ in L_2-norm, and then we attain a contradiction.

1 Introduction and a History of Nodal Line Conjecture

Topology of nodal lines of eigenfunctions is not yet well known. In a global concept Courant nodal domain theorem may be regarded as almost the only remarkable achievement. Courant nodal domain theorem [3] states that the nodal set of any k-th eigenfunction of C^∞ domain divides the domain into at most k subregions. A corollary of this theorem tells us that the second eigenfunction has precisely two nodal domains. As a simple case, study on the closed nodal line of the second eigenfunction of Δ_e has been pursued since L. E. Payne's conjecture [13] : In 1967 he conjectured that in any simply connected bounded domain in $\mathbb{R}^2$ any second eigenfunction of Δ_e with Dirichlet homogeneous boundary condition cannot have a closed nodal line.

Throughout the paper, Ω is any simply connected bounded (possibly non-convex) domain in $\mathbb{R}^2$ with C^∞ boundary. An *eigenfunction* of the Laplace-Beltrami operator Δ_g in Ω with C^∞ Riemannian metric g is meant to be a solution $u \not\equiv 0$ satisfying the homogeneous Dirichlet eigenvalue problem

$$\begin{cases} \Delta_g u + \lambda u &= 0 \quad \text{in } \Omega, \\ u &\equiv 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_g = \sum_{i,j} \frac{1}{\sqrt{|g|}} D_i\big(\sqrt{|g|} g^{ij} D_j\big)$, $i,j = 1,2$, $g = \big(g_{ij}\big)$, $|g| = \det\big(g_{ij}\big)$. A positive λ for which the above Dirichlet condition (1) possesses a solution u is called an *eigenvalue* of Δ_g in Ω. Each associated eigenspace is finite dimensional, distinct eigenspaces are orthogonal each other in $L^2(\Omega)$, and $L^2(\Omega)$ is the direct sum of all the eigenspaces. We will denote the volume element associated with g by $\underline{d}_g$. Let us denote by e the standard euclidean metric. Each eigenfunction of Δ_g is C^∞ in $\overline{\Omega}$, and eigenfunction of Δ_e is analytic in Ω. It is known that Δ_e is analytic-hypoelliptic operator. The *nodal line* of an (eigen)function u is defined to be the set

$$\overline{\{z \in \Omega |\ u(z) = 0\}},$$

and a *nodal domain* of u is a component of $\Omega \setminus u^{-1}(0)$. According to [4], the nodal sets of eigenfunctions of C^∞-Riemannian manifold in $\mathbb{R}^2$ (if they exist) consist of a number of C^2-immersed one-dimensional closed submanifolds.

It is known that for generically many metric g eigenfunctions of Δ_g in a manifold with boundary vanish of the second order where their nodal lines intersect boundary [15]. C.-S. Lin [8] showed any convex smooth boundary bounded domain in $\mathbb{R}^2$ with symmetry under rational rotation with respect one point has no second eigenfunction of Δ_e whose nodal line is closed. A rational rotation means a rotation with angle $2\pi p/q$ for positive integers p and q. He also proves that if $\phi_2(\Omega)$ is one of the normalized second eigenfunctions of Δ_e in a bounded smooth convex domain $\Omega \subset \mathbb{R}^2$ such that $\frac{\partial \phi_2(\Omega)}{\partial \nu} \geq 0$, $\frac{\partial}{\partial \nu}$ the outward normal derivative, on $\partial\Omega$, then $\phi_2(\Omega)$ is the only normalized second eigenfunction of Δ_e in Ω. David Jerison [7] showed that for convex bounded domain $\Omega \subset \mathbb{R}^2$ there exists an absolute constant C such that if $\text{diameter}(\Omega)\big/\text{inradius}(\Omega) \geq C$, then the nodal line for the second eigenfunction of Ω touches the boundary. Recently A. D. Melas [11](1992) has proved that the Payne's conjecture is true for bounded convex smooth (boundary) domain in $\mathbb{R}^2$, and G. Alessandrini [1] for bounded convex domain. On the other hand, as to a multiply connected domain, M. H.-Ostenhof, T. H.-Ostenhof and N. Nadirashvili [12] have given a counterexample to Payne's conjecture in a non-simply connected domain. Also they give an example of domain in $\mathbb{R}^2$ whose the second eigenvalue is of multiplicity three.

It is well known that the first eigenvalue is simple. As for the second eigenvalue, from S. Y. Cheng [4] it can be shown that the multiplicity of second eigenvalue of bounded smooth boundary domain is at most three. Later Z. Liqun [10] verifies the dimension of the second eigenspace of Δ_e in bounded smooth boundary simply connected domain of $\mathbb{R}^2$ is at most two. In this paper, according to the result of Z. Liqun we assume that the second eigenspace of Δ_e in any bounded smooth boundary simply connected domain of $\mathbb{R}^2$ is at most two dimension.

2 Deformations whose Eigenfunctions form a C^1-Path which passes through a Given Eigenfunction

2.1 a Regularity of Paths composed of Eigenvalues and Eigenfunctions of Deformations

Let $J : \Omega \times [0,1] \to \mathbb{R}^2$, $J_t(P) := J(p,t)$, be a smooth deformation of Ω. According to [5], p.419, p.421, Theorem 10, the n-th eigenvalue of a C^∞ deformation $J_t(\Omega)$ varies continuously to the n-th eigenvalue of Ω as $t \to 0$. Also Courant and Hilbert [5] showed that the r-th ordered eigenfunctions as well as eigenvalues of perturbed domains which are C^k-diffeomorphic to Ω converge to those of Ω, if the perturbed domains converge to Ω. Their proof consisted of reducing the problem to study of a family of differential operators on Ω obtained from C^k-diffeomorphisms. The coefficients of operators becomes to differ arbitrarily little from the original coefficients. The section 13 of chapter V [5] is devoted to find the first and second approximations of perturbations of domain. [5] shows a counter example with irregular perturbation and Neumann boundary problem in which the continuous property of eigenvalues failed.

A linear integral equation corresponding to elliptic equation $(\Delta_e + \mu)\phi = 0$ is represented by a homogeneous functional equation

$$\phi(x,y) - \mu \int_\Omega K(x,y,\zeta,\tau)\phi(\zeta,\tau)d\zeta d\tau = 0.$$

Let $K_0(x,y;\zeta,\tau)$ and $K_t(x,y;\zeta,\tau)$ be symmetric kernels associated with the above eigenvalue problem of domain Ω and $J_t(\Omega)$, respectively. According to [5] (Chapter III, §8 p.151, and §9 p.152), one comprehends the following continuity property of eigenfunctions: Let the n-th eigenspace of Δ_e in Ω, $n \geq 2$, be of m-dimensional, and let the kernel K_t converge to K_0 uniformly as $t \to 0$. Then, given any n-th eigenfunction ϕ of Δ_e in Ω, there exists a linear combination

$$a_1(t)\psi_{i+1}(t) + \cdots + a_m(t)\psi_{i+m}(t) \tag{2}$$

which converges uniformly to ϕ as $t \to +0$, where $a_k(t) \in \mathbb{R}$, and $\psi_{i+k}(t)$ is an eigenfunction, $k = 1, 2, ..., m$, of Δ_e in $J_t(\Omega)$. Note that the above linear sum is not necessarily eigenfunction, since all eigenfunctions $\psi_{i+k}(t)$ may not belong to the same eigenspace. But it is said from the above fact that each $\psi_{i+k}(t)$ converges to an n-th eigenfunction $\psi_{i+k}(0)$ of Ω, as $t \to 0$, and moreover the limit of each linear combination (2) composes the n-th eigenspace of Δ_e in Ω. Thus, there may be eigenfunctions of Ω to which no eigenfunction of $J_t(\Omega)$ converges.

Throughout paper it is assumed that a second eigenfunction of Ω has a closed nodal line. We will find a smooth deformation J_t of Ω such that C^1-path $\{\phi_{2,J_t(\Omega)}\}$ which composes of the normalized second eigenfunction of $J_t(\Omega)$ passes through a given second eigenfunction in Ω and each $\phi_{2,J_t(\Omega)}$, $t \in (0,1]$, has a closed nodal line separated from boundary, and if $\lambda_i(0)$ is double, $i = 2, 3$, then derivatives of path $\{\lambda_i(J_t(\Omega))\}$ composed of the second and third eigenvalues of $J_t(\Omega)$ at $t = 0$ are equivalent with prescribed values.

Firstly we are to show regularity of these paths. Let us denote by

$$\begin{cases} \phi_i(t) := \phi_{i,J_t(\Omega)} := \phi_i\big(J_t(\Omega)\big) \text{ the } i\text{-th normalized eigenfunction in } J_t(\Omega), \\ \phi^*_{i,t_0}(t) := \big(J_t \circ J_{t_0}^{-1}\big)^* \phi_i(t) \text{ the eigenfunction } \phi_i(t) \text{ pulled back to } J_{t_0}(\Omega), \\ \lambda_i(t) := \lambda_i\big(J_t(\Omega)\big) \text{ the } i\text{-th eigenvalue of } J_t(\Omega) \text{ associated with } \phi_i(t), \end{cases} \tag{3}$$

for $i \in \mathbb{N}$, and $0 \leq t_0, t \leq 1$. For an orthonormal complete system $\{\phi_i(t_0) : i \in \mathbb{N}\}$ in $J_t(\Omega)$ let us expand $\phi^*_{i,t_0}(t)$ as follows, and call this expansion $\phi^*_{i,t_0}(t)$ *expanded with respect to* $\phi_i(t_0)$*;*

$$\phi^*_{i,t_0}(t) = \phi_i(t_0) + \sum_{k\in\mathbb{N}} \beta_{t_0,i,k}(t)\phi_k(t_0), \quad \beta_{t_0,i,k}(t) \in \mathbb{R}, \quad i \in \mathbb{N}. \tag{4}$$

From identity $\Delta_{(J_t\circ J_{t_0}^{-1})^* e}\phi^*_{i,t_0}(t) = -\lambda_i(t)\phi^*_{i,t_0}(t)$, $i\in\mathbb{N}$, one can induce for $0\le t, t_0\le 1$

$$\begin{aligned}&\Big(\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e} - \Delta_e\Big)\phi^*_{i,t_0}(t)\\ &= -\big(\lambda_i(t)-\lambda_i(t_0)\big)\phi^*_{i,t_0}(t) - \Big(\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e} + \lambda_i(t_0)\Big)\big(\phi^*_{i,t_0}(t)-\phi_i(t_0)\big).\end{aligned} \tag{5}$$

Remark 2.1. The operator $\Delta_e + \lambda_2(0) : W^{k,2}(\Omega)\cap W^{1,2}_0(\Omega)\to W^{k-2,2}(\Omega)$, $k\ge 2$, is a Fredholm of index zero, that is,

$$\dim\Big\{W^{k-2,2}(\Omega)\Big/\mathrm{Im}\big(\Delta_e+\lambda_2(0)\big)\Big\} = \dim\mathrm{Ker}\big(\Delta_e+\lambda_2(0)\big),$$

where $W^{k,p}$ denotes Sovolev space. If $\lambda_2(0)$ is double, then $\int_\Omega\big\{\big(\Delta_e+\lambda_2(0)\big)\mathfrak{f}\big\}\phi_i(0) = \int_\Omega\big\{\big(\Delta_e+\lambda_2(0)\big)\phi_2(0)\big\}\mathfrak{f} = 0$, $i=2,3$, $\mathfrak{f}\in W^{k,2}(\Omega)\cap W^{1,2}_0$. Thus $W^{k-2,2}(\Omega)\Big/\mathrm{Im}\big(\Delta_e + \lambda_2(0)\big)\cong\langle\phi_2(0),\phi_2(0)\rangle$. Thus, the operator $\Delta_e+\lambda_2(0) : W^{k,2}(\Omega)\cap W^{1,2}_0(\Omega)\Big/\langle\phi_2(0),\phi_2(0)\rangle\to W^{k-2,2}(\Omega)\Big/\langle\phi_2(0),\phi_2(0)\rangle$ is bijective. The inverse operator $\big(\Delta_e+\lambda_2(0)\big)^{-1} : \mathrm{Im}\big(\Delta_e + \lambda_2(0)\big)\to W^{k,2}(\Omega)\cap W^{1,2}_0(\Omega)/\mathrm{Ker}\big(\Delta_e+\lambda_2(0)\big)$ given by

$$\sum_{j\notin\mathcal{K}}^{j\in\mathbb{N}}\beta_j\phi_j(0)\ \mapsto\ \sum_{j\notin\mathcal{K}}^{j\in\mathbb{N}}\frac{1}{-\lambda_j(0)+\lambda_2(0)}\beta_j\phi_j(0)$$

is also bounded operator since

$$\sup_{\beta_j}\sum_{j\notin\mathcal{K}}^{j\in\mathbb{N}}\left\|\frac{1}{-\lambda_j(0)+\lambda_2(0)}\beta_j\phi_j(0)\right\|_2^2\le\frac{1}{\big\{-\lambda_{j_0}+\lambda_2(0)\big\}^2}<\infty, \tag{6}$$

where $\{\phi_j : j\in\mathcal{K}\}$ is an orthonormal basis of the second eigenspace in Ω, $\sum_{j\notin\mathfrak{B}}\beta_j^2 = 1$, and $|-\lambda_{j_0}(0)+\lambda_2(0)|$ is the smallest number among $\big\{|-\lambda_j(0)+\lambda_2(0)| : j\notin\mathcal{K}\big\}$. //

Proposition 2.1 *The value*

$$\sup_{\{j,l\}}\sup_{(x,y)\in\Omega}\frac{\partial^{j+l}}{\partial x^j\partial y^l}\phi^*_{i,t_0}(t)(x,y),\quad 1\le j+l\le 2,\quad j,l\in\{0,1,2\},$$

is bounded on $t\in[0,\varepsilon]$, $0<\varepsilon\le 1$. *The left hand of (5),* $\Big(\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e} - \Delta_e\Big)\phi^*_{i,t_0}(t)$, *converges in* L^2*-norm to zero as* $t\to t_0$, $t_0\in[0,\varepsilon]$. $\Big\{\frac{d}{dt}\Delta_{(J_t\circ J_{t_0}^{-1})^* e}\Big\}\phi^*_{i,t_0}(t)$ *is bounded in* Ω *on* $t\in[0,\varepsilon]$.

Proof. Theorem 3.7 ([6] Theorem 9.26, p.250) says that $\sup_\Omega|\phi^*_{i,t_0}(t)| < C$ for a constant C over all $t\in[0,\varepsilon]$ for a ε, $0<\varepsilon\le 1$. Then, according to global estimate for solutions of elliptic equation, Theorem 3.6 ([6] Theorem 6.6, p.98), if we set $Lu = -\lambda_i(t)\phi^*_{i,t_0}(t)$, then the solution is $u=\phi^*_{i,t_0}(t)$, and then $\frac{\partial^{j+l}}{\partial x^j\partial y^l}\phi^*_{i,t_0}(t)$ is bounded over all $t\in[0,\varepsilon]$. For this note that the constant $C = C_t$ in Theorem 3.6 is dependent on $J_t(\Omega)$, but $\max_{t\in[0,\varepsilon]} C_t$ exists.

Let us set

$$\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e} = \sum_{1\le j+l\le 2}^{j,l=0,1,2} a_{j,l}(t)(x,y)\frac{\partial^{j+l}}{\partial x^j\partial y^l}. \tag{7}$$

Each entry $a_{j,l}(t)(x,y) - a_{j,l}(t_0)(x,y)$ of differential operator $\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e} - \Delta_e$ converges to zero in $\|\cdot\|_{k,2;\Omega}$-norm in $W^{k,2}(\Omega)$, $k\ge 2$, as $t\to t_0$, since J_t is a smooth deformation. Thus, the first claim follows. Since each derivative $\frac{d}{dt}a_{j,l}(t)$ exists at any $t\in[0,1]$, from Theorem 3.6 the last claim is valid. Theorem 3.7 and Theorem 3.6 will be introduced in the next section without proof. □

Remark 2.2. Let us assume that $\int_{J_{t_0}(\Omega)} \phi^*_{2,t_0}(t)\cdot\phi_2(t_0)d_e$ converges to a non-zero number as $t\to t_0$. Then, since the left hand of (5) converges to zero as $t\to t_0$ from Proposition 2.1, and since $\mathrm{Im}\big(\Delta_{J^*_{t_0}e}+\lambda_2(t_0)\big)$ is orthogonal in metric $J^*_{t_0}e=e$ to the second eigenspace of Δ_e in $J_{t_0}(\Omega)$, we can conclude that $\lambda_2(t)$ converges to $\lambda_2(t_0)$, and furthermore, then $\big(\Delta_{J^*_{t_0}e}+\lambda_2(t_0)\big)\phi^*_2(t)$ which is naturally orthogonal in metric e to the second eigenspace of Δ_e in $J_{t_0}(\Omega)$ also converges to zero as $t\to t_0$. Consequently, by boundedness of $\big(\Delta_{J^*_{t_0}e}+\lambda_2(t_0)\big)^{-1}$ (refer to Remark 2.1) the component of $\phi^*_2(t)$ orthogonal to the second eigenspace of Δ_e in $J_{t_0}(\Omega)$ denoted by $\big(\phi^*_2(t)\big)^\circ_{t_0}$ also converges (in L^2-norm) to zero as $t\to t_0$.

But whether $\phi^*_2(t)$ converges to an element in the second eigenspace of Δ_e in $J_{t_0}(\Omega)$ or oscillates as $t\to t_0$, the result of preceding paragraph is valid provided $\lambda_2(t)$converges to $\lambda_2(t_0)$ as $t\to t_0$. //

Proposition 2.2. *Let $\lambda_2(t)$ be simple at $t=t_0\in(0,\varepsilon)$. Then, $\lambda_2(t)$ is continuous at $t=t_0$, and $\phi^*_{2,t_0}(t)$ converges in L^2-norm to the unique second eigenfunction $\phi_2(t_0)$ as $t\to t_0$.*

(**Note.**) If $\lambda_2(t_0)$ is simple, then there is an $\varepsilon>0$ such that if $|t-t_0|<\varepsilon$, then $\lambda_2(t)$ is also simple. From Theorem 3.6 and by convergence of $\Delta_{J^*_te}$ to Δ_e one can verify convergence of $\phi^*_2(t)$ in L^2-norm to $\phi_2(t_0)$ assures convergence in $\|\cdot\|_{2,2,\Omega}$-norm.

Proof. Firstly, we consider energy of the pushed-ahead first eigenfunction $\big(J_{t_0}\circ J_t^{-1}\big)^*\phi_1(t_0)$ to $J_t(\Omega)$. We have

$$\begin{aligned}&\frac{\int_{J_t(\Omega)}\nabla\phi_1(t)\nabla\phi_1(t)d_e}{\|\phi_1(t)\|_2^2}\\ &\le\frac{\int_{J_t(\Omega)}\nabla\big(J_{t_0}\circ J_t^{-1}\big)^*\phi_1(t_0)\nabla\big(J_{t_0}\circ J_t^{-1}\big)^*\phi_1(t_0)d_e}{\big\|\big(J_{t_0}\circ J_t^{-1}\big)^*\phi_1(t_0)\big\|_2^2}.\end{aligned}\tag{8}$$

On the contrary we have

$$\begin{aligned}&\frac{\int_{J_{t_0}(\Omega)}\nabla\phi_1(t_0)\nabla\phi_1(t_0)d_e}{\|\phi_1(t_0)\|_2^2}\\ &\le\frac{\int_{J_{t_0}(\Omega)}\nabla\big(J_t\circ J_{t_0}^{-1}\big)^*\phi_1(t)\nabla\big(J_t\circ J_{t_0}^{-1}\big)^*\phi_1(t)d_e}{\big\|\big(J_t\circ J_{t_0}^{-1}\big)^*\phi_1(t)\big\|_2^2}.\end{aligned}\tag{9}$$

If $t\to t_0$, the right hand of (8) converges to $\lambda_1(t_0)$, and the right hand of (9) converges to $\lambda_1(t)$. Therefore, if $t\to t_0$, we have $\lambda_1(t)\le\lambda_1(t_0)+\epsilon_1(t)$ and $\lambda_1(t_0)\le\lambda_1(t)+\epsilon_2(t)$, where $\epsilon_i(t)>0$, $i=1,2$, converges to zero as $t\to t_0$. Consequently, $\lambda_1(t)\to\lambda_1(t_0)$.

Then, since from Remark 2.1 the inverse operator of $\Delta_{J^*_{t_0}e}+\lambda_1(t_0)$ defined in $\mathrm{Im}\big(\Delta_{J^*_{t_0}e}+\lambda_1(t_0)\big)$ is bounded, and since $\lambda_1(t)-\lambda_1(t_0)$ converges to zero, and since from Proposition 2.1 the left hand of (5) converges to zero, we can conclude that the component of $\phi^*_{1,t_0}(t)-\phi_1(t_0)$ orthogonal to $\langle\phi_1(t_0)\rangle$ with respect to metric e converges to zero function as $t\to t_0$. Therefore, $\phi^*_{1,t_0}(t)$ expanded with respect to $\{\phi_k(t_0):k\in\mathbb{N}\}$, then $\sum_{k\in\mathbb{N}\setminus\{1\}}\beta^2_{t_0,1,k}(t)$ converges to zero. Since $\|\phi^*_{1,t_0}(t)\|^2_{(J_t\circ J_{t_0}^{-1})^*e}=1=\|\phi_1(t_0)\|_e^2$, and each function of entries of the smooth metric tensor $\big(J_t\circ J_{t_0}^{-1}\big)^*e$ converges (in $\|\cdot\|_{2,2;\Omega}$-sense) to that of e as $t\to t_0$, we may conclude that $\|\phi^*_{1,t_0}(t)\|_e^2$ converges to one. Thus, $\big(1+\beta_{t_0,1,1}(t)\big)^2$ converges to one, and then $\phi^*_{1,t_0}(t)$ converges to $\phi_1(t_0)$ as $t\to t_0$.

Secondly, we are to verify proposition for the second eigenfunction. From minimum energy property of eigenfunction we have

$$
\begin{aligned}
&\frac{\int_{J_t(\Omega)} \nabla\phi_2(t)\nabla\phi_2(t)}{\|\phi_2(t)\|_2^2} \\
&\leq \Big[\int_{J_t(\Omega)} \nabla\Big\{ \big(J_{t_0}\circ J_t^{-1}\big)^*\phi_2(t_0) - \delta_{2,1}(t_0,t)\phi_1(t) \Big\} \\
&\qquad\qquad \cdot \nabla\Big\{ \big(J_{t_0}\circ J_t^{-1}\big)^*\phi_2(t_0) - \delta_{2,1}(t_0,t)\phi_1(t) \Big\} \Big] \\
&\quad \cdot \Big\| \big(J_{t_0}\circ J_t^{-1}\big)^*\phi_2(t_0) - \delta_{2,1}(t_0,t)\phi_1(t) \Big\|_2^{-2},
\end{aligned} \tag{10}
$$

where $\delta_{2,1}(t_0,t) = \int_{J_t(\Omega)} \{ (J_{t_0}\circ J_t^{-1})^*\phi_2(t_0)\}\phi_1(t)$. On the contrary

$$
\begin{aligned}
&\frac{\int_{J_{t_0}(\Omega)} \nabla\phi_2(t_0)\nabla\phi_2(t_0)}{\|\phi_2(t_0)\|_2^2} \\
&\leq \Big[\int_{J_{t_0}(\Omega)} \nabla\Big\{ \big(J_t\circ J_{t_0}^{-1}\big)^*\phi_2(t) - \delta_{2,1}(t,t_0)\phi_1(t_0) \Big\} \\
&\qquad\qquad \cdot \nabla\Big\{ \big(J_t\circ J_{t_0}^{-1}\big)^*\phi_2(t) - \delta_{2,1}(t,t_0)\phi_1(t_0) \Big\} \Big] \\
&\quad \cdot \Big\| \big(J_t\circ J_{t_0}^{-1}\big)^*\phi_2(t) - \delta_{2,1}(t,t_0)\phi_1(t_0) \Big\|_2^{-2},
\end{aligned} \tag{11}
$$

where $\delta_{2,1}(t,t_0) = \int_{J_{t_0}(\Omega)} \{ (J_t\circ J_{t_0}^{-1})^*\phi_2(t)\}\phi_1(t_0)$. Since $\phi_1^*(t) \to \phi_1^*(t_0)$, we have

$$
\lim_{t\to t_0} \delta_{2,1}(t_0,t) = 0 = \lim_{t\to t_0} \delta_{2,1}(t,t_0). \tag{12}
$$

According to the argument verifying convergence of the first eigenvalues, from (10) and (11) $\lambda_2(t) \to \lambda_2(t_0)$ as $t \to t_0$.

Therefore, since the inverse operator of $\Delta_{J_{t_0}^* e} + \lambda_2(t_0)$ defined in $\mathrm{Im}\big(\Delta_{J_{t_0}^* e} + \lambda_2(t_0)\big)$ is bounded operator from Proposition 2.1, formula (5) says the component of $\phi_2^*(t) - \phi_2^*(t_0)$ orthogonal to $\langle\phi_2^*(t_0)\rangle$ with respect to metric $J_{t_0}^* e$ converges (in L^2-norm) to zero. That is, $\sum_{k\in\mathbb{N}\setminus\{2\}} \big(\beta_{t_0,2,k}(t) - \beta_{t_0,2,k}(t_0)\big)^2$ converges to zero. Since $\|\phi_2^*(t)\|_{J_t^* e}^2 = 1 = \|\phi_2^*(t_0)\|_{J_{t_0}^* e}^2$ and since each function of entries of metric tensor $J_t^* e$ converges to that of $J_{t_0}^* e$ as $t \to t_0$, we may conclude that $\|\phi_2^*(t)\|_{J_{t_0}^* e}$ converges to one. Therefore, $\big(1 + \beta_{t_0,2,2}(t)\big)^2$ converges to one, and then $\phi_2^*(t)$ converges (in L^2-norm) to $\phi_2^*(t_0)$ as $t \to t_0$. □

Proposition 2.3. *Let us suppose that $\lambda_2(t)$ is simple at $t = t_0 \in (0,\varepsilon)$. Then, derivatives $\frac{d}{dt}_{t=t_0}\lambda_2(t)$, and $\frac{d}{dt}_{t=t_0}\phi_{2,t_0}^*(t)$ exist, and we have a unique representation*

$$
\frac{d}{dt}_{t=t_0} \Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e}\phi_2(t_0) = -\frac{d}{dt}_{t=t_0}\lambda_2(t)\phi_2(t_0) - \big(\Delta_e + \lambda_2(t_0)\big)\frac{d}{dt}_{t=t_0}\phi_{2,t_0}^*(t). \tag{13}
$$

Proof. The limit

$$
\lim_{t\to t_0} \left\{ \frac{1}{t-t_0}\Big(\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e} - \Delta_e\Big)\phi_{2,t_0}^*(t) \right\}
$$

exists, since from Proposition 2.2 $\phi_{2,t_0}^*(t)$ converges to $\phi_2(t_0)$ and $\frac{d}{dt}_{t=t_0}\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e}$ exists. Then this limit equals to the left hand of equality (13). Thus, from (5) we get (13). In formula (13) the image of $\Delta_e + \lambda_2(t_0)$ is orthogonal under the metric e to the second eigenspace $\langle\phi_2(t_0)\rangle$ of Δ_e in $J_{t_0}(\Omega)$. Therefore, a unique existence of $\frac{d}{dt}_{t=t_0}\lambda_2(t)$ and a unique existence of the component of $\frac{d}{dt}_{t=t_0}\phi_2^*(t)$ which is orthogonal to $\langle\phi_2(t_0)\rangle$ denoted by $\Big(\frac{d}{dt}_{t=t_0}\phi_{2,t_0}^*(t)\Big)^\circ$ are shown.

To show existence of $\phi_2(t_0)$-component of $\frac{d}{dt}_{t=t_0}\phi^*_{2,t_0}(t)$ we expand $\phi^*_{2,t_0}(t)$ with respect to $\phi_2(t_0)$. Then, it suffices to show the existence of coefficient $\frac{d}{dt}_{t=t_0}\beta_{t_0,2,2}(t)$. We introduce a notation;

Notation 2.1. Let us suppose that $\phi^*_{2,t_0}(t)$ converge to $\phi_2(t_0)$ as $t \to t_0$. Expanding $\phi^*_{2,t_0}(t)$ with respect to $\phi_2(t_0)$, we define a value $\chi_{t_0}\big(\phi_2(t)\big)$, $t_0 \in (0, \varepsilon)$, by

$$\begin{aligned}
&\chi^2_{t_0}\big(\phi_2(t)\big)\\
:=&\big\|\phi^*_{2,t_0}(t)(x,y)\big\|^2_{2,e}\\
=&\int_{J_{t_0}(\Omega)}\Big(\phi_2(t_0)+\sum_{k=1}^{\infty}\beta_{t_0,2,k}(t)\phi_k(t_0)\Big)\Big(\phi_2(t_0)+\sum_{k=1}^{\infty}\beta_{t_0,2,k}(t)\phi_k(t_0)\Big)d_e\\
=&1+2\beta_{t_0,2,2}(t)+\sum_{k=1}^{\infty}\beta^2_{t_0,2,k}(t). \qquad (14)
\end{aligned}$$

Note that $\big\|\phi^*_{2,t_0}(t)(x,y)\big\|^2_{2,\left(J_t\circ J^{-1}_{t_0}\right)^*e} = 1$ for all $t \in [0,1]$, but $\big\|\phi^*_2(t)(x,y)\big\|^2_{2,e}$ may not equal to one for $t_0 \neq t$. //

We are to show from smoothness of J_t that $\frac{d}{dt}_{t=t_0}\chi_{t_0}\big(\phi_2(t)\big)$ exists at $t_0 \in (0,\varepsilon)$. From equality $\big\|\phi^*_{2,t_0}(t)(x,y)\big\|^2_{2,\left(J_t\circ J^{-1}_{t_0}\right)^*e} = 1$, by differentiating a product of function and metric tensor, we have

$$\begin{aligned}
&\frac{d}{dt}_{t=t_0}\int_{J_{t_0}(\Omega)}\big|\phi^*_{2,t_0}(t)(x,y)\big|^2 d_{\left(J_t\circ J^{-1}_{t_0}\right)^*e}\\
=&\frac{d}{dt}_{t=t_0}\chi^2_{t_0}\big(\phi_2(t)\big)+\frac{d}{dt}_{t=t_0}\int_{J_{t_0}(\Omega)}\big|\phi_2(t_0)(x,y)\big|^2 d_{\left(J_t\circ J^{-1}_{t_0}\right)^*e}=0,
\end{aligned}$$

and then

$$\frac{d}{dt}_{t=t_0}\chi^2_{t_0}\big(\phi_2(t)\big) = -\frac{d}{dt}_{t=t_0}\int_{J_{t_0}(\Omega)}\big|\phi_2(t_0)(x,y)\big|^2 d_{\left(J_t\circ J^{-1}_{t_0}\right)^*e}. \qquad (15)$$

The right side of (15) exists at t_0, that is, $\frac{d}{dt}_{t=t_0}\chi^2_{t_0}\big(\phi_2(t)\big)$ exists. Differentiating (14), we have

$$\begin{aligned}
&\frac{d}{dt}_{t=t_0}\chi^2_{t_0}\big(\phi_2(t)\big)\\
=&2\frac{d}{dt}_{t=t_0}\beta_{t_0,2,2}(t)\big(1+\beta_{t_0,2,2}(t_0)\big)+2\sum_{k\in\mathbb{N}\setminus\{2\}}\beta_{t_0,2,k}(t_0)\frac{d}{dt}_{t=t_0}\beta_{t_0,2,k}(t).
\end{aligned}$$

Since $\lim_{t\to t_0}\phi^*_{2,t_0}(t) = \phi_2(t_0)$ and then $\beta_{t_0,2,k}(t_0) = 0$ for any $k \in \mathbb{N}$, we have

$$\frac{d}{dt}_{t=t_0}\chi^2_{t_0}\big(\phi_2(t)\big) = 2\frac{d}{dt}_{t=t_0}\beta_{t_0,2,2}(t). \qquad (16)$$

Thus, $\frac{d}{dt}\beta_{t_0,2,2}(t)$ exists at $t = t_0$. □

Remark 2.3. If $\lambda_2(t_0)$ is simple, then obviously $\frac{d}{dt}_{t=t_0}\phi^*_{2,t_0}(t)$ is represented by

$$\sum_{k\in\mathbb{N}}\frac{d}{dt}_{t=t_0}\beta_{t_0,2,k}(t),$$

and since $\lambda_k(t_0) - \lambda_2(t_0) \neq 0$ for $k \in \mathbb{N}\setminus\{2\}$, identity (13) implies that each $\frac{d}{dt}\beta_{t_0,2,k}(t)$, $k \neq 2$, converges as $t \to t_0$. One can represent the component of $\frac{d}{dt}_{t=t_0}\phi^*_{2,t_0}(t)$ orthogonal to $\langle\phi_2(t_0)\rangle$ by

$$\Big(\frac{d}{dt}_{t=t_0}\phi^*_{2,t_0}(t)\Big)^{\circ} = -\int_{J_{t_0}(\Omega)}\Big\{\frac{d}{dt}_{t=t_0}\Delta_{\left(J_t\circ J^{-1}_{t_0}\right)^*e}\phi_2(t_0)(x,y)\Big\}K_{\lambda_2(t_0)}(x,y;\zeta,\tau)d_e, \quad (17)$$

where $K_{\lambda_2(t_0)}$ stands for Green's function of $\Delta_e + \lambda_2(t_0)$ in $J_{t_0}(\Omega)$. Green's function will be introduced in this section later. //

Proposition 2.4. *Let us suppose $\lambda_2(t_0)$ be simple for $t_0 \in (0,\varepsilon)$. Then, paths of t-variable functions $\frac{d}{dt}\lambda_2(t)$ and $\frac{d}{dt}\phi^*_{2,t_0}(t)$ are continuous at $t = t_0$.*

Proof. From the formula (15) $\frac{d}{dt}\chi_{t_0}\big(\phi_2(t)\big)$ is continuous at $t = t_0$, since J is smooth. Then, from (16) $\frac{d}{dt}\beta_{t_0,2,2}(t)$ is also continuous at $t = t_0$. By a similar way to getting (5) from (1), we obtain the following second order identity from (13);

$$
\begin{aligned}
&\Big\{\frac{d}{dt}\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e} - \frac{d}{dt}_{t=t_0}\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e}\Big\}\phi^*_{2,t_0}(t)\\
&\quad + \frac{d}{dt}_{t=t_0}\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e}\{\phi^*_{2,t_0}(t) - \phi_2(t_0)\}\\
=& - \Big\{\frac{d}{dt}\lambda_2(t) - \frac{d}{dt}_{t=t_0}\lambda_2(t)\Big\}\phi^*_{2,t_0}(t)\\
&\quad - \frac{d}{dt}_{t=t_0}\lambda_2(t)\{\phi^*_{2,t_0}(t) - \phi_2(t_0)\}\\
&\quad - \Big\{\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e} + \lambda_2(t) - \big(\Delta_e + \lambda_2(t_0)\big)\Big\}\frac{d}{dt}\phi^*_{2,t_0}(t)\\
&\quad - \big(\Delta_e + \lambda_2(t_0)\big)\Big\{\frac{d}{dt}\phi^*_{2,t_0}(t) - \frac{d}{dt}_{t=t_0}\phi^*_{2,t_0}(t)\Big\}. \qquad (18)
\end{aligned}
$$

Since $\lambda_2(t) \to \lambda_2(t_0)$, and $\phi^*_{2,t_0}(t) \to \phi_2(t_0)$, identity (18) shows that $\frac{d}{dt}\lambda_2(t) - \frac{d}{dt}_{t=t_0}\lambda_2(t)$ and $\Big(\frac{d}{dt}\phi^*_{2,t_0}(t) - \frac{d}{dt}_{t=t_0}\phi^*_{2,t_0}(t)\Big)^\circ$ converge to zero and zero function, respectively, as $t \to t_0$. We obtain from (18)

$$
\begin{aligned}
&\frac{d^2}{dt^2}_{t=t_0}\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e}\phi_2(t_0)\\
=& -2\frac{d}{dt}_{t=t_0}\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e}\frac{d}{dt}_{t=t_0}\phi^*_{2,t_0}(t)\\
&\quad -2\frac{d}{dt}_{t=t_0}\lambda_2(t)\frac{d}{dt}_{t=t_0}\phi^*_{2,t_0}(t)\\
&\quad - \frac{d^2}{dt^2}_{t=t_0}\lambda_2(t)\phi_2(t_0)\\
&\quad - \big(\Delta_e + \lambda_2(t_0)\big)\frac{d^2}{dt^2}_{t=t_0}\phi^*_{2,t_0}(t). \qquad (19)
\end{aligned}
$$

Proposition 2.3 and identity (19) show a unique existences of $\frac{d^2}{dt^2}_{t=t_0}\lambda_2(t)$ and a unique existence of the component $\big(\frac{d^2}{dt^2}_{t=t_0}\phi^*_{2,t_0}(t)\big)^\circ$ of $\frac{d^2}{dt^2}_{t=t_0}\phi^*_{2,t_0}(t)$ which is orthogonal in metric e to $\langle\phi_2(t_0)\rangle$. Therefore, particularly, $\frac{d}{dt}\lambda_2(t)$ and $\big(\frac{d}{dt}\phi^*_{2,t_0}(t)\big)^\circ$ are continuous at $t = t_0$. Since $\frac{d}{dt}\beta_{t_0,2,2}(t)$ is continuous at $t = t_0$, the continuity of $\frac{d}{dt}\phi^*_{2,t_0}(t)$ at $t = t_0$ is shown. □

Proposition 2.5. *Let $\lambda_2(t)$ be double at $t = 0$.*
(i) Then, regardless of the existence of $\lim_{t\to 0}\phi^*_{i,0}(t)$, and regardless of multiplicity of $\lambda_2(t)$, $t > 0$, $i = 2,3$, we have $\lim_{t\to 0}\lambda_i(t) = \lambda_2(0)$, and the function $\big(\phi^*_{i,0}(t) - \phi_i(0)\big)^\circ$ which stands for the component of $\phi^*_{i,0}(t) - \phi_i(0)$ orthogonal to $\langle\phi_2(0), \phi_3(0)\rangle$ also converges (in L^2-norm) to zero-function as $t \to 0$.
(ii) Regardless of multiplicity of $\lambda_2(t)$, $t > 0$, if $\lim_{t\to 0}\phi^*_{2,0}(t)$ exists, then $\lambda_2(t)$ and $\big(\phi^*_{2,0}(t)\big)^\circ$, the component of $\phi^*_{2,0}(t)$ which is orthogonal to the second eigenspace of Ω, are continuously differentiable at $t = 0$.

Proof. (i) Let us assume there is a sequence $\{t_n \in (0,\varepsilon)\}$ such that t_n converges to 0, and assume $\int_\Omega \psi_{2,n}(0)\cdot\phi^*_{2,0}(t_n)$ converges to zero for any second eigenfunction $\psi_{2,n}(0) \in$

$\langle\phi_2(0),\phi_3(0)\rangle$ corresponding to each t_n as $n\to\infty$. Since $\phi^*_{1,0}(t_n)$ converges to $\phi_1(0)$, $\int_\Omega \phi_1(0)\cdot \phi^*_{2,0}(t_n)$ converges to zero as $t_n\to 0$, and then we have $\lambda_2(t_n)\to\lambda_2(0)+\epsilon_1$ for a number $\epsilon_1>0$. But minimal energy property of eigenfunction implies

$$\Big\{\int_{J_{t_n}(\Omega)}\nabla\Big(J^{-1*}_{t_n}\phi_2(0)-\delta_{2,1}(0,t_n)\phi_1(t_n)\Big)\cdot\nabla\Big(J^{-1*}_{t_n}\phi_2(0)-\delta_{2,1}(0,t_n)\phi_1(t_n)\Big)\Big\}\cdot\Big\{\int_{J_{t_n}(\Omega)}\big|J^{-1*}_{t_n}\phi_2(0)-\delta_{2,1}(0,t_n)\phi_1(t_n)\big|^2\Big\}^{-1}\geq\lambda_2(t_n),$$

where $\delta_{2,1}(0,t_n)=\int_{J_{t_n}(\Omega)}J^{-1*}_{t_n}\phi_2(0)\cdot\phi_1(t_n)$. Since $\phi^*_{1,0}(t_n)$ converges to $\phi_1(0)$, $\delta_{2,1}(0,t_n)$ converges to zero as $t_n\to 0$. Observe the left hand of this inequality converges to $\lambda_2(0)$ as $t_n\to 0$. It implies $\lambda_2(0)\geq\lambda_2(t_n)+\epsilon_2$ for sufficiently large all n and for an $\epsilon_2>0$. It is impossible under the consequence such that $\lambda_2(t_n)\to\lambda_2(0)+\epsilon_1$. Consequently, $\int_\Omega\tilde\phi_{2,m}(0)\cdot\phi^*_{2,0}(t_m)$ does not converge to zero for any sequence $\{t_m\in(0,\varepsilon)\}$ converging to zero, and for a second eigenfunction $\tilde\phi_{2,m}(0)\in\langle\phi_2(0),\phi_3(0)\rangle$ corresponding to each given t_m. Then, from formula (5) $\lim_{t\to 0}\lambda_2(t)=\lambda_2(0)$, and $\big(\phi^*_{2,0}(t)-\phi_2(0)\big)^\circ$ converges to zero.

In the same way let us assume there is a sequence $\{t_n\in(0,\varepsilon)\}$ such that t_n converges to 0, and assume $\int_\Omega\psi_{2,n}(0)\cdot\phi^*_{3,0}(t_n)$ converges to zero for any the second eigenfunctin $\psi_{2,n}(0)\in\langle\phi_2(0),\phi_3(0)\rangle$. Then, since $\int_\Omega\phi^*_{3,0}(t_n)\cdot\phi_1(0)$ also converges to zero as $t_n\to 0$, we have $\lambda_3(t_n)\to\lambda_2(0)+\epsilon_3$ for a number $\epsilon_3>0$ as $t_n\to 0$. But

$$\Big\{\int_{J_{t_n}(\Omega)}\nabla\Big(J^{-1*}_{t_n}\phi_3(0)-\delta_{3,1}(0,t_n)\phi_1(t_n)-\delta_{3,2}(0,t_n)\phi_2(t_n)\Big)\cdot\nabla\Big(J^{-1*}_{t_n}\phi_3(0)-\delta_{3,1}(0,t_n)\phi_1(t_n)-\delta_{3,2}(0,t_n)\phi_2(t_n)\Big)\Big\}\cdot\Big\{\int_{J_{t_n}(\Omega)}\Big|J^{-1*}_{t_n}\phi_3(0)-\delta_{3,1}(0,t_n)\phi_1(t_n)-\delta_{3,2}(0,t_n)\phi_2(t_n)\Big|^2\Big\}^{-1}\geq\lambda_3(t_n),\qquad(20)$$

where $\delta_{3,i}(0,t_n):=\int_{J_{t_n}(\Omega)}J^{-1*}_{t_n}\phi_3(0)\cdot\phi_i(t_n)$, $i=1,2$. Since $\phi^*_{1,0}(t_n)\to\phi_1(0)$, we have $\delta_{3,1}(0,t_n)\to 0$, and by the assumption $\int_\Omega\psi_{2,n}(0)\cdot\phi^*_{3,0}(t_n)\to 0$ for any second eigenfunction $\psi_{2,n}(0)$ in Ω, we have $\delta_{3,2}(0,t_n)\to 0$ as $n\to\infty$. It implies the left hand of (20) converges to $\lambda_2(0)$, and therefore $\lambda_2(0)\geq\lambda_3(t_n)+\epsilon_4$ for suffciently large all n and for an $\epsilon_4>0$. It contradicts to the consequence $\lambda_3(t_n)\to\lambda_2(0)+\epsilon_3$. Consequently, from (5) $\lim_{t\to 0}\lambda_3(t)=\lambda_3(0)$, and $\big(\phi^*_{3,0}(t)-\phi_3(0)\big)^\circ$ converges to zero as $t\to 0$.

(ii) When we replace t_0 by 0, the formula (19) as well as (13) holds in spite of double multiplicity of $\lambda_2(0)$. For this, consider the argument in the proofs of Proposition 2.3 and Proposition 2.4. □

Remark 2.4. Let $\lambda_2(t)$ be double at $t=0$, and simple on $t\in(0,\varepsilon)$. Let $\{\phi_2(0),\phi_3(0)\}$ be an orthonormal basis of the second eigenspace in Ω, and let $\phi^*_{2,0}(t)$ be expanded with respect to $\phi_2(0)$. Let us assume that $\phi^*_{2,0}(t)$ converges to $\phi_2(0)$ as $t\to 0$. Then, from (13) we have

$$\left\{\begin{array}{l}\Big(\frac{d}{dt}_{t=0}\phi^*_{2,0}(t)\Big)^\circ(x,y)=-\int_\Omega\frac{d}{dt}_{t=0}\Delta_{J^*_te}\phi_2(0)(\zeta,\tau)\cdot K_{\lambda_2(0)}(x,y;\zeta,\tau)d\zeta d\tau,\\ \frac{d}{dt}_{t=0}\lambda_2(t)=-\int_\Omega\frac{d}{dt}_{t=0}\Delta_{J^*_te}\phi_2(0)\cdot\phi_2(0),\\ \frac{d}{dt}_{t=0}\beta_{0,2,k}(t)=\frac{-1}{-\lambda_k(0)+\lambda_2(0)}\int_\Omega\frac{d}{dt}_{t=0}\Delta_{J^*_te}\phi_2(0)\cdot\phi_k(0),\ k\in\mathbb{N}\setminus\{2,3\},\end{array}\right.\qquad(21)$$

where $K_{\lambda_2(0)}$ is Green's function of $\Delta_e+\lambda_2(0)$ in Ω, and $\big(\frac{d}{dt}_{t=0}\phi^*_{2,0}(t)\big)^\circ$ denotes the component of $\frac{d}{dt}_{t=0}\phi^*_{2,0}(t)$ orthogonal to $\langle\phi_2(0),\phi_3(0)\rangle$. Thus, $\lambda_2(t)$ and $\big(\phi^*_{2,0}(t)\big)^\circ$ are continuously differentiable on $t\in[0,\varepsilon)$. Even though we assume that $\phi^*_{i,0}(t)$, $i=2,3$, converges to $\phi_i(0)$, respectively, we can not assure existence of $\frac{d}{dt}_{t=0}\phi^*_{i,0}(t)$, since existences of $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\beta_{0,i,j}(t)$, $i,j=2,3$, are not yet verified.

Now let us suppose that we do not know whether $\phi^*_{2,0}(t)$ converges to $\phi_2(0)$ or not. From (13) we represent at $t_0 \in (0,\varepsilon)$, for $i = 2,3$, $k \in \mathbb{N}$,

$$\frac{d}{dt}_{t=t_0}\beta_{t_0,i,k}(t) = \frac{-1}{-\lambda_k(t_0)+\lambda_i(t_0)}\int_{J_{t_0}(\Omega)}\Big\{\frac{d}{dt}_{t=t_0}\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e}\phi_i(t_0)\Big\}\cdot\phi_k(t_0)dxdy.$$

Since $\frac{-1}{-\lambda_3(t_0)+\lambda_2(t_0)}$ diverges as $t_0 \to 0$, one can not exclude the possibility of either divergence or oscillation of $\frac{d}{dt}_{t=t_0}\beta_{t_0,2,3}(t)$ as $t_0 \to 0$. Thus, $\langle\phi_2(t_0),\phi_3(t_0)\rangle$ component of $\frac{d}{dt}_{t=t_0}\phi^*_{i,t_0}(t)$ might oscillate as $t_0 \to 0$ like the derivative of function $t^2\sin\frac{1}{t}$. But in the case of $k \in \mathbb{N}\setminus\{2,3\}$, Proposition 2.1 implies $\frac{d}{dt}_{t=t_0}\beta_{t_0,i,k}(t)$ is bounded on $0 < t_0 < \varepsilon$. //

Proposition 2.6. *Let J_t be a smooth deformation of Ω such that $\lambda_2(0) = \lambda_3(0)$ is double second eigenvalue, and $\lambda_2(t)$ and $\lambda_3(t)$ are simple on $t \in (0,\varepsilon)$. Then, for $i = 2,3$, $\frac{d}{dt}_{t=t_0}\lambda_i(t)$ and $\big(\frac{d}{dt}_{t=t_0}\phi^*_{i,t_0}(t)\big)^\circ$ which denotes the component of $\frac{d}{dt}_{t=t_0}\phi^*_{i,t_0}(t)$ orthogonal to $\langle\phi_2(t_0),\phi_3(t_0)\rangle = \langle\phi_2(t_0)\rangle\oplus\langle\phi_3(t_0)\rangle$ under the metric e are bounded on $0 \lneq t_0 < \varepsilon$.*

Proof. From Proposition 2.5 $\lambda_i(t_0)$ is bounded on $0 \lneq t_0 < \varepsilon$. Considering Proposition 2.1, regardless of existence of $\lim_{t_0\to 0}\phi^*_{i,0}(t_0)$ (even though $\phi^*_{i,0}(t_0)$ oscillates as $t_0 \to 0$), one can infer that in the identity (13)

$$\frac{d}{dt}_{t=t_0}\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e}\phi_i(t_0) = -\frac{d}{dt}_{t=t_0}\lambda_2(t)\phi_i(t_0) - \big(\Delta_e + \lambda_i(t_0)\big)\frac{d}{dt}_{t=t_0}\phi^*_{i,t_0}(t),$$

$\frac{d}{dt}_{t=t_0}\Delta_{\left(J_t\circ J_{t_0}^{-1}\right)^* e}\phi_i(t_0)$ is bounded on $0 \lneq t_0 < \varepsilon$. Then, since $\mathrm{Im}\big(\Delta_e+\lambda_i(t_0)\big)$ is orthogonal to $\langle\phi_i(t_0)\rangle$ under metric e, two summands $\frac{d}{dt}_{t=t_0}\lambda_i(t)\phi_i(t_0)$ and $\big(\Delta_e+\lambda_i(t_0)\big)\frac{d}{dt}_{t=t_0}\phi^*_{i,t_0}(t)$ which are orthogonal to each other are bounded on $0 \lneq t_0 < \varepsilon$. Consequently, from Remark 2.1 $\big(\frac{d}{dt}_{t=t_0}\phi^*_{i,t_0}(t)\big)^\circ$ which denotes the component of $\frac{d}{dt}_{t=t_0}\phi^*_{i,t_0}(t)$ orthogonal to $\langle\phi_2(t_0),\phi_3(t_0)\rangle$ under metric e is also bounded on $0 \lneq t_0 < \varepsilon$. □

Remark 2.5. Under hypotheses of Proposition 2.6 in order to show the existence of $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\phi^*_{2,0}(t)$ we are to verify that $\frac{d}{dt}_{t=t_0}\beta_{t_0,2,j}(t)$, $j = 2,3$, $0 \lneq t_0 < \varepsilon$, converge as $t_0 \to 0$. In this case $\frac{d}{dt}_{t=t_0}\phi^*_{2,t_0}(t)$ exists from Proposition 2.4 and is represented by

$$\frac{d}{dt}_{t=t_0}\Big(\phi_2(t_0) + \sum_{k\in\mathbb{N}}\beta_{t_0,2,k}(t)\phi_k(t_0)\Big).$$

If we expand $J^*_{t_0}\phi^*_{2,t_0}(t) = \phi^*_{2,0}(t)$, $0 \lneq t_0$, with respect to $\phi_2(0)$, then from Proposition 2.3 the following identity holds;

$$\frac{d}{dt}_{t=t_0}\Delta_{J^*_t e}\phi^*_{2,0}(t_0) = -\frac{d}{dt}_{t=t_0}\lambda_2(t)\phi^*_{2,0}(t_0) - \big(\Delta_{J^*_{t_0}e} + \lambda_2(t_0)\big)\frac{d}{dt}_{t=t_0}\phi^*_{2,0}(t).$$

In this case since $\Delta_{J^*_{t_0}e}\phi_k(0) \neq \lambda_k(0)\phi_k(0)$, the coefficient of $\phi_k(0)$-component of $\big(\Delta_{J^*_{t_0}e} + \lambda_2(t_0)\big)\frac{d}{dt}_{t=t_0}\phi^*_{2,0}(t)$ may differ from $\big(-\lambda_k(t_0)+\lambda_2(k_0)\big)\frac{d}{dt}_{t=t_0}\beta_{0,2,k}(t)\phi_k(0)$. Thus, the coefficient $\frac{d}{dt}_{t=t_0}\beta_{0,i,k}(t)$ can not be represented by the same formula as $\frac{d}{dt}_{t=t_0}\beta_{t_0,i,k}(t)$ expressed in Remark 2.4.

If we just show $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\beta_{t_0,i,j}(t)$, $i,j = 2,3$, exist, then $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\beta_{0,i,j}(t)$ also exists, and vice versa. For this, note that since $J^*_{t_0}\circ(J_t\circ J_{t_0}^{-1})^* = J^*_t$, we have

$$J^*_{t_0}\Big(\frac{d}{dt}_{t=t_0}\phi^*_{i,t_0}(t)\Big) = \frac{d}{dt}_{t=t_0}\phi^*_{i,0}(t).$$

We may define $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\beta_{0,i,j}(t)$ by $\frac{d}{dt}_{t=0}\beta_{0,i,j}(t)$. Then, $\lim_{t_0\to 0}\phi^*_{i,0}(t_0)$ exists from Proposition 2.5, and

$$\lim_{t_0\to 0}\phi^*_{i,0}(t_0) := \tilde{\phi}_i(0) = \big(\phi^*_{i,0}(t_0)\big)^\circ + \sum_{j=2}^{3}\beta_{0,i,j}(t_1)\phi_j(0) + \sum_{j=2}^{3}\int_{t_1}^{0}\frac{d}{dt}\beta_{0,i,j}(t)\phi_j(0)dt.$$

Consequently, according to Remark 2.4, it is confirmed that $\frac{d}{dt}_{t=t_0}\phi^*_{i,t_0}(t)$ converges as $t_0 \to 0$. //

Before proving existences of $\lim_{t_0\to 0}\phi^*_{i,0}(t_0)$ and $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\phi^*_{i,0}(t)$ we are to describe concretely a deformation of Ω.

Definition 2.1. Let $J^{-1}:\mathbb{R}^2\times[0,1]\to\mathbb{R}^2$ be a C^∞ deformation defined by

$$\begin{aligned} &J_t^{-1}(\xi,\eta)=(x,y)\\ =&\big(J_x^{-1}(\xi,\eta,t),\ \ J_y^{-1}(\xi,\eta,t)\big)\\ =&\big(\xi-G_t(\xi,\eta),\ \ \eta-H_t(\xi,\eta)\big),\qquad 0\le t\le 1, \end{aligned} \tag{22}$$

where $J_t^{-1}:\mathbb{R}^2\to\mathbb{R}^2$, $J_t^{-1}(\xi,\eta):=J^{-1}(\xi,\eta,t)$, is a diffeomorphism for each $t\in[0,1]$. Thus, $G:\mathbb{R}^2\times[0,1]\to\mathbb{R}$ and $H:\mathbb{R}^2\times[0,1]\to\mathbb{R}$, $G_t(\xi,\eta):=G(\xi,\eta,t)$, $H_t(\xi,\eta):=H(\xi,\eta,t)$, are C^∞-function such that Jacobian determinant of J_t^{-1},

$$\begin{vmatrix} 1-\frac{\partial G_t}{\partial\xi} & -\frac{\partial H_t}{\partial\xi}\\ -\frac{\partial G_t}{\partial\eta} & 1-\frac{\partial H_t}{\partial\eta}\end{vmatrix},$$

must not vanish at each $t\in[0,1]$. Also, Then, by inverse mapping theorem the inverse map $J_t:\mathbb{R}^2\to\mathbb{R}^2$ exists and is of $C^\infty(\mathbb{R}^2)$ at each $t\in[0,1]$. Since J_t^{-1} is of C^∞ for t-variable, J_t is also of $C^\infty([0,1])$. Thus by definition the following constraint equation holds;

$$J_t\big(J_t^{-1}(\xi,\eta)\big)=(\xi,\eta),\qquad J_t^{-1}\big(J_t(x,y)\big)=(x,y).$$

We call J^{-1} the *inverse deformation* of J. Let us call the restriction of J_t to Ω a *simple deformation* of Ω. The smoothness of $\partial J_t(\Omega)$ is also assumed. Let us define the *support* of C^∞ deformation $J:\mathbb{R}^2\times[0,1]\to\mathbb{R}^2$ by the closure of set

$$\Big\{p\in\mathbb{R}^2 : J(p,t)\neq p,\ \text{for all } t,\ 0\lneq t\le 1\Big\}.$$

Let us call the intersection of support of J with $\overline{\Omega}$ the *support* of a simple deformation J of Ω, and denote the support by $\mathrm{supp}(J)$. Equivalently, it holds that $\mathbb{R}^2\setminus\big(\mathrm{supp}(J)\big)^\circ=\big\{p\in\Omega : J(p,t)=p,\ \text{for a } t, 0\lneq t\le 1\big\}$. One can show that $\mathrm{supp}(J^{-1})=\mathrm{supp}(J)$. For this, if and only if $J(p,t_1)=p$ for a t_1, from the equality $J^{-1}\big(J(p,t_1),t_1\big)=p=J^{-1}(p,t_1)$, we can conclude $p\notin\mathrm{supp}(J^{-1})$. Let us call *boundary support* of $J_{|\Omega}$ the set $\mathrm{supp}(J)\cap\partial\Omega$. //

Let us set $J(x,y,t)=(\xi,\eta)=\big(J_\xi(x,y,t),J_\eta(x,y,t)\big)$. Then the following equations hold;

$$\begin{aligned} 0=\frac{d}{dt}_{t=0}x=&\frac{d}{dt}_{t=0}J_x^{-1}\big(J(x,y,t),t\big)\\ =&\Big[\frac{\partial J_x^{-1}}{\partial\xi}\frac{\partial\xi}{\partial t}+\frac{\partial J_x^{-1}}{\partial\eta}\frac{\partial\eta}{\partial t}+\frac{\partial J_x^{-1}}{\partial t}\Big]_{t=0}\\ =&\Big[\frac{\partial J_x^{-1}}{\partial\xi}\frac{\partial J_\xi}{\partial t}+\frac{\partial J_x^{-1}}{\partial\eta}\frac{\partial J_\eta}{\partial t}+\frac{\partial J_x^{-1}}{\partial t}\Big]_{t=0}\\ =&\frac{\partial J_\xi}{\partial t}_{t=0}+\frac{\partial J_x^{-1}}{\partial t}_{t=0}. \end{aligned}$$

For this, note that $\frac{\partial J_x^{-1}}{\partial\xi}_{t=0}=1$, and $\frac{\partial J_x^{-1}}{\partial\eta}_{t=0}=0$. About variable η the same result occurs, and then we have

$$\frac{\partial J}{\partial t}_{t=0}=-\frac{\partial J^{-1}}{\partial t}_{t=0}.$$

Let us consider the co-ordinates change $(\xi,\eta)\mapsto(x,y)$. The metric $J_t^*e(x,y)=\big(g_{ij}\big)(x,y)$ is given by

$$\begin{cases} g_{11}=\frac{\partial\xi}{\partial x}\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial x},\\ g_{12}=g_{21}=0,\\ g_{22}=\frac{\partial\xi}{\partial y}\frac{\partial\xi}{\partial y}+\frac{\partial\eta}{\partial y}\frac{\partial\eta}{\partial y}. \end{cases} \tag{23}$$

From the fact $\Delta_g = \sum_{i,j} \frac{1}{\sqrt{|g|}} D_i(\sqrt{|g|} g^{ij} D_j)$, $i,j = 1,2$, $g = (g_{ij})$, $|g| = \det(g_{ij})$ one can calculate $\frac{d}{dt}_{t=t_0}\{\Delta_{J_t^* e} J_{t_0}^* \psi(t_0)\}(x,y)$ for any $\psi(t) \in C^2\big(J_t(\Omega)\big)$. One has $\Delta_{J_t^* e}\big(J_{t_0}^* \psi(t_0)\big) = J_t^* \Delta_e\big(J_t^{-1*} J_{t_0}^* \psi(t_0)\big)$. From this equality one can obtain the following;

$$\frac{d}{dt}_{t=t_0}\Big\{\Delta_{J_t^* e} J_{t_0}^* \psi(t_0)\Big\} = \frac{d}{dt}_{t=t_0}\Big[\Big\{\Delta_e\big(J_{t_0}^* \psi(t_0) \circ J_t{}^{-1}\big)\Big\} \circ J_t\Big].$$

Calculations using the chain rule on this equality also yields the following formula;

$$\begin{aligned}
&\frac{d}{dt}_{t=t_0}\Big\{\Delta_{J_t^* e} J_{t_0}^* \psi(t_0)\Big\}(x,y) \\
=&\frac{d}{dt}_{t=t_0}\Big[\Big\{\Big(\frac{\partial J_x^{-1}{}_t}{\partial \xi}\Big)^2 + \Big(\frac{\partial J_x^{-1}{}_t}{\partial \eta}\Big)^2\Big\} \circ J_t \cdot \frac{\partial^2 J_{t_0}^* \psi(t_0)}{\partial x^2} \\
&\quad + \Big\{\Big(\frac{\partial J_y^{-1}{}_t}{\partial \xi}\Big)^2 + \Big(\frac{\partial J_y^{-1}{}_t}{\partial \eta}\Big)^2\Big\} \circ J_t \cdot \frac{\partial^2 J_{t_0}^* \psi(t_0)}{\partial y^2} \\
&\quad + 2\Big\{\Big(\frac{\partial J_x^{-1}{}_t}{\partial \xi}\Big) \cdot \Big(\frac{\partial J_y^{-1}{}_t}{\partial \xi}\Big) + \Big(\frac{\partial J_x^{-1}{}_t}{\partial \eta}\Big) \cdot \Big(\frac{\partial J_y^{-1}{}_t}{\partial \eta}\Big)\Big\} \circ J_t \cdot \frac{\partial^2 J_{t_0}^* \psi(t_0)}{\partial x \partial y} \\
&\quad + \Big\{\Big(\frac{\partial^2 J_x^{-1}{}_t}{\partial \xi^2}\Big) + \Big(\frac{\partial^2 J_x^{-1}{}_t}{\partial \eta^2}\Big)\Big\} \circ J_t \cdot \frac{\partial J_{t_0}^* \psi(t_0)}{\partial x} \\
&\quad + \Big\{\Big(\frac{\partial^2 J_y^{-1}{}_t}{\partial \xi^2}\Big) + \Big(\frac{\partial^2 J_y^{-1}{}_t}{\partial \eta^2}\Big)\Big\} \circ J_t \cdot \frac{\partial J_{t_0}^* \psi(t_0)}{\partial y}\ \Big](x,y).
\end{aligned} \tag{24}$$

Let us denote $\dot{G}_{t_1} := \frac{d}{dt}_{t=t_1} G_t$. From now on, particularly at $t = 0$, without inconsistency we will identify $\Big[\Big\{\frac{\partial^{k_1+k_2}}{\partial \xi^{k_1} \partial \eta^{k_2}} \dot{G}_t\Big\} \circ J_t(x,y)\Big]_{t=0}$ with $\frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} \dot{G}_0(x,y)$. //

Proposition 2.7. *Let a simple deformation J_t of Ω be given by (22), and $\psi(0)$ be any $C^2(\Omega)$-function. Then, we have*

$$\begin{aligned}
&\frac{d}{dt}_{t=0}\Big\{\Delta_{J_t^* e} \psi(0)(x,y)\Big\} \\
=&-2\frac{\partial \dot{G}_0}{\partial x} \cdot \frac{\partial^2 \psi(0)}{\partial x^2} - 2\frac{\partial \dot{G}_0}{\partial y} \cdot \frac{\partial^2 \psi(0)}{\partial x \partial y} - \Big\{\frac{\partial^2 \dot{G}_0}{\partial x^2} + \frac{\partial^2 \dot{G}_0}{\partial y^2}\Big\} \cdot \frac{\partial \psi(0)}{\partial x} \\
&-2\frac{\partial \dot{H}_0}{\partial y} \cdot \frac{\partial^2 \psi(0)}{\partial y^2} - 2\frac{\partial \dot{H}_0}{\partial x} \cdot \frac{\partial^2 \psi(0)}{\partial x \partial y} - \Big\{\frac{\partial^2 \dot{H}_0}{\partial x^2} + \frac{\partial^2 \dot{H}_0}{\partial y^2}\Big\} \cdot \frac{\partial \psi(0)}{\partial y}.
\end{aligned} \tag{25}$$

where $\dot{G}_0 := \frac{d}{dt}_{t=0} G_t$.

Proof. Direct calculation using (24) yields the equality. Note that for the first summand in (24)

$$\begin{aligned}
&\frac{d}{dt}_{t=0}\Big\{\Big(\frac{\partial J_x^{-1}{}_t}{\partial \xi}\Big)^2 \circ J_t\Big\} = \frac{d}{dt}_{t=0}\Big\{\frac{\partial}{\partial \xi}\Big(\xi - G_t(\xi,\eta)\Big)\Big\}^2 \\
&= \frac{d}{dt}_{t=0}\Big\{1 - \frac{\partial G_t}{\partial \xi}\big(J_\xi(x,y,t), J_\eta(x,y,t)\big)\Big\}^2 \\
&= -2\frac{d}{dt}_{t=0}\Big\{\frac{\partial G_t}{\partial \xi}\big(J_\xi(x,y,t), J_\eta(x,y,t)\big)\Big\} + \frac{d}{dt}_{t=0}\Big\{\frac{\partial G_t}{\partial \xi}\big(J_\xi(x,y,t), J_\eta(x,y,t)\big)\Big\}^2 \\
&= -2\Big[\Big(\frac{d}{dt}\frac{\partial G_t}{\partial \xi}\Big)(\xi,\eta) + \frac{\partial^2 G_t}{\partial \xi^2}\frac{dJ_\xi}{dt} + \frac{\partial^2 G_t}{\partial \xi \partial \eta}\frac{dJ_\eta}{dt}\Big]_{t=0} \\
&\qquad + 2\Big\{\frac{\partial G_t}{\partial \xi}\big(J_\xi(x,y,t), J_\eta(x,y,t)\big)\Big\}_{t=0}\frac{d}{dt}_{t=0}\Big\{\frac{\partial G_t}{\partial \xi}\big(J_\xi(x,y,t), J_\eta(x,y,t)\big)\Big\}.
\end{aligned}$$

Since $\left[\frac{\partial G_t}{\partial \xi}\right]_{t=0} = \left[\frac{\partial^2 G_t}{\partial \xi^2}\right]_{t=0} = \left[\frac{\partial^2 G_t}{\partial \xi \partial \eta}\right]_{t=0} \equiv 0$, we have

$$\frac{d}{dt}_{t=0}\Big\{\Big(\frac{\partial {J_x^{-1}}_t}{\partial \xi}\Big)^2 \circ J_t\Big\} = -2\Big[\Big(\frac{d}{dt}\frac{\partial G_t}{\partial \xi}\Big)\big(J_\xi(x,y,t), J_\eta(x,y,t)\big)\Big]_{t=0}$$
$$= -2\frac{\partial \dot{G}_0}{\partial \xi} = \Big(-2\frac{\partial \dot{G}_0}{\partial x}\Big).$$

By similar calculations, for the other summands it is also easily shown. □

Proposition 2.8. *Let a simple deformation J_t be given by (22), and $\{\phi_2, \phi_3\}$ be a basis of the second eigenspace of Ω. Then we have*

$$\int_\Omega \Big\{\frac{d}{dt}_{t=0}\Delta_{J_t^* e}\phi_i(x,y)\Big\}\phi_j(x,y)dxdy$$
$$= \int_{\partial\Omega} \dot{G}_0(x,y)\frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial \nu}dA$$
$$+ \int_{\partial\Omega} \dot{H}_0(x,y)\frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial \nu}dA, \qquad i,j \in \{2,3\}. \tag{26}$$

Proof. Using property $\big(\Delta_e + \lambda_2(0)\big)\phi_i = 0 = \big(\Delta_e + \lambda_2(0)\big)\phi_j$ as a solution of eigenvalue problem, from Proposition 2.7 one can write the component of $\int_\Omega \{\frac{d}{dt}_{t=0}\Delta_{J_t^* e}\phi_i\}\phi_j$ induced by y-directional deformation $(\xi, \eta + H_t)$ as follows;

$$\int_\Omega -\Big(2\frac{\partial \dot{H}_0}{\partial y}\cdot\frac{\partial^2 \phi_i}{\partial y^2}\phi_j + 2\frac{\partial \dot{H}_0}{\partial x}\cdot\frac{\partial^2 \phi_i}{\partial x \partial y}\phi_j + \Big\{\frac{\partial^2 \dot{H}_0}{\partial x^2} + \frac{\partial^2 \dot{H}_0}{\partial y^2}\Big\}\cdot\frac{\partial \phi_i}{\partial y}\phi_j\Big)$$
$$= -\int_\Omega \frac{\partial}{\partial y}\Big(\dot{H}_0\frac{\partial^2 \phi_i}{\partial y^2}\phi_j + \frac{\partial \dot{H}_0}{\partial y}\frac{\partial \phi_i}{\partial y}\phi_j - \dot{H}_0\frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y}\Big)dxdy$$
$$-\int_\Omega \frac{\partial}{\partial x}\Big(\dot{H}_0\frac{\partial^2 \phi_i}{\partial x \partial y}\phi_j + \frac{\partial \dot{H}_0}{\partial x}\frac{\partial \phi_i}{\partial y}\phi_j - \dot{H}_0\frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial x}\Big)dxdy. \tag{27}$$

For each y such that $\overline{\Omega} \cap \{y \times \mathbb{R}\} \neq \emptyset$, denote the union of components of $\overline{\Omega} \cap \{y \times \mathbb{R}\}$ by $\bigcup_k [r_k(y), r_{k+1}(y)]$, $k = 1, 2, ...,$ $r_k(y) \le r_{k+1}(y)$, and for each x such that $\overline{\Omega} \cap \{\mathbb{R} \times x\} \neq \emptyset$, denote the union of components of $\overline{\Omega} \cap \{\mathbb{R} \times x\}$ by $\bigcup_l [s_l(x), s_{l+1}(x)]$, $l = 1, 2, ...,$ $s_l(x) \le s_{l+1}(x)$. Since $\phi_j = 0$ on $\partial\Omega$, (27) attains to

$$\int_{\partial\Omega} \sum_l \Big[\dot{H}_0\frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y}\Big]_{s_l(x)}^{s_{l+1}(x)}dx + \int_{\partial\Omega} \sum_k \Big[\dot{H}_0\frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial x}\Big]_{r_k(y)}^{r_{k+1}(y)}dy$$
$$= \int_{\partial\Omega} \dot{H}_0\frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial \nu}dA. \tag{28}$$

One exchanging the coordinate y for x, (28) for $\dot{G}_0$ also holds by the same arguments. □

Remark 2.6. Although $\lambda_2(0)$ is double, if $\phi_{0,2}^*(t)$ converges to $\phi_2(0)$, then from (13) one can represent as follows;

$$\frac{d}{dt}_{t=0}\lambda_2(t) = -\int \Big\{\frac{d}{dt}_{t=0}\Delta_{J_t^* e}\phi_2(0)\Big\}\phi_2(0)d_e,$$
$$\frac{\partial}{\partial \nu}_P\Big(\frac{d}{dt}_{t=0}\phi_2^*(t)\Big)^\circ = -\int \frac{\partial}{\partial \nu}_{(\zeta,\tau)=P} K_{\lambda_2(0)}(x,y;\zeta,\tau)\Big\{\frac{d}{dt}_{t=0}\Delta_{J_t^* e}\phi_2(0)\Big\}dxdy,$$

where $K_{\lambda_2(0)}$ stands for modified Green's function of $\Delta_e + \lambda_2(0)$ in $J_0(\Omega)$ which will be mentioned later. Let us review the equality (27) replacing ϕ_j by $\frac{\partial}{\partial \nu}_{(\zeta,\tau)=P}K_{\lambda_2(0)}$. Considering induction of equality (27), we can infer that the component of $\int_\Omega \left\{ \frac{d}{dt}_{t=0} \Delta_{J_t^* e} \phi_i \right\} \frac{\partial}{\partial \nu}_P K_{\lambda_2(0)}$ induced by y-directional deformation $(\xi, \eta) \mapsto (\xi, \eta - H_t)$ is read as

$$
\begin{aligned}
&- \int_\Omega \frac{\partial}{\partial y} \Big(\dot{H}_0 \frac{\partial^2 \phi_i}{\partial y^2} \phi_j + \frac{\partial \dot{H}_0}{\partial y} \frac{\partial \phi_i}{\partial y} \phi_j - \dot{H}_0 \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \Big) dxdy \\
&- \int_\Omega \frac{\partial}{\partial x} \Big(\dot{H}_0 \frac{\partial^2 \phi_i}{\partial x \partial y} \phi_j + \frac{\partial \dot{H}_0}{\partial x} \frac{\partial \phi_i}{\partial y} \phi_j - \dot{H}_0 \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial x} \Big) dxdy \\
&- \int_\Omega \frac{\partial}{\partial \nu}_{(\zeta,\tau)=P} \Big(\dot{H}_0 \frac{\partial \phi_i}{\partial y} \Big) \delta_{(\zeta,\tau)}(x,y) dxdy, \qquad (29)
\end{aligned}
$$

where $\int_\Omega \frac{\partial}{\partial \nu}_P \Big(\dot{H}_0 \frac{\partial \phi_i}{\partial y} \Big) \delta = \frac{\partial}{\partial \nu}_{(\zeta,\tau)=P} \Big(\dot{H}_0(\zeta,\tau) \frac{\partial \phi_i(\zeta,\tau)}{\partial \tau} \Big)$. The last term results from the definition $\big(\Delta_e + \lambda_i(0)\big) K_{\lambda_i(0)} = \delta$ as follows;

$$
\begin{aligned}
&- \int \dot{H}_0 \frac{\partial^3 \phi_i}{\partial y^3} \phi_j - \int \dot{H}_0 \frac{\partial^3 \phi_i}{\partial x^2 \partial y} \phi_j + \Big(\int \dot{H}_0 \frac{\partial \phi_i}{\partial y} \frac{\partial^2 \phi_j}{\partial y^2} + \int \dot{H}_0 \frac{\partial \phi_i}{\partial y} \frac{\partial^2 \phi_j}{\partial x^2} \Big) \\
&= \int \dot{H}_0 \lambda_i(0) \frac{\partial \phi_i}{\partial y} \phi_j - \int \dot{H}_0 \lambda_i(0) \frac{\partial \phi_i}{\partial y} \phi_j + \int \frac{\partial}{\partial \nu}_P \Big(\dot{H}_0 \frac{\partial \phi_i}{\partial y} \delta \Big). \qquad (30)
\end{aligned}
$$

Since ϕ_i is $O(|P - (x,y)|^3)$ from hypothesis, one has $\frac{\partial}{\partial \nu}_P \Big(\dot{H}_0 \frac{\partial \phi_i}{\partial \tau} \Big) = 0$. Therefore, since $\frac{\partial}{\partial \nu}_{(\zeta,\tau)=P} K_{\lambda_2(0)}$ vanishes on $\partial\Omega \setminus P$ for the measure zero point set $\{P\}$ on the boundary $\partial\Omega$, (26) also holds when $\frac{\partial}{\partial \nu}_{(\zeta,\tau)=P} K_{\lambda_2(0)}$ substitutes for ϕ_j. //

Proposition 2.9. *For any distinct eigenfunctions ϕ_{k_1} and ϕ_{k_2} in the second eigenspace $\langle \phi_2(0), \phi_3(0) \rangle$ of Δ_e in Ω, we have*

$$
\int_{\partial\Omega} \frac{\partial \phi_{k_1}}{\partial x} \frac{\partial \phi_{k_2}}{\partial \nu} dA = 0.
$$

Proof. Set $G_t = t$, and $H_t = 0$. Then, J_t is a translation, and therefore $J_t^* e = e$, and $\frac{d}{dt}_{t=0} \Delta_{J_t^* e} \phi_k(x,y) \equiv 0$. From (26) our claim follows. □

Definition 2.2 Let us set

$$
\frac{\partial}{\partial \nu}_{(\xi,\eta)} = \alpha(\xi,\eta) \frac{\partial}{\partial \xi} + \beta(\xi,\eta) \frac{\partial}{\partial \eta},
$$

for C^∞ functions α and β on $\partial\Omega$, and let G_t and H_t satisfy

$$
\begin{cases}
\frac{dG_t}{dt}_{t=0}(\xi,\eta) = \mathfrak{G}(\xi,\eta)\alpha(\xi,\eta), \\
\frac{dH_t}{dt}_{t=0}(\xi,\eta) = \mathfrak{G}(\xi,\eta)\beta(\xi,\eta),
\end{cases} \qquad (31)
$$

where $(\xi,\eta) \in \partial\Omega$, and $\mathfrak{G}$ is a C^∞ function of $\partial\Omega$ called a *boundary function* of J_t. //

The following equality with self adjointness of operator $\frac{d}{dt}_{t=0} \Delta_{J_t^* e}$ on the second eigenspace in Ω holds;

Proposition 2.10. *Let $\{\phi_2(0), \phi_3(0)\}$ be a basis of the second eigenspace in Ω, and let J_t be a simple deformation of Ω given by (22) and (31). Then,*

$$
\begin{aligned}
&\int_\Omega \Big\{ \frac{d}{dt}_{t=0} \Delta_{J_t^* e} \phi_i(0) \Big\} \phi_j(0) d_e \\
&= \int_{\partial\Omega} \mathfrak{G} \frac{\partial \phi_i(0)}{\partial \nu} \frac{\partial \phi_j(0)}{\partial \nu} d_e A \\
&= \int_\Omega \phi_i(0) \Big\{ \frac{d}{dt}_{t=0} \Delta_{J_t^* e} \phi_j(0) \Big\} d_e, \quad i,j \in \{2,3\}. \qquad (32)
\end{aligned}
$$

where d_eA is the volume element on the boundary with respect to standard euclidean metric e.

Proof. It is verified from Proposition 2.8 and from the preceding definition 2.2. $\square$

When $k \notin \{2,3\}$, the following related to adjointness holds;

Proposition 2.11. *Let J_t be a simple deformation of Ω given by (22) and (31), and let $\phi_k(t_0)$, $t_0 \in (0,\varepsilon)$, the k-th eigenfunction of $J_{t_0}(\Omega)$. Suppose that $\lambda_2(t_0)$ and $\lambda_3(t_0)$ are simple. We have for $k \in \mathbb{N} \setminus \{2,3\}$*

$$\begin{aligned}
&\int_{J_{t_0}(\Omega)} \Big\{ \frac{d}{dt}_{t=t_0} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_k(t_0) \Big\} \phi_3(t_0) d_e \\
= &\int_{J_{t_0}(\Omega)} \Big\{ \frac{d}{dt}_{t=t_0} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_3(t_0) \Big\} \phi_k(t_0) de \\
&- \big(\lambda_3(t_0) - \lambda_k(t_0)\big) \int_{J_{t_0}(\Omega)} \Big\{ \dot{G}_{t_0} \frac{\partial \phi_3(t_0)}{\partial x} + \dot{H}_{t_0} \frac{\partial \phi_3(t_0)}{\partial y} \Big\} \phi_k(t_0) de \\
&+ \big(\lambda_k(t_0) - \lambda_3(t_0)\big) \int_{J_{t_0}(\Omega)} \Big\{ \dot{G}_{t_0} \frac{\partial \phi_k(t_0)}{\partial x} + \dot{H}_{t_0} \frac{\partial \phi_k(t_0)}{\partial y} \Big\} \phi_3(t_0) de \\
= &\int_{J_{t_0}(\Omega)} \Big\{ \frac{d}{dt}_{t=t_0} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_3(t_0) \Big\} \phi_k(t_0) de \\
&- \big(\lambda_k(t_0) - \lambda_3(t_0)\big) \int_{J_{t_0}(\Omega)} \Big\{ \frac{\partial \dot{G}_{t_0}}{\partial x} + \frac{\partial \dot{H}_{t_0}}{\partial y} \Big\} \phi_3(t_0) \phi_k(t_0) de.
\end{aligned} \tag{33}$$

Proof. When $\lambda_i(t_0) \neq \lambda_j(t_0)$, equality (27) turns into the following formula;

$$\begin{aligned}
&\int_{J_{t_0}(\Omega)} -\Big(2 \frac{\partial \dot{H}_{t_0}}{\partial y} \cdot \frac{\partial^2 \phi_i(t_0)}{\partial y^2} \phi_j(t_0) + 2 \frac{\partial \dot{H}_{t_0}}{\partial x} \cdot \frac{\partial^2 \phi_i(t_0)}{\partial x \partial y} \phi_j(t_0) de \\
&\qquad\qquad + \Big\{ \frac{\partial^2 \dot{H}_{t_0}}{\partial x^2} + \frac{\partial^2 \dot{H}_{t_0}}{\partial y^2} \Big\} \cdot \frac{\partial \phi_i(t_0)}{\partial y} \phi_j(t_0) \Big) de \\
= &- \int \frac{\partial}{\partial y} \Big(\dot{H}_{t_0} \frac{\partial^2 \phi_i(t_0)}{\partial y^2} \phi_j(t_0) + \frac{\partial \dot{H}_{t_0}}{\partial y} \frac{\partial \phi_i(t_0)}{\partial y} \phi_j(t_0) - \dot{H}_{t_0} \frac{\partial \phi_i(t_0)}{\partial y} \frac{\partial \phi_j(t_0)}{\partial y} \Big) de \\
&- \int \frac{\partial}{\partial x} \Big(\dot{H}_{t_0} \frac{\partial^2 \phi_i(t_0)}{\partial x \partial y} \phi_j(t_0) + \frac{\partial \dot{H}_{t_0}}{\partial x} \frac{\partial \phi_i(t_0)}{\partial y} \phi_j(t_0) - \dot{H}_{t_0} \frac{\partial \phi_i(t_0)}{\partial y} \frac{\partial \phi_j(t_0)}{\partial x} \Big) de \\
&- \big(\lambda_i(t_0) - \lambda_j(t_0)\big) \int_{J_{t_0}(\Omega)} \dot{H}_{t_0} \frac{\partial \phi_i(t_0)}{\partial y} \phi_j(t_0) de.
\end{aligned} \tag{34}$$

Notice the equality below is commutative with respect to sub-indices i and j when $\lambda_i(t_0) \neq \lambda_j(t_0)$;

$$\begin{aligned}
&- \int \frac{\partial}{\partial y} \Big(\dot{H}_{t_0} \frac{\partial^2 \phi_i(t_0)}{\partial y^2} \phi_j(t_0) + \frac{\partial \dot{H}_{t_0}}{\partial y} \frac{\partial \phi_i(t_0)}{\partial y} \phi_j(t_0) - \dot{H}_{t_0} \frac{\partial \phi_i(t_0)}{\partial y} \frac{\partial \phi_j(t_0)}{\partial y} \Big) dxdy \\
&- \int \frac{\partial}{\partial x} \Big(\dot{H}_{t_0} \frac{\partial^2 \phi_i(t_0)}{\partial x \partial y} \phi_j(t_0) + \frac{\partial \dot{H}_{t_0}}{\partial x} \frac{\partial \phi_i(t_0)}{\partial y} \phi_j(t_0) - \dot{H}_{t_0} \frac{\partial \phi_i(t_0)}{\partial y} \frac{\partial \phi_j(t_0)}{\partial x} \Big) dxdy \\
= &\int_{\partial J_{t_0}(\Omega)} \mathfrak{G} \frac{\partial \phi_i(t_0)}{\partial \nu} \frac{\partial \phi_j(t_0)}{\partial \nu} d_eA.
\end{aligned} \tag{35}$$

Thus

$$\int\Big\{\frac{d}{dt}_{t=t_0}\Delta_{J_t^*e}\phi_k(t_0)\Big\}\phi_3(t_0)-\big(\lambda_k(t_0)-\lambda_3(t_0)\big)\int\Big\{\dot{G}_{t_0}\frac{\partial\phi_k(t_0)}{\partial x}+\dot{H}_{t_0}\frac{\partial\phi_k(t_0)}{\partial y}\Big\}\phi_3(t_0)$$
$$=\int_{\partial J_{t_0}(\Omega)}\mathfrak{G}\frac{\partial\phi_k(t_0)}{\partial\nu}\frac{\partial\phi_3(t_0)}{\partial\nu}d_eA$$
$$=\int_{J_{t_0}(\Omega)}\Big\{\frac{d}{dt}_{t=t_0}\Delta_{J_t^*e}\phi_3(t_0)\Big\}\phi_k(t_0)d_e$$
$$-\big(\lambda_3(t_0)-\lambda_k(t_0)\big)\int_{J_{t_0}(\Omega)}\Big\{\dot{G}_{t_0}\frac{\partial\phi_3(t_0)}{\partial x}+\dot{H}_{t_0}\frac{\partial\phi_3(t_0)}{\partial y}\Big\}\phi_k(t_0)d_e. \tag{36}$$

Note

$$\int\dot{G}_{t_0}\frac{\partial\phi_k(t_0)}{\partial x}\phi_3(t_0)=-\int\dot{G}_{t_0}\frac{\partial\phi_3(t_0)}{\partial x}\phi_k(t_0)-\int\frac{\partial\dot{G}_{t_0}}{\partial x}\phi_k(t_0)\phi_3(t_0).$$ □

Proposition 2.12. *Let J_t be a simple deformation of Ω such that $\lambda_2(0)=\lambda_3(0)$ is double second eigenvalue, and $\lambda_2(t)$ and $\lambda_3(t)$ are simple on $(0,\varepsilon)$. Furthermore, for $t_0\in(0,\varepsilon)$ let $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\lambda_i(t)$, $i=2,3$, exist and*

$$\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\lambda_3(t)\neq\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\lambda_2(t). \tag{37}$$

Then, $\frac{d}{dt}_{t=t_0}\beta_{t_0,i,j}(t)$, $i,j=2,3$, converge as $t_0\to 0$. Then, $\phi^*_{i,0}(t_0)$ converges to one element $\tilde{\phi}_i(0)\in\langle\phi_2(0),\phi_3(0)\rangle$. Therefore, formulae (13) and (19) hold provided $\phi_2(t_0)$ and t_0 are replaced by a second eigenfunction $\tilde{\phi}_2(0)$ of Ω and 0, respectively. Consequently, $\lambda_i(t)$ is of $C^1\big([0,\varepsilon)\big)$, and defining by

$$\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\beta_{t_0,i,k}(t):=\frac{d}{dt}_{t=0}\beta_{0,i,k}(t),\ \ i,k=2,3, \tag{38}$$

$\phi^*_{i,0}(t)\in C^1\big([0,\varepsilon)\big)$.

(**Note**) When $\phi^*_{i,0}(t)$ is expanded with respect to $\phi_i(0)$, $i=2,3$, Proposition 2.12 implies that one can represent

$$\begin{cases}\tilde{\phi}_i(0)=\phi_i(0)+\lim_{t_0\to 0}\beta_{0,i,2}(t_0)\phi_2(0)+\lim_{t_0\to 0}\beta_{0,i,3}(t_0)\phi_3(0),\\ \beta_{0,i,k}(t_0)=\beta_{0,i,k}(t_1)+\int_{t=t_1}^{t=t_0}\frac{d}{ds}_{s=t}\beta_{0,i,k}(s)dt.\end{cases}$$

Proof. Let us suppose that $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\beta_{t_0,2,3}(t)$ exists. Then, according to Remark 2.5, $\lim_{t_0\to 0}\beta_{0,2,3}(t_0)$ exists, and then from Proposition 2.5 and from the fact that $\|\phi^*_{2,0}(t_0)\|_{2,J^*_{t_0}e}$ converges to one as $t_0\to 0$ one can infer that $\lim_{t_0\to 0}\beta_{0,2,2}(t_0)$ also exists. Thus from Proposition 2.5 $\phi^*_{2,0}(t_0)$ converges to a second eigenfunction in Ω as $t_0\to 0$. Therefore, according to the argument succeeding to Notation 2.1, regardless of multiplicity of $\lambda_2(0)$, (16) shows the existence of $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\beta_{0,2,2}(t)$ and the continuity of $\frac{d}{dt}_{t=t_0}\beta_{0,2,2}(t)$ on $[0,\varepsilon)$. Consequently, according to the argument of Remark 2.5, $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\phi^*_{2,0}(t)$ exists and $\phi^*_{2,0}(t)\in C^1\big([0,\varepsilon)\big)$.

The proof for $\beta_{t_0,3,j}(t)$, $j=2,3$, can be accomplished in the same way as $\beta_{t_0,2,j}(t)$. Also in a different way, one can infer that when existence of $\lim_{t_0\to 0}\phi^*_{2,0}(t_0)$ is shown, the limit of $\phi^*_{3,0}(t_0)$ which is orthogonal to $\phi^*_{2,0}(t_0)$ also exist. Then, existence of $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\phi^*_{3,0}(t)$ can be verified. Consequently, it suffices to prove only existence of $\lim_{t_0\to 0}\frac{d}{dt}_{t=t_0}\beta_{t_0,2,3}(t)$.

To obtain an equation for coefficient $\frac{d}{dt}_{t=t_0}\beta_{t_0,2,3}(t)$ of $\phi_3(t_0)$-component of $\frac{d}{dt}_{t=t_0}\phi^*_{2,t_0}(t)$

we take inner product of each side of (19) and $\phi_3(t_0)$. Then, we have

$$
\begin{aligned}
&\int_{J_t(\Omega)} \Big(\frac{d^2}{dt^2}_{t=t_0} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_2(t_0)\Big) \phi_3(t_0) d_e \\
=&2 \frac{d}{dt}_{t=t_0} \lambda_3(t) \frac{d}{dt}_{t=t_0} \beta_{t_0,2,3}(t) - 2 \frac{d}{dt}_{t=t_0} \lambda_2(t) \frac{d}{dt}_{t=t_0} \beta_{t_0,2,3}(t) \\
&+ 2 \sum_{k \in \mathbb{Z}^+ \setminus \{3\}} \big(- \lambda_3(t_0) + \lambda_k(t_0)\big) \frac{d}{dt}_{t=t_0} \beta_{t_0,k,3}(t) \frac{d}{dt}_{t=t_0} \beta_{t_0,2,k}(t) \\
&- \big(- \lambda_3(t_0) + \lambda_2(t_0)\big) \frac{d^2}{dt^2}_{t=t_0} \beta_{t_0,2,3}(t). \qquad (39)
\end{aligned}
$$

From (39) we obtain

$$
\begin{aligned}
&\frac{d}{dt}_{t=t_0} \beta_{t_0,2,3}(t) \Big\{ 2\Big(\frac{d}{dt}_{t=t_0} \lambda_3(t) - \frac{d}{dt}_{t=t_0} \lambda_2(t)\Big) \\
&\qquad + 2\big(- \lambda_3(t_0) + \lambda_2(t_0)\big) \frac{d}{dt}_{t=t_0} \beta_{t_0,2,2}(t) \Big\} \\
=&\int_{\Omega} \Big\{ \frac{d^2}{dt^2}_{t=t_0} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_2(t_0) \Big\} \phi_3(t_0) d_e \\
&- 2 \sum_{k \in \mathbb{Z}^+ \setminus \{2,3\}} \big(- \lambda_3(t_0) + \lambda_k(t_0)\big) \frac{d}{dt}_{t=t_0} \beta_{t_0,k,3}(t) \frac{d}{dt}_{t=t_0} \beta_{t_0,2,k}(t) \\
&+ \big(- \lambda_3(t_0) + \lambda_2(t_0)\big) \frac{d^2}{dt^2}_{t=t_0} \beta_{t_0,2,3}(t). \qquad (40)
\end{aligned}
$$

We do not know yet whether the second derivative $\frac{d^2}{dt^2}_{t=t_0} \beta_{0,2,3}(t)$ and the second summand in the right hand side of equality (40) converge or not as $t_0 \to 0$. So firstly we will describe a formula of $\frac{d}{dt}_{t=t_0} \beta_{0,2,3}(t)$ by solving ordinary linear differential equation (40). One can write (40) equivalently as follows;

$$
\begin{aligned}
&\lim_{t \to t_0} \frac{d}{dt} \beta_{t_0,2,3}(t) =: \frac{d}{dt}_{t=t_0} \beta_{t_0,2,3}(t) \\
=&\lim_{t \to t_0} \frac{1}{2\big(\frac{d}{dt}\lambda_3(t) - \frac{d}{dt}\lambda_2(t)\big) + 2\big(- \lambda_3(t) + \lambda_2(t)\big) \frac{d}{dt}\beta_{t_0,2,2}(t)} \\
&\qquad \Big[\int_{\Omega} \Big\{ \frac{d^2}{dt^2} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_2(t_0) \Big\} \phi_3(t_0) d_e \\
&\qquad + \big(- \lambda_3(t) + \lambda_2(t)\big) \frac{d^2}{dt^2} \beta_{t_0,2,3}(t) \\
&\qquad - 2 \sum_{k \in \mathbb{Z}^+ \setminus \{2,3\}} \big(- \lambda_3(t) + \lambda_k(t)\big) \frac{d}{dt} \beta_{t_0,k,3}(t) \frac{d}{dt} \beta_{t_0,2,k}(t) \Big]. \qquad (41)
\end{aligned}
$$

From ordinary linear differential equation (41) we obtain a primitive function $\frac{d}{dt}\beta_{t_0,2,3}(t)$ of t-variable

$$
\frac{d}{dt}\beta_{t_0,2,3}(t) \exp\Big(\int P_{t_0}(t) dt \Big) = \int \frac{Q_{t_0}(t)}{\lambda_3(t) - \lambda_2(t)} \cdot \exp\Big(\int P_{t_0}(t) dt \Big) dt + C,
$$

where C is an integration constant,

$$
P_{t_0}(t) = \frac{2\big(\frac{d}{dt}\lambda_3(t) - \frac{d}{dt}\lambda_2(t)\big) + 2\big(- \lambda_3(t) + \lambda_2(t)\big) \frac{d}{dt}\beta_{t_0,2,2}(t)}{\lambda_3(t) - \lambda_2(t)},
$$

and

$$Q_{t_0}(t) = \int \Big\{ \frac{d^2}{dt^2} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_2(t_0) \Big\} \phi_3(t_0) d_e$$
$$- 2 \sum_{k \in \mathbb{Z}^+ \setminus \{2,3\}} \big(- \lambda_3(t) + \lambda_k(t) \big) \frac{d}{dt} \beta_{t_0,k,3}(t) \frac{d}{dt} \beta_{t_0,2,k}(t).$$

A simple calculation shows

$$\exp\Big(\int P_{t_0}(t) dt \Big) = \exp\Big(\int \frac{d}{dt} 2 \log\big(\lambda_3(t) - \lambda_2(t)\big) \Big) \exp\Big(- 2 \int \frac{d}{dt} \beta_{t_0,2,2}(t) dt \Big)$$
$$= C_1' \big(\lambda_3(t) - \lambda_2(t)\big)^2 \cdot \exp\Big(- 2\beta_{t_0,2,2}(t) + C_2' \Big), \quad C_1', C_2' \in \mathbb{R}.$$

Then, we have

$$\begin{aligned}
&\frac{d}{dt}_{t=t_0} \beta_{t_0,2,3}(t) \\
=& \frac{1}{C_3' \big(\lambda_3(t_0) - \lambda_2(t_0)\big)^2 \cdot \exp\big(- 2\beta_{t_0,2,2}(t)\big)} \int \big(\lambda_3(t) - \lambda_2(t)\big) \cdot \exp\big(- 2\beta_{t_0,2,2}(t)\big) \\
&\cdot \Big[\int \Big\{ \frac{d^2}{dt^2} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_2(t_0) \Big\} \phi_3(t_0) dx dy \\
&\quad - 2 \sum_{k \in \mathbb{Z}^+ \setminus \{2,3\}} \big(- \lambda_3(t) + \lambda_k(t) \big) \frac{d}{dt} \beta_{t_0,k,3}(t) \frac{d}{dt} \beta_{t_0,2,k}(t) \Big] dt \Big|_{t=t_0} \\
&+ \frac{C}{C_3' \big(\lambda_3(t) - \lambda_2(t)\big)^2 \cdot \exp\big(- 2\beta_{t_0,2,2}(t)\big)} \Big|_{t=t_0}, \quad C_3' \in \mathbb{R}.
\end{aligned} \tag{42}$$

We are to show that the value

$$-2 \sum_{k \in \mathbb{Z}^+ \setminus \{2,3\}} \big(- \lambda_3(t_0) + \lambda_k(t_0) \big) \frac{d}{dt}_{t=t_0} \beta_{t_0,k,3}(t) \frac{d}{dt}_{t=t_0} \beta_{t_0,2,k}(t) \tag{43}$$

which is a summand of integrand in (42) converges as $t_0 \to 0$.

Lemma 2.13. *Let $\lambda_2(0) = \lambda_3(0)$ be the double second eigenvalue, and let $\lambda_2(t)$ and $\lambda_3(t)$ be simple eigenvalues of $J_t(\Omega)$ for $t \in (0, \varepsilon)$, where J_t is a simple deformation of Ω given by (22). Let $k \in \mathbb{N} \setminus \{2, 3\}$. Then, for the coefficients $\beta_{t_0,i,j}(t)$ defined by (4) the series (43) converges as $t_0 \to 0$.*

Proof. We will show the series is bounded on $0 \lneq t_0 < \varepsilon$. Each factor $\big(- \lambda_3(t_0) + \lambda_k(t_0)\big) \frac{d}{dt}_{t=t_0} \beta_{t_0,k,3}(t)$ is bounded on $t_0 \in (0, \varepsilon)$ from Proposition 2.6. Then, $\frac{d}{dt}_{t=t_0} \beta_{t_0,2,k}(t)$ will converge to zero as $k \to \infty$ at any t_0. That is, each summand term converges to zero as $k \to \infty$. Consequently, the series will converge as $t_0 \to 0$.

Let $t_0 \gneq 0$. From Proposition 2.11

$$\begin{aligned}
&\frac{d}{dt}_{t=t_0} \beta_{t_0,k,3}(t) \\
=& - \frac{1}{-\lambda_3(t_0) + \lambda_k(t_0)} \int \frac{d}{dt}_{t=t_0} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_k(t_0) \phi_3(t_0) \\
=& - \frac{1}{-\lambda_3(t_0) + \lambda_k(t_0)} \int \frac{d}{dt}_{t=t_0} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_3(t_0) \phi_k(t_0) + \int \Big\{ \frac{\partial \dot{G}_{t_0}}{\partial x} + \frac{\partial \dot{H}_{t_0}}{\partial y} \Big\} \phi_3(t_0) \phi_k(t_0) \\
=& \frac{d}{dt}_{t=t_0} \beta_{t_0,3,k}(t) + \int \Big\{ \frac{\partial \dot{G}_{t_0}}{\partial x} + \frac{\partial \dot{H}_{t_0}}{\partial y} \Big\} \phi_3(t_0) \phi_k(t_0).
\end{aligned} \tag{44}$$

Thus, (43) is equivalent with

$$\sum_{k\in\mathbb{N}\setminus\{2,3\}} \big(-\lambda_3(t_0)+\lambda_k(t_0)\big)\bigg[\frac{d}{dt}_{t=t_0}\beta_{t_0,3,k}(t) + \int \Big\{\frac{\partial \dot G_{t_0}}{\partial x} + \frac{\partial \dot H_{t_0}}{\partial y}\Big\}\phi_3(t_0)\phi_k(t_0)\bigg]\frac{d}{dt}_{t=t_0}\beta_{t_0,2,k}(t). \tag{45}$$

The following equality holds;

$$\begin{aligned}&\sum_{k\in\mathbb{N}\setminus\{2,3\}} \big(-\lambda_3(t_0)+\lambda_k(t_0)\big)\frac{d}{dt}_{t=t_0}\beta_{t_0,3,k}(t) \cdot \big(-\lambda_2(t_0)+\lambda_k(t_0)\big)\frac{d}{dt}_{t=t_0}\beta_{t_0,2,k}(t)\\ &=\int \Big[\frac{d}{dt}_{t=t_0}\Delta_{(J_t\circ J_{t_0}^{-1})^*e}\phi_3(t_0)\cdot \frac{d}{dt}_{t=t_0}\Delta_{(J_t\circ J_{t_0}^{-1})^*e}\phi_2(t_0)\Big]\\ &\quad-\sum_{k\in\{2,3\}}\int \Big\{\frac{d}{dt}_{t=t_0}\Delta_{(J_t\circ J_{t_0}^{-1})^*e}\phi_3(t_0)\Big\}\phi_k(t_0)\cdot\int \Big\{\frac{d}{dt}_{t=t_0}\Delta_{(J_t\circ J_{t_0}^{-1})^*e}\phi_2(t_0)\Big\}\phi_k(t_0).\end{aligned} \tag{46}$$

Then,

$$\begin{aligned}&\Big|\sum_{k\in\mathbb{N}\setminus\{2,3\}} \big(-\lambda_3(t_0)+\lambda_k(t_0)\big)\frac{d}{dt}_{t=t_0}\beta_{t_0,3,k}(t) \cdot \big(-\lambda_2(t_0)+\lambda_k(t_0)\big)\frac{d}{dt}_{t=t_0}\beta_{t_0,2,k}(t)\Big|\\ &\leq\Big\{\int \Big|\frac{d}{dt}_{t=t_0}\Delta_{(J_t\circ J_{t_0}^{-1})^*e}\phi_3(t_0)\Big|^2\Big\}^{\frac{1}{2}}\cdot \Big\{\int \Big|\frac{d}{dt}_{t=t_0}\Delta_{(J_t\circ J_{t_0}^{-1})^*e}\phi_2(t_0)\Big|^2\Big\}^{\frac{1}{2}}\\ &\quad-\sum_{k\in\{2,3\}}\int \Big\{\frac{d}{dt}_{t=t_0}\Delta_{(J_t\circ J_{t_0}^{-1})^*e}\phi_3(t_0)\Big\}\phi_k(t_0)\cdot\int \Big\{\frac{d}{dt}_{t=t_0}\Delta_{(J_t\circ J_{t_0}^{-1})^*e}\phi_2(t_0)\Big\}\phi_k(t_0).\end{aligned} \tag{47}$$

Even though $\frac{d}{dt}_{t=t_0}\beta_{t_0,i,j}(t)$, $i,j\in\{2,3\}$, oscillates as $t_0\to 0$, (47) is bounded on $t_0\in[0,\varepsilon)$, because $\int\big|\big\{\frac{d}{dt}_{t=t_0}\Delta_{(J_t\circ J_{t_0}^{-1})^*e}\phi_i(t_0)\big\}\big|$, $i\in\{2,3\}$, is bounded on $t_0\in[0,\varepsilon)$ from Proposition 2.1. Consequently, $\sum_{k\in\mathbb{N}\setminus\{2,3\}}\big(-\lambda_3(t_0)+\lambda_k(t_0)\big)\frac{d}{dt}_{t=t_0}\beta_{t_0,3,k}(t)\frac{d}{dt}_{t=t_0}\beta_{t_0,2,k}(t)$ is also bounded on $t_0\in[0,\varepsilon)$. On the other hand the series

$$\sum_{k\in\mathbb{N}\setminus\{2,3\}} \big(-\lambda_3(t_0)+\lambda_k(t_0)\big)\bigg[\int \Big\{\frac{\partial \dot G_{t_0}}{\partial x} + \frac{\partial \dot H_{t_0}}{\partial y}\Big\}\phi_3(t_0)\phi_k(t_0)\bigg]\frac{d}{dt}_{t=t_0}\beta_{t_0,2,k}(t) \tag{48}$$

is bounded on $t_0\in[0,\varepsilon)$. For this, note that

$$\begin{aligned}&\sum_{k\in\mathbb{N}\setminus\{2,3\}} \big(-\lambda_2(t_0)+\lambda_k(t_0)\big)\bigg[\int \Big\{\frac{\partial \dot G_{t_0}}{\partial x} + \frac{\partial \dot H_{t_0}}{\partial y}\Big\}\phi_3(t_0)\phi_k(t_0)\bigg]\frac{d}{dt}_{t=t_0}\beta_{t_0,2,k}(t)\\ &=\int\Big[\Big\{\frac{\partial \dot G_{t_0}}{\partial x} + \frac{\partial \dot H_{t_0}}{\partial y}\Big\}\phi_3(t_0)\cdot\frac{d}{dt}_{t=t_0}\Delta_{J_t^*e}\phi_2(t_0)\Big]\\ &\quad+\int\Big[\Big\{\frac{\partial \dot G_{t_0}}{\partial x} + \frac{\partial \dot H_{t_0}}{\partial y}\Big\}\phi_3(t_0)\frac{d}{dt}_{t=t_0}\lambda_2(t)\phi_2(t_0)\Big]\\ &\quad+\int\Big[\Big\{\frac{\partial \dot G_{t_0}}{\partial x} + \frac{\partial \dot H_{t_0}}{\partial y}\Big\}\phi_3(t_0)\big(-\lambda_3(t_0)+\lambda_2(t_0)\big)\frac{d}{dt}_{t=t_0}\beta_{t_0,2,3}(t)\phi_3(t_0)\Big]\end{aligned} \tag{49}$$

is bounded on $t_0 \in [0, \varepsilon)$ from the fact that

$$\frac{d}{dt}_{t=t_0} \lambda_2(t) = -\int \Big\{ \frac{d}{dt}_{t=t_0} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \phi_2^*(t_0) \Big\} \phi_2^*(t_0)$$

is bounded on $t_0 \in [0, \varepsilon)$ and from Proposition 2.1. Inequality

$$\frac{-\lambda_3(t_0) + \lambda_{k+1}(t_0)}{-\lambda_3(t_0) + \lambda_k(t_0)} \le \frac{-\lambda_2(t_0) + \lambda_{k+1}(t_0)}{-\lambda_2(t_0) + \lambda_k(t_0)}$$

holds for sufficiently large all k. Then, by the comparison test for the serieses we can show (48) is also bounded on $t_0 \in [0, \varepsilon)$. □

To show whether the right hand of (42) at $t = t_0$ converges as $t_0 \to 0$ one may set approximately $\lambda_3(t) - \lambda_2(t)$ up to equal to

$$\Big(\frac{d}{dt}_{t=0} \lambda_3(t) - \frac{d}{dt}_{t=0} \lambda_2(t) \Big) t \tag{50}$$

for t in a sufficiently small deleted neighborhood of zero. Let us denote a factor of integrand of (42) by

$$F_{t_0}(t) := \int \Big\{ \frac{d^2}{dt^2} \Delta_{\left(J_t \circ J_{t_0}^{-1}\right)^* e} \Big\} \phi_2(t_0) \phi_3(t_0) dx dy$$
$$- 2 \sum_{k \in \mathbb{N} \setminus \{2,3\}} \big(- \lambda_3(t) + \lambda_k(t) \big) \frac{d}{dt} \beta_{t_0,k,3}(t) \frac{d}{dt} \beta_{t_0,2,k}(t).$$

From Lemma 2.13 $F_{t_0}(t_0)$ converges as $t_0 \to 0$. The integration $\int \big(\lambda_3(t) - \lambda_2(t)\big) \exp\big(-2\beta_{t_0,2,2}(t)\big) F_{t_0}(t) dt$ vanishes in the second order as $t \to 0$. Thus, if $C = 0$, one can affirm that function (42) at $t = t_0$ converges as $t_0 \to 0$. Let us assume that the constant C is not zero. Then, the last summand of (42) at $t = t_0$ diverges in $\frac{1}{(t_0)^2}$ degree as $t_0 \to 0$. That is, $\frac{d}{dt}_{t=t_0} \beta_{t_0,2,3}(t)$ diverges in $\frac{1}{(t_0)^2}$ degree as $t_0 \to 0$. Then, $\beta_{t_0,2,3}(t_0)$ must be diverge in $\frac{1}{t_0}$ degree. It is not true, since $\beta_{t_0,2,3}(t_0)$ must be bounded around $t_0 = 0$. Thus, the integration constant C must be zero. Consequently, $\frac{d}{dt}_{t=t_0} \beta_{t_0,2,3}(t)$ converges as $t_0 \to 0$. □

2.2 Another Approach to Regularity of Path $\phi^*_{2,J_t(\Omega)} := \phi^*_{2,0}(t)$ with respective to t-variable when $\lambda_2(\Omega)$ is Double

Firstly, we wish to find a criterion for a simple deformation J_t under which an orthonormal basis $\{\phi_2(0), \phi_3(0)\}$ of the second eigenspace of Ω is given, there are pulled-back eigenfunctions $\phi^*_{2,0}(t)$ and $\phi^*_{3,0}(t)$ of $J_t(\Omega)$ which converge respectively to $\phi_2(0)$ and $\phi_3(0)$. Let us consider the following equalities for $\rho_2, \rho_3 \in \mathbb{R}$;

$$\frac{d}{dt}_{t=0} \Delta_{J_t^* e} \phi_2(0) = -\rho_2 \phi_2(0) - \big(\Delta_e + \lambda_2(0)\big) g_2(0), \tag{51}$$

$$\frac{d}{dt}_{t=0} \Delta_{J_t^* e} \phi_3(0) = -\rho_3 \phi_3(0) - \big(\Delta_e + \lambda_3(0)\big) g_3(0), \tag{52}$$

$$g_i(0) \equiv 0, \text{ on } \partial\Omega, \quad i = 2, 3.$$

Proposition 2.14. *Let $\lambda_i(0)$, $i = 2, 3$, be double eigenvalues. Let us assume that hypotheses (51) and (53) hold for J_t. If*

$$\rho_2 \neq \rho_3, \tag{53}$$

then pulled-back eigenfunction $\phi^*_{i,0}(t)$ in $J_t(\Omega)$ to Ω and the associated eigenvalue $\lambda_i(t)$ of $J_t(\Omega)$, $i = 2, 3$, have derivatives, respectively, on $t \in [0, \varepsilon)$ which satisfy

$$\lim_{t \to 0} \phi^*_{i,0}(t) = \phi_i(0), \quad \lim_{t \to 0} \frac{d}{dt} \lambda_i(t) = \rho_i, \quad \textit{and} \quad \lim_{t \to 0} \Big(\frac{d}{dt} \phi^*_{i,0}(t) \Big)^\circ = g_i(0), \tag{54}$$

where $\left(\frac{d}{dt}\phi^*_{i,0}(t)\right)^\circ$ *stands for the component of* $\frac{d}{dt}\phi^*_{i,0}(t)$ *orthogonal to* $\langle\phi_2(0),\phi_3(0)\rangle$*.*

Conversely, if $\phi^*_{2,0}(t)$ *converges to* $\phi_2(0)$ *as* $t\to 0$*, then there is a pulled-back eigenfunction* $\phi^*_{3,0}(t)$ *to* Ω *which converges to* $\phi_3(0)$*. Thus, conditions (51) and (52) hold.*

Proof. First, if $\lambda_i(t)$ is double on $t\in[0,\varepsilon)$, then obviously given any $\phi_i(0)$, there exist a pair $\big(a(t),b(t)\big)$ of real numbers such that $a(t)\phi^*_{2,0}(t)+b(t)\phi^*_{3,0}(t)$ which is a pulled-back function of the second eigenfunction $a(t)\phi_2(t)+b(t)\phi_3(t)$ of $J_t(\Omega)$ converges to $\phi_i(0)$ as $t\to 0$. Then, we considering (13), (51) and (52) hold with $\rho_2=\rho_3$. That is, hypothesis (53) is not kept. Consequently, we may verify Proposition only in the case that $\lambda_i(t)$ is simple on $(0,\varepsilon)$ for an $\varepsilon>0$.

Let us assume $\phi^*_{2,0}(t)$ does not converge to $\phi_2(0)$. Let us denote

$$\phi^*_{2,0}(t_0)=\phi_2(0)+\beta_{0,2,2}(t_0)\phi_2(0)+\beta_{0,2,3}(t_0)\phi_3(0)+\sum_{k\in\mathbb{N}\setminus\{2,3\}}\beta_{0,2,k}(t_0)\phi_i(0).$$

Since $\sum_{k\in\mathbb{N}\setminus\{2,3\}}\beta_{0,2,k}(t_0)\phi_i(0)$ converges to zero as $t\to t_0$ from Proposition 2.5, although $\epsilon>0$ assumes any sufficiently small value, there are a fixed constant $\sigma>0$ and $0<t_0<\epsilon$ such that $\beta^2_{0,2,3}(t_0)>\sigma$. Note that from Proposition 2.3 for $t_0\in(0,\varepsilon)$

$$\begin{aligned}&\frac{d}{dt}_{t=t_0}\Delta_{J_t^*e}\phi^*_{2,0}(t_0)\\ &=-\frac{d}{dt}_{t=t_0}\lambda_2(t)\phi^*_{2,0}(t_0)-\big(\Delta_{J^*_{t_0}e}+\lambda_2(t_0)\big)\frac{d}{dt}_{t=t_0}\sum_{k\in\mathbb{N}\setminus\{2,3\}}\beta_{0,2,k}(t)\phi_k(0).\end{aligned}\tag{55}$$

On the other hand from (51) and (52)

$$\begin{aligned}&\frac{d}{dt}_{t=t_0}\Delta_{J_t^*e}\phi^*_{2,0}(t_0)\\ =&\frac{d}{dt}_{t=t_0}\Delta_{J_t^*e}\Big(\big(1+\beta_{0,2,2}(t_0)\big)\phi_2(0)+\beta_{0,2,3}(t_0)\phi_3(0)+\big(\phi^*_{2,0}(t_0)\big)^\circ\Big)\\ =&-\big(1+\beta_{0,2,2}(t_0)\big)\rho_2\phi_2(0)-\beta_{0,2,3}(t_0)\rho_3\phi_3(0)\\ &\quad-\big(\Delta_{J^*_{t_0}e}+\lambda_2(t_0)\big)\big\{\big(1+\beta_{0,2,2}(t_0)\big)g_2(0)+\beta_{0,2,3}(t_0)g_3(0)\big\}\\ &\quad+\frac{d}{dt}_{t=t_0}\Delta_{J_t^*e}\Big\{\sum_{k\in\mathbb{N}\setminus\{2,3\}}\beta_{0,2,k}(t_0)\phi_k(0)\Big\}.\end{aligned}$$

We are to show the $\phi_2(0)$ and $\phi_3(0)$-component of the summand

$$\frac{d}{dt}_{t=t_0}\Delta_{J_t^*e}\Big\{\sum_{k\in\mathbb{N}\setminus\{2,3\}}\beta_{0,2,k}(t_0)\phi_k(0)\Big\}$$

converge to zero as $t_0\to 0$. Referring to (7), for $i=2,3$ the following value is bounded over all $k\in\mathbb{N}\setminus\{2,3\}$;

$$\begin{aligned}&\int\Big\{\frac{d}{dt}_{t=t_0}\Delta_{J_t^*e}\phi_k(0)\Big\}\phi_i(0)dxdy\\ =&\int\Big\{\sum_{1\le j+l\le 2}^{j,l=0,1,2}\frac{d}{dt}_{t=t_0}b_{j,l}(t)(x,y)\frac{\partial^{j+l}}{\partial x^j\partial y^l}\phi_k(0)(x,y)\Big\}\phi_i(0)(x,y)\\ =&\int\phi_k(0)(x,y)\Big\{\sum_{1\le j+l\le 2}^{j,l=0,1,2}\frac{\partial^{j+l}}{\partial x^j\partial y^l}\Big(\frac{d}{dt}_{t=t_0}b_{j,l}(t)(x,y)(-1)^{j+l}\phi_i(0)(x,y)\Big)\Big\}.\end{aligned}$$

Since $\sum_{k\in\mathbb{N}\setminus\{2,3\}}\beta_{0,2,k}(t_0)\phi_k(0)=\big(\phi^*_{2,0}(t_0)\big)^\circ$ converges to zero in L_2-norm as $t_0\to 0$, the norm $\sum_{k\in\mathbb{N}\setminus\{2,3\}}\beta^2_{0,2,k}(t_0)$ converges to zero, and it verifies our claim. Consequently, from (55) $-\big(1+\beta_{0,2,2}(t_0)\big)\rho_2\phi_2(0)-\beta_{0,2,3}(t_0)\rho_3\phi_3(0)$ must be equivalent to the formula

$$\frac{d}{dt}_{t=t_0}\lambda_2(t)\big\{-\big(1+\beta_{0,2,2}(t_0)\big)\phi_2(0)-\beta_{0,2,3}(t_0)\phi_3(0)\big\},$$

and since $|\beta_{0,2,3}(t_0)| > \sqrt{\sigma}$, we have $\rho_2 = \rho_3$. It is a contradiction. For $\phi^*_{3,0}(t)$, the same argument can be applied.

Conversely, if $\phi^*_{2,0}(t)$ converges to $\phi_2(0)$, then from Proposition 2.5 formula (51) holds, and then the converse statement can be verified from the adjointness in Proposition 2.10. The converse is also shown by the fact that the third eigenfunction $\phi^*_{3,0}(t_0)$ which is orthogonal to $\phi^*_{2,0}(t_0)$ under metric $J^*_{t_0}e$ is uniquely determined, and therefore $\phi^*_{3,0}(t_0)$ also converges, and then (52) holds. □

Remark 2.7. We have shown in Proposition 2.10 that $\int_\Omega \frac{d}{dt}_{t=0}\Delta_{J^*_te}\phi_2(0)\cdot\phi_3(0) = \int_\Omega \frac{d}{dt}_{t=0}\Delta_{J^*_te}\phi_3(0)\cdot\phi_2(0)$. Therefore, if $\phi^*_{2,0}(t)$ converges to $\phi_2(0)$, then $\int_\Omega \frac{d}{dt}_{t=0}\Delta_{J^*_te}\phi_2(0)\cdot\phi_3(0) = 0$, that is, if (51) holds, and then (52) which does not have $\phi_2(0)$-component naturally follows, since image of $\Delta_e + \lambda_i(0)$ is orthogonal to $\langle\phi_2(0), \phi_3(0)\rangle$ which is kernel of $\Delta_e + \lambda_i(0)$. Then, setting $\psi_2 = a_{2,2}\phi_2(0) + a_{2,3}\phi_3(0)$, we can ascertain Proposition 2.10 as follows;

$$\begin{aligned}
&\int_\Omega \frac{d}{dt}_{t=0}\Delta_{J^*_te}\phi_2(0)\cdot\psi_2\\
=&\int_\Omega \frac{d}{dt}_{t=0}\Delta_{J^*_te}\phi_2(0)\cdot\big(a_{2,2}\phi_2(0)+a_{2,3}\phi_3(0)\big)\\
=&-\int_\Omega \rho_2\phi_2(0)\cdot\big(a_{2,2}\phi_2(0)+a_{2,3}\phi_3(0)\big)\\
=&-a_{2,2}\rho_2 = \int_\Omega \frac{d}{dt}_{t=0}\Delta_{J^*_te}\psi_2(0)\cdot\phi_2(0).
\end{aligned}$$

The number $\rho_i(0)$ is uniquely determined, and $g_i(0) \in C^2_0(\Omega)$ is also uniquely determined modulo $\langle\phi_2(0), \phi_3(0)\rangle$. //

Definition 2.3. Let us call the following deformation an *inflation or deflation of* Ω *in c-rate*;

$$I_{c,t} : (x, y) \in \Omega \mapsto (x + ctx,\ y + cty), \quad -\infty < c < \infty, \quad t \in \{t \in [0, 1] : -1 < ct\}. \tag{56}$$

Let $\{\phi_2(0), \phi_3(0)\}$ be the orthonormal basis of the second eigenspace of Ω. Define

$$\begin{aligned}
&\phi_i(t)(x,y)\\
:=&\frac{1}{1+ct}\{I^{-1*}_{c,t}\phi_i(0)\}(x,y)\\
=&\frac{1}{1+ct}\phi_i(0)\left(\frac{1}{1+ct}x, \frac{1}{1+ct}y\right), \quad (x,y)\in I_{c,t}(\Omega), \quad i=2,3.
\end{aligned}$$

Then, $\phi_i(t)$ is a normalized second eigenfunction of $I_{c,t}(\Omega)$, and $\lambda_i\big(I_{c,t}(\Omega)\big) = \frac{1}{(1+ct)^2}\lambda_i(\Omega)$. Then, we have

$$\begin{cases}
\phi^*_{i,0}(t) = I^*_{c,t}\left\{\frac{1}{1+ct}I^{-1*}_{c,t}\phi_i(0)\right\} = \frac{\phi_i(0)}{1+ct},\\
\frac{d}{dt}_{t=0}\lambda_i(t) =: \lambda_i\big(I_{c,t}(\Omega)\big) = -2c\lambda_i(0),\\
\frac{d}{dt}_{t=0}\phi^*_{i,0}(t) = -c\phi_i(0), \quad \text{and}\\
\frac{d}{dt}_{t=0}\beta_{0,i,i}(t) = -c = \frac{d}{dt}_{t=0}\chi_0\big(\phi_i(t)\big), \quad i=2,3.
\end{cases} \tag{57}$$

For a more general form consider the following eigenfunction in $I_{c,t}(\Omega)$ for real smooth functions $\alpha_2(t)$ and $\alpha_3(t)$ such that $\alpha_i(0) = 0$ for $i = 2, 3$;

$$\begin{aligned}
&\left\{I^{-1*}_{c,t}\Big(\phi_2(0)+\alpha_2(t)\phi_2(0)+\alpha_3(t)\phi_3(0)\Big)\right\}(x,y)\\
=&\Big\{\phi_2(0)+\alpha_2(t)\phi_2(0)+\alpha_3(t)\phi_3(0)\Big\}\Big(\frac{x}{1+ct}, \frac{y}{1+ct}\Big), \quad (x,y)\in I_{c,t}(\Omega).
\end{aligned} \tag{58}$$

Define a normalized second eigenfunction $\tilde{\phi}_2(t)$ in $I_{c,t}(\Omega)$ by

$$\frac{1}{1+ct}\cdot\frac{1}{\sqrt{\big(1+\alpha_2(t)\big)^2+\alpha_3^2(t)}}\Big\{\phi_2(0)+\alpha_2(t)\phi_2(0)+\alpha_3(t)\phi_3(0)\Big\}\Big(\frac{x}{1+ct}, \frac{y}{1+ct}\Big).$$

Then, $\tilde{\phi}_2^*(t) \to \phi_2(0)$ as $t \to 0$, and

$$\tilde{\phi}_2^*(t) = \frac{1}{1+ct} \cdot \frac{1}{\sqrt{\big(1+\alpha_2(t)\big)^2 + \alpha_3^2(t)}} \Big\{ \phi_2(0) + \alpha_2(t)\phi_2(0) + \alpha_3(t)\phi_3(0) \Big\} \Big(x, y\Big).$$

Then,

$$\begin{cases} \frac{d}{dt}_{t=0}\lambda_2(t) = \frac{d}{dt}_{t=0}\Bigg[\frac{1}{(1+ct)\sqrt{\left(1+\alpha_2(t)\right)^2+\alpha_3^2(t)}} \\ \qquad\qquad \cdot \Big\{ \big(1+\alpha_2(t)\big)\lambda_2(0) + \alpha_3(t)\lambda_3(0) \Big\} \Bigg], \\ \qquad = -\{c + \alpha_2'(0) + \alpha_3'(0)\}\lambda_2(0) + \{\alpha_2'(0)\lambda_2(0) + \alpha_3'(0)\lambda_3(0)\} \\ \qquad = -c\lambda_2(0) - \alpha_3'(0)\{\lambda_2(0) - \lambda_3(0)\} = -c\lambda_2(0), \\ \frac{d}{dt}_{t=0}\beta_{0,2,2}(t) = \frac{d}{dt}_{t=0}\Bigg[\frac{1}{(1+ct)\sqrt{\left(1+\alpha_2(t)\right)^2+\alpha_3^2(t)}} \cdot \big(1+\alpha_2(t)\big) \Bigg] \\ \qquad = -\{c + \alpha_2'(0) + \alpha_3'(0)\} + \alpha_2'(0) = -c - \alpha_3'(0), \\ \frac{d}{dt}_{t=0}\beta_{0,2,3}(t) = \frac{d}{dt}_{t=0}\Bigg[\frac{1}{(1+ct)\sqrt{\left(1+\alpha_2(t)\right)^2+\alpha_3^2(t)}} \cdot \alpha_3(t)\big) \Bigg] = \alpha_3'(0), \end{cases} \tag{59}$$

where $\lambda_2(t)$ and $\beta_{t_0,i,k}(t)$ defined by (4) are associated with the pulled-back eigenfunction $\tilde{\phi}_2^*(t)$. Thus three derivatives can be set at random by selecting $\frac{d}{dt}_{t=0}\alpha_i(t)$, $i = 2, 3$, and the constant c at our disposal. We may define $\tilde{\phi}_3^*(t)$ as follows; for real smooth functions $\gamma_i(t)$ such that $\gamma_i(0) = 0$, $i = 2, 3$, and $\sum_{i=2,3} \alpha_i(t)\gamma_i(t) = 0$,

$$\tilde{\phi}_3^*(t)(x, y) = \frac{1}{1+ct} \cdot \frac{1}{\sqrt{\big(1+\gamma_2(t)\big)^2 + \gamma_3^2(t)}} \Big\{ \phi_3(0) + \gamma_2(t)\phi_2(0) + \gamma_3(t)\phi_3(0) \Big\} \Big(x, y\Big). \qquad //$$

Definition 2.4. Two C^∞ deformations J^1 and J^2 of Ω given, we define the sum $J^1 \uplus J^2$ by the deformation of Ω given by

$$\begin{aligned} &(J^1 \uplus J^2)_t(x, y) \\ =&(x, y) + \{J_t^1(x, y) - (x, y)\} + \{J_t^2(x, y) - (x, y)\} \\ =&J_t^1(x, y) + J_t^2(x, y) - (x, y) \\ =&\big(J_\xi^1(x, y, t) + J_\xi^2(x, y, t) - x,\ \ J_\eta^1(x, y, t) + J_\eta^2(x, y, t) - y\big), \quad (x, y) \in \Omega. \end{aligned}$$

Obviously $\uplus$ is commutative, associative. //

We can show a linearity for $\uplus$ sum of simple deformations J^k of Ω;

Proposition 2.15. *Let J^k be a simple deformation of Ω given by formula (22) for each $k = 1, 2, 3, ..., l$. For any $C^2(\Omega)$-function $\psi(0)$ in Ω and for $-\infty < \zeta_k < \infty$* we have

$$\begin{aligned} &\frac{d}{dt}_{t=0}\Big\{ \Delta_{\left(\uplus_{k=1}^l J_{\zeta_k t}^k\right)^* e}\psi(0) \Big\} \\ =&\sum_{k=1}^{l} \zeta_k \Big\{ \frac{d}{dt}_{t=0} \Delta_{J_t^{k*} e}\psi(0) \Big\}. \end{aligned} \tag{60}$$

Proof. For $\zeta_k \in \mathbb{R}$, $J_{\zeta_k t}^k$ satisfies the equality

$$\frac{d}{dt}_{t=0} J_{\zeta_k t}^k(x, y) = \zeta_k \frac{d}{dt}_{t=0} J_t(x, y), \quad (x, y) \in \Omega.$$

Denote $J_{\zeta_k t}^{k^{-1}}(\xi, \eta) := \big(\xi - G_{\zeta_k t}^k(\xi, \eta), \eta - H_{\zeta_k t}^k(\xi, \eta)\big)$. Then, $J_{\zeta_1 t}^{1^{-1}} \uplus J_{\zeta_2 t}^{2^{-1}}(\xi, \eta) = \big(\xi - \sum_{k=1}^2 G_{\zeta_k t}^k(\xi, \eta), \eta - \sum_{k=1}^2 H_{\zeta_k t}^k(\xi, \eta)\big)$. Proposition 2.7 (25) shows (60). □

Thus from (60) we may state the following linearities;

Proposition 2.16. *Let $\phi_2^*\big(J_t^k(\Omega)\big)$, $k=1,2$, converge to the same eigenfunction $\phi_2(\Omega)$ as t tends to zero. Then, we have*

$$\begin{cases} \frac{d}{dt}_{t=0}\phi_2^*\big(J_t^1 \uplus J_t^2(\Omega)\big) = \frac{d}{dt}_{t=0}\phi_2^*\big(J_t^1(\Omega)\big) + \frac{d}{dt}_{t=0}\phi_2^*\big(J_t^2(\Omega)\big), \\ \frac{d}{dt}_{t=0}\lambda_2\big(J_t^1 \uplus J_t^2(\Omega)\big) = \frac{d}{dt}_{t=0}\lambda_2\big(J_t^1(\Omega)\big) + \frac{d}{dt}_{t=0}\lambda_2\big(J_t^2(\Omega)\big). \end{cases} \tag{61}$$

Therefore, $\phi_2^*\big(J_t^1 \uplus J_t^2(\Omega)\big)$ converges to $\phi_2(\Omega)$ provided $\frac{d}{dt}_{t=0}\lambda_2\big(J_t^1 \uplus J_t^2(\Omega)\big) \neq \frac{d}{dt}_{t=0}\lambda_3\big(J_t^1 \uplus J_t^2(\Omega)\big)$.

Proof. Let us denote

$$\Delta_{J_t^{k*}e} - \Delta_e := \mathfrak{D}\Delta_{J_t^{k*}e},$$
$$\lambda_2\big(J_t^k(\Omega)\big) - \lambda_2(\Omega) := \mathfrak{D}\lambda_2\big(J_t^k(\Omega)\big),$$
$$\phi_2^*\big(J_t^k(\Omega)\big) - \phi_2(\Omega) := \mathfrak{D}\phi_2^*\big(J_t^k(\Omega)\big).$$

By the first hypothesis the limits of these differences exist as $t \to 0$. Thus

$$\Big(\Delta_e + \mathfrak{D}\Delta_{J_t^{k*}e}\Big)\Big(\phi_2(\Omega) + \mathfrak{D}\phi_2^*\big(J_t^k(\Omega)\big)\Big)$$
$$= -\Big\{\lambda_2(\Omega) + \mathfrak{D}\lambda_2\big(J_t^k(\Omega)\big)\Big\}\Big(\phi_2(\Omega) + \mathfrak{D}\phi_2^*\big(J_t^k(\Omega)\big)\Big)$$

holds for all $t \in [0,\epsilon]$. Discarding the second degree terms $\mathfrak{D}\Delta_{J_t^{k*}e} \cdot \mathfrak{D}\phi_2^*\big(J_t^k(\Omega)\big)$ and $\mathfrak{D}\lambda_2\big(J_t^k(\Omega)\big) \cdot \mathfrak{D}\phi_2^*\big(J_t^k(\Omega)\big)$, we have for each $k=1,2$

$$\lim_{t\to 0}\frac{1}{t}\Big\{\Delta_e\mathfrak{D}\phi_2^*\big(J_t^k(\Omega)\big) + \mathfrak{D}\Delta_{J_t^{k*}e}\phi_2(\Omega)\Big\}$$
$$= -\lim_{t\to 0}\frac{1}{t}\Big\{\mathfrak{D}\lambda_2\big(J_t^k(\Omega)\big)\phi_2(\Omega) + \lambda_2(\Omega)\mathfrak{D}\phi_2^*\big(J_t^k(\Omega)\big)\Big\}. \tag{62}$$

Thus, we have

$$\lim_{t\to 0}\frac{1}{t}\Big\{\Delta_e\sum_{k=1,2}\mathfrak{D}\phi_2^*\big(J_t^k(\Omega)\big) + \sum_{k=1,2}\mathfrak{D}\Delta_{J_t^{k*}e}\phi_2(\Omega)\Big\}$$
$$= -\lim_{t\to 0}\frac{1}{t}\Big\{\sum_{k=1,2}\mathfrak{D}\lambda_2\big(J_t^k(\Omega)\big)\phi_2(\Omega) + \lambda_2(\Omega)\sum_{k=1,2}\mathfrak{D}\phi_2^*\big(J_t^k(\Omega)\big)\Big\}. \tag{63}$$

Equality (62) is no other than (13). In fact from (60) equality (63) implies (61).

From the first assumption $\phi_3^*\big(J_t^k(\Omega)\big)$, $k=1,2$, also converge to the same eigenfunction $\phi_3(\Omega)$. Then, from the last hypothesis, Proposition 2.14, and from (60), equality (61) implies the last statement. □

Remark 2.8. Let $\{\phi_2(0), \phi_3(0)\}$ be an orthonormal basis of the second eigenspace in Ω. For inflation or deflation $I_{c,t}$ we have $\frac{\partial \dot{G}_0}{\partial x} = c = \frac{\partial \dot{H}_0}{\partial y}$, and $\frac{\partial \dot{G}_0}{\partial x} \cdot \frac{\partial^2 \phi_i(0)}{\partial x^2} + \frac{\partial \dot{H}_0}{\partial y} \cdot \frac{\partial^2 \phi_i(0)}{\partial y^2} = -c\lambda_i(0)\phi_i(0)$, $i=2,3$, but the other summands are all zero. Then, from Proposition 2.7 $\frac{d}{dt}_{t=0}\Big\{\Delta_{I_{c,t}^*e}\phi_i(0)\Big\} = +c\lambda_i(0)\phi_i(0)$. Thus, $\frac{d}{dt}_{t=0}\lambda_i(t) = -c\lambda_i(0)$.

One referring to an example of Definition 2.3, $\tilde{\phi}_i^*(t)$ converge to $\phi_i(\Omega)$, $i=2,3$, respectively. For the pulled-back eigenfuncion $\tilde{\phi}_2^*(t)$ in Definition 2.3, the following equalities holds;

$$\frac{d}{dt}_{t=0}\tilde{\phi}_2^*(t)$$
$$= \frac{d}{dt}_{t=0}\beta_{0,2,2}(t)\phi_2(0) + \frac{d}{dt}_{t=0}\beta_{0,2,3}(t)\phi_3(0),$$

where coefficients $\frac{d}{dt}_{t=0}\beta_{0,2,i}(t)$, $i=2,3$, can be determined by selecting a suitable c-rate of inflation or deflation and by adding suitable second eigenfunctions $\alpha_i(t)\phi_i(0)$, $i=2,3$, in Ω to the second eigenfunctions in $I_{c,t}(\Omega)$ at our disposal. For this, we may refer to (57) and (58).

A simple deformation J_t given, and let $\phi_2(t)$ be the second eigenfunction of $J_t(\Omega)$ such that $\phi^*_{2,0}(t) \to \phi_2(0)$, and let us denote by $\psi_2(t)$ the second eigenfunction in $J_t \uplus I_{c,t}(\Omega)$ converging to $\phi_2(0)$. Denote $(J_t \uplus I_{c,t})^*\psi_2(t) = \psi^*_2(t)$. Considering the linearity (60) in Proposition 2.16, from (57) and (58) one can define coefficient $-\frac{d}{dt}_{t=0}\alpha_3(t)$ to be equal to the coefficient of $\phi_3(0)$-component of $\frac{d}{dt}_{t=0}\phi^*_2(t)$ so that the coefficient of $\phi_3(0)$-component of $\frac{d}{dt}_{t=0}\psi^*_2(t)$ may vanishes.

Let us suppose closed nodal line of $\phi_2(0)$ meets $\partial\Omega$ at P, and $\phi_2(0)$ is positive in the inner nodal domain. Then, nodal line of $\psi_2(t)$ separates from boundary at sufficiently small all $t \in (0,1]$ provided $\frac{\partial}{\partial\nu}_P\big(\frac{d}{dt}_{t=0}\phi^*_{2,0}(t)\big)^\circ \gneq 0$ and provided the coefficient of $\phi_3(0)$-component of $\frac{d}{dt}_{t=0}\psi^*_2(t)$ vanishes. Since $\frac{\partial}{\partial\nu}_P\phi_2(0) = 0$, it does not matter what change of the coefficient of $\phi_2(0)$-component of $\frac{d}{dt}_{t=0}\psi^*_2(t)$ is. //

Proposition 2.17. *Suppose that $\lambda_2(0) = \lambda_3(0)$ is double, and $\lambda_2(t)$ and $\lambda_3(t)$ are simple on $t \in (0,\varepsilon)$. Let $\{\phi_2(0), \phi_3(0)\}$ be an orthonormal basis of the second eigenspace in Ω. Let J_t be a simple deformation of Ω and let us suppose that $\phi^*_{2,0}(t) \to \phi_2(0)$, and $\frac{d}{dt}_{t=0}\lambda_2(t) \neq \frac{d}{dt}_{t=0}\lambda_3(t)$. Given any real number $c \lneq 0$, one can select an inflation or deflation $I_{c,t}$ and coefficients $\alpha_3(t)$ defined by (57) so that the pulled-back second eigenfunction $\psi^*_2(t)$ in $J_t \uplus I_{c,t}(\Omega)$ to Ω may converge to $\phi_2(0)$ as $t \to 0$, and*

$$\begin{cases} \int_\Omega \frac{d}{dt}_{t=0}\psi^*_2(t)\cdot\phi_3(0) = 0, \\ \frac{d}{dt}_{t=0}\lambda_2\big(J_t \uplus I_{c,t}(\Omega)\big) = \frac{d}{dt}_{t=0}\lambda_2(t) - c\lambda_2(0), \\ \Big(\frac{d}{dt}_{t=0}\psi^*_2(t)\Big)^\circ = \Big(\frac{d}{dt}_{t=0}\phi^*_{2,0}(t)\Big)^\circ, \end{cases}$$

where $\big(\frac{d}{dt}_{t=0}\psi^*_2(t)\big)^\circ$ stands for the component of $\frac{d}{dt}_{t=0}\psi^*_2(t)$ orthogonal to $\langle\phi_2(0), \phi_3(0)\rangle$.

Proof. Considering (57), (58), (59), and Proposition 2.16, we can select $\alpha_3(t)$ satisfying our proposition. □

Definition 2.5. A *deformation with $\phi_3(\Omega_m)$-component eliminated in c-rate* means a deformation $J_t \uplus I_{c,t}$ described in Poposition 2.17. //

2.3 Green's Function of $\Delta_e + \lambda_2(0)$ in Ω

Green's function of $\Delta_e + \lambda_i(0)$ in Ω is the kernel $K_{\lambda_i(0)}(x,y;\zeta,\tau)$ of an integral operator represented by

$$u(\zeta,\tau) = \int_\Omega K_{\lambda_i(0)}(x,y;\zeta,\tau)k(x,y)dxdy,$$

which corresponds to the linear inhomogeneous equation $\big(\Delta_e + \lambda_i(0)\big)u = k$. Recalling Remark 2.1, such integral operator may be accepted as an inverse of $\Delta_e + \lambda_i(0)$ and bounded in a sense

$$\sup_{\|k\|_2=1}\Big\|\int K_{\lambda_i(0)}(x,y;\zeta,\tau)k(x,y)\Big\|_2 < \infty. \tag{64}$$

Green's function $K_{\lambda_i(0)}$ acting on distributions over Ω is a weak solution of

$$\big(\Delta_e + \lambda_i(0)\big)_{(x,y)}K_{\lambda_i(0)}(\zeta,\tau;x,y) = \delta_{(\zeta,\tau)}(x,y), \tag{65}$$

where δ is the Dirac distribution. From this we have

$$\int_\Omega K_{\lambda_i(0)}\Big\{\big(\Delta_e + \lambda_i(0)\big)\mathfrak{f}\Big\}\,dxdy = \int_\Omega \Big\{\big(\Delta_e + \lambda_i(0)\big)K_{\lambda_i(0)}\Big\}\mathfrak{f}\,dxdy = \mathfrak{f}(\zeta,\tau), \tag{66}$$

if $\int_\Omega \psi \mathfrak{f} = 0$ for all $\psi \in W_0^{1,2}(\Omega)$ such that $\big(\Delta_e + \lambda_i(0)\big)\psi \equiv 0$, where $\mathfrak{f} \in C^2(\Omega) \cap W_0^{1,2}(\Omega)$. According to [5] p. 370, Green's function satisfies

$$\begin{cases} K_{\lambda_2(0)}(x,y;\zeta,\tau) = K_{\lambda_2(0)}(\zeta,\tau;x,y), \\ K_{\lambda_2(0)}(x,y;\zeta,\tau) = -\frac{\mathfrak{b}(x,y;\zeta,\tau)}{2\pi}\log r + \gamma(x,y;\zeta,\tau), \\ K_{\lambda_2(0)}(x,y;\zeta,\tau) = 0, \quad \text{for } (\zeta,\tau) \in \partial\Omega,\ (x,y) \in \Omega, \end{cases} \tag{67}$$

where

$$r = \sqrt{(x-\zeta)^2 + (y-\tau)^2},$$

$\gamma(x,y;\zeta,\tau)$ and its derivatives up to the second order are continuous in a neighborhood of the singular point, and where $\mathfrak{b}(x,y;\zeta,\tau)$ denotes a function having continuous derivatives up to the second order such that $\mathfrak{b}(\zeta,\tau;\zeta,\tau)$, the value at source point, is identical to one.

To find symmetric functions $\mathfrak{b}(x,y;\zeta,\tau)$ and γ for which (65) and (67) are satisfied we consider the equalities;

$$\begin{aligned} &\big(\Delta_{(\zeta,\tau)} + \lambda_2(0)\big)K_{\lambda_2(0)} \\ &= -\frac{\log r}{2\pi}(\Delta\mathfrak{b}) - \mathfrak{b}\frac{1}{2\pi}\Delta\log r \\ &\quad - \frac{2}{2\pi}\frac{\partial \log r}{\partial\zeta}\frac{\partial\mathfrak{b}}{\partial\zeta} - \frac{2}{2\pi}\frac{\partial\log r}{\partial\tau}\frac{\partial\mathfrak{b}}{\partial\tau} \\ &\quad - \lambda_2(0)(\frac{1}{2\pi}\log r)\mathfrak{b} + \big(\Delta_{(\zeta,\tau)} + \lambda_2(0)\big)\gamma \\ &= -\frac{\log r}{2\pi}(\Delta_{(\zeta,\tau)}\mathfrak{b}) - \frac{2}{2\pi}\frac{\partial\log r}{\partial\zeta}\frac{\partial\mathfrak{b}}{\partial\zeta} - \frac{2}{2\pi}\frac{\partial\log r}{\partial\tau}\frac{\partial\mathfrak{b}}{\partial\tau} - \lambda_2(0)(\frac{\log r}{2\pi})\mathfrak{b} \\ &\quad + \mathfrak{b}\delta_{(x,y)}(\zeta,\tau) + \big(\Delta_{(\zeta,\tau)} + \lambda_2(0)\big)\gamma = \delta_{(x,y)}(\zeta,\tau). \end{aligned} \tag{68}$$

Let us denote by

$$g_{\mathfrak{b}_i}(x,y;\zeta,\tau) := \frac{\log r}{2\pi}(\Delta_{(\zeta,\tau)}\mathfrak{b}_i) + \frac{2}{2\pi}\frac{\partial\log r}{\partial\zeta}\frac{\partial\mathfrak{b}_i}{\partial\zeta} + \frac{2}{2\pi}\frac{\partial\log r}{\partial\tau}\frac{\partial\mathfrak{b}_i}{\partial\tau} + \lambda_2(0)\Big(\frac{\log r}{2\pi}\Big)\mathfrak{b}_i,$$

for some symmetric functions $\mathfrak{b}_i(x,y;\zeta,\tau)$ which belongs to $C^2(\Omega\times\Omega)$. Defining $\mathfrak{b} = \sum_i \sigma_i\mathfrak{b}_i$, for real constants σ_i, one can select σ_i so that $\mathfrak{b}(\zeta,\tau;\zeta,\tau) = 1$, $\mathfrak{b} \in C^2(\Omega\times\Omega)$ and $g_{\mathfrak{b}} := \sum_i g_{\mathfrak{b}_i} \in C^1(\Omega\times\Omega)$. Such belongingness can be achieved by selecting σ_i so that $\Delta_{(\zeta,\tau)}\mathfrak{b} + \lambda_2(0)\mathfrak{b} \in o(r)$, $\frac{\partial}{\partial\zeta}\mathfrak{b} \in o(r)$, and $\frac{\partial}{\partial\tau}\mathfrak{b} \in o(r)$. Let us consider equations

$$\begin{cases} \big(\Delta_e + \lambda_2(0)\big)_{(\zeta,\tau)}\gamma(x,y;\zeta,\tau) = g_{\mathfrak{b}}(x,y;\zeta,\tau), \quad \text{in } \Omega, \\ \gamma(x,y;\zeta,\tau) = \frac{\mathfrak{b}(x,y;\zeta,\tau)}{2\pi}\log\sqrt{(x-\zeta)^2+(y-\tau)^2} := \varphi(x,y;\zeta,\tau), \ \text{ on } (\zeta,\tau) \in \partial\Omega \end{cases} \tag{69}$$

for $\varphi \in C^2(\overline{\Omega})$. Such a symmetric solution γ will satisfy (67). Let us set

$$\gamma(x,y;\zeta,\tau) = u + h(x,y;\zeta,\tau),$$

where $u \in W^{k,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $k \geq 2$, and h is a harmonic function, that is, $\Delta_e h = 0$ in Ω, and $h \equiv \varphi$ on $\partial\Omega$. Thus,

$$\big(\Delta_e + \lambda_2(0)\big)u = g_{\mathfrak{b}}(x,y;\zeta,\tau) - \big(\Delta_e + \lambda_2(0)\big)h. \tag{70}$$

Therefore, from Remark 2.1 u exists, and then γ also exists. The uniqueness of Green's function is stated later with some assumption.

Let $P \in \partial\Omega$. By symmetric property we may regard $(x,y) = P$ as a unique singular point of $K_{\lambda_2(0)}(x,y;P)$ in $\overline{\Omega}$. From (65) we have

$$\big(\Delta + \lambda_2(0)\big)_{(x,y)}\bigg(\frac{\partial}{\partial\nu}_{(\zeta,\tau)=P}K_{\lambda_2(0)}(x,y;\zeta,\tau)\bigg) = 0, \quad \text{for all } (x,y) \in \Omega. \tag{71}$$

Since $K_{\lambda_2(0)}(x,y;\zeta,\tau) = 0$ for $(x,y) \in \partial\Omega \setminus P$ and for any $(\zeta,\tau) \in \overline{\Omega} \setminus P$, we also have

$$\frac{\partial}{\partial\nu}_{(\zeta,\tau)=P}K_{\lambda_2(0)}(x,y;\zeta,\tau) \equiv 0 \quad \text{at any } (x,y) \in \partial\Omega \setminus P. \tag{72}$$

Let us denote by $\mathcal{N}$ the closure of the set

$$\Big\{(x,y) \in \Omega : \frac{\partial}{\partial\nu}_{(\zeta,\tau)=P} K_{\lambda_2(0)}(x,y;\zeta,\tau) = 0\Big\}. \tag{73}$$

No matter how path points (x,y) moves to P, and no matter what the following value is;

$$\lim_{(x,y)\in\mathcal{P}\to P} \frac{\partial}{\partial\nu}_{(\zeta,\tau)=P} K_{\lambda_2(0)}(x,y;\zeta,\tau) \in \mathbb{R}\cup\{\infty\}\setminus\{0\},$$

the line $\mathcal{N}$ divides Ω at most into two domains. To verify this assertion we refer to [3]. Let us assume $\Omega\setminus\mathcal{N}$ consists of domains G_1, G_2, and G_3. For each $j = 1,2$, define

$$\psi_j = \begin{cases} \frac{\partial}{\partial\nu}_P K_{\lambda_2(0)} & \text{in } G_j, \\ 0 & \text{in } \Omega\setminus G_j. \end{cases}$$

Also let $P \in \partial G_3$. One can find a nontrivial function

$$f = \sum_{j=1}^{2} \alpha_j\psi_j, \qquad \alpha_j \in \mathbb{R}$$

satisfying

$$0 = \big(f, \phi_1(\Omega)\big) := \int_\Omega \langle f, \phi_1(\Omega)\rangle dV,$$

where $\phi_{1,\Omega}$ is the first eigenfunction of Ω. One can verify that each ψ_j, $j = 1,2$, belongs to the space of admissible functions in spite of existence of a singular point P. Space of admissible functions is the completion of C^∞ functions compactly supported on Ω under the metric induced by $\|f\|_{(1)}^2 := \|f\|_2^2 + \|\text{Grad f}\|_2^2$, where $\|\text{Grad f}\|_2^2 = \int\langle\text{Grad f}, \text{Grad f}\rangle dV$, and Grad is defined by using a concept of weak derivative for functions in $L^2(\Omega)$ under a given metric on the domain, that is, Grad f is a weak derivative of $f$satisfying $(\text{Grad f}, X) = -(f, \text{div}X)$ for all C^1 vector fields X with compact support on Ω. Since $\frac{\partial}{\partial\nu}_{(\zeta,\tau)=P} K_{\lambda_2(0)}(x,y;\zeta,\tau) \equiv 0$ at all $(x,y) \in \partial\Omega$ except the unique singularity $(x,y) = P \in \partial G_3$, the following equality holds;

$$\big(\text{Grad}\,\phi, \text{Grad}\,f\big) = -\big(\Delta\phi, f\big). \tag{74}$$

Then, $\big(\text{Grad}\,f, \text{Grad}\,f\big)/\|f\|_2^2 = \sum_{j=1}^2 \alpha_j^2\lambda_2(0)/\|f\|_2^2 = \lambda_2(0)$. Therefore, Rayleigh's theorem implies f is an eigenfunction in Ω with $g = e$ and $\lambda = \lambda_2(0)$. Thus, f is analytic in Ω. So, since $f \equiv 0$ in G_3, from maximum principle [3] p.329, chapter XII, section 11, (or from the unique analytic continuation theorem,) $f \equiv 0$ in Ω. It is impossible.

For the first eigenvalue $\lambda_1(0)$ one can show $\frac{\partial}{\partial\nu}_{(\zeta,\tau)=P} K_{\lambda_1(0)}$ has no line $\mathcal{N}$ which is defined in the same way as (73). The line $\mathcal{N}$ defined for $\frac{\partial}{\partial\nu}_{(\zeta,\tau)=P} K_{\lambda_2(0)}$ divides Ω precisely into two domains. We state a generalized Courant's nodal domain theorem;

Proposition 2.18. *The line $\mathcal{N}$ defined by (73) divides Ω exactly into two sub-domains.*

Proof. This assertion has been shown in the previous paragraph. □

Definition 2.6. Let us call the line $\mathcal{N}$ defined by (73) a *nodal line*, and call a component of $\Omega\setminus\mathcal{N}$ a *nodal domain* of $\frac{\partial}{\partial\nu}_{(\zeta,\tau)=P} K_{\lambda_2(0)}$. //

Remark 2.9. Let us consider the function $\frac{y}{x}$. This function assumes to be zero on $\{(x,0) : x \in \mathbb{R}\setminus\{0\}\}$, converges to zero as (x,y) tends to $(0,0)$ along path $\{(x,y) = (x, cx^2) : x \in \mathbb{R}\}$, and converges to a constant c as (x,y) tends to $(0,0)$ along the path $\{(x,y) = (x, cx) : x \in \mathbb{R}\}$ for any constant $c \in \mathbb{R}$. But this function diverges as (x,y) converges to $(0,0)$ along the path $\{(x,y) = (x, \pm\sqrt{x}) : x > 0\}$. //

Proposition 2.19. *Let $\{\phi_2(0), \phi_3(0)\}$ be an orthonormal basis of the second eigenspace in Ω. There are at most two points on $\partial\Omega$ at which the nodal line of*

$$a_2\phi_2(x,y) + a_3\phi_3(x,y) + a_4 \frac{\partial}{\partial\nu}_{(\zeta,\tau)=P} K_{\lambda_2(0)}(x,y;\zeta,\tau)$$

meets $\partial\Omega$, where $a_i \in \mathbb{R}$ are not all zero.

Proof. It can be proven according to the proof of Courant's nodal domain theorem. The argument was shown in the paragraph preceding to Propsoition 2.18. □

Green's function may be required to satisfy

$$(\Delta_e + \lambda_2)K_{\lambda_i(0)} = \delta_{(x,y)}(\zeta,\tau) - \sum_{i=2,3} \phi_i(0)(\zeta,\tau)\phi_i(0)(x,y), \quad i = 2,3, \tag{75}$$

where $\{\phi_2(0), \phi_3(0)\}$ is an orthonormal basis of the second eigenspace in Ω. From (75) Green's function satisfies the following consistency of a distribution;

$$\int_\Omega K_{\lambda_i(0)}\Big\{\big(\Delta_e + \lambda_i(0)\big)\phi_i(0)\Big\} = 0 = \int_\Omega \Big\{\big(\Delta_e + \lambda_2(0)\big)K_{\lambda_i(0)}\Big\}\phi_i(0),\ i = 2,3. \tag{76}$$

In contrast to this, for Green's function which is not defined by (75) we have

$$\int_\Omega K_{\lambda_i(0)}\Big\{\big(\Delta_e + \lambda_i(0)\big)u\Big\} = \int_\Omega \Big\{\big(\Delta_e + \lambda_2(0)\big)K_{\lambda_i(0)}\Big\}u^\circ, \tag{77}$$

where u° denotes the component of u which is orthogonal to $\langle\phi_2(0), \phi_3(0)\rangle$. According to [15], Green's function is uniquely determined if we assume that

$$\int_\Omega K_{\lambda_i(0)}(x,y;\zeta,\tau)\phi_i(0)(x,y)dxdy = 0, \quad i = 2,3. \tag{78}$$

We call this uniquely determined Green's function a *modified* Green's function. It is obtained by subtracting the eigenspace associated with $\lambda_i(0)$. We are to represent modified Green function K_{λ_2} by bilinear eigenfunction expansion. Let $\phi_k(0)$, $k = 1,2,3,...$, be the k-th orthonormal eigenfunctions of Δ_e in Ω associated with eigenvalue $\lambda_k(0)$, $\lambda_2(0) = \lambda_3(0)$. Then, we have

$$(\Delta_e + \lambda_2)\phi_k(0) = -\big(\lambda_k(0) - \lambda_2(0)\big)\phi_k(0),$$

that is, $\phi_k(0)$ is an eigenfunction of $\Delta_e + \lambda_2(0)$ in Ω associated with eigenvalue $\lambda_k(0) - \lambda_2(0)$. Thus, $\{\phi_2(0), \phi_3(0)\}$ is the ortho-normal basis of the eigenspace of operator $\Delta_e + \lambda_2(0)$ in Ω associated with eigenvalue equal to zero. Modified Green's function is constructed in the manner of bilinear series converging in L_2-norm;

$$K_{\lambda_2}(x,y;\zeta,\tau) = \sum_{k\neq 2,3}^{\infty} \frac{\phi_k(0)(\zeta,\tau)\cdot\phi_k(0)(x,y)}{\lambda_k(0) - \lambda_2(0)}.$$

2.4 Deformations such that nodal line of $\phi_2^*\big(J_t(\Omega)\big)$ is split from boundary, and $\lambda_2\big(J_t(\Omega)\big)$ which is multiple at $t = 0$ changes into simple eigenvalue as t grows from zero

The following lemma implies that if ϕ satisfies the Dirichlet eigenvalue problem (1), and if nodal line of ϕ does not intersect a boundary point q, then $\frac{\partial\phi}{\partial\nu}(q) \neq 0$, that is, on a neighborhood of boundary around q ϕ vanishes in the first order.

Lemma 2.20. ([6] p.34, Hopf's boundary point lemma in a limited sense) *Suppose*

$$\Delta_e\phi + c(x,y)\phi \geq 0,$$

in an open set $D \subset \Omega$ such that ∂D is smooth and $\partial D \cap \partial\Omega$ is non-empty. Let q_0 be a point on $\partial D \cap \partial\Omega$ such that
1) ϕ is continuous at q_0;
2) $\phi(x,y) < \phi(q_0)$ for all $(x,y) \in D$ and $\phi(q_0) = 0$.
Then the outer normal derivative of ϕ at q_0 satisfies the strict inequality

$$\frac{\partial\phi}{\partial\nu}(q_0) > 0.$$

Definition 2.7. Let Ω be a bounded smooth domain in $\mathbb{R}^2$ and f be an eigenfunction on Ω and $p \in \partial\Omega$. We say that f *has equi-angular K-system at p* if and only if the nodal line of f divides $\Omega \cap B(p,r)$ into K sectors of equal amplitude by theirs tangent lines at p for sufficiently small all $r > 0, K = 1,2,3,\cdots$. //

If f has equi-angular K-system at p, then f vanishes of the K-th order. The equi-angularity of K-system of an eigenfunction at boundary of a convex Euclidean domain in $\mathbb{R}^2$ with no smoothness assumption was verified by Alessandrini [1]. A. D. Melas [11] also showed that nodal line of the second eigenfunction approaches to boundary point nontangentially with respect to the boundary. Previously Cheng [4] showed that the eigenfunction of a Riemannian manifold has equi-angular K-system at interior points where the nodal line meets.

The following Proposition is regarded as a concrete version of Hopf's boundary point lemma. From this proposition one can perceive eigenfunction ϕ vanishes as homogeneous spherical harmonic polynomial in the lowest degree near each $p \in \partial\Omega$;

Proposition 2.21. *Any eigenfunction ϕ of Δ_e in Ω has an equi-angular system at every $p \in \partial\Omega$.*

Proof. To show that ϕ vanishes only up to finite order around p, we follow the arguments of A. D. Melas [11]. Let $H = \{(x,y) : y > 0\}$ be the upper half plane and let h be a conformal mapping of H onto $\Omega \cap B(p,r)$ with $h((0,0)) = p$, where r is small enough for $\Omega \cap B(p,r)$ to be a simple domain. From the boundary regularity of elliptic differential equations it follows that ϕ is C^∞ up to the boundary near p. By a theorem of D. Kellogg [8] h extends C^∞ to the boundary of H. Let

$$\hat{\phi} = \phi \circ h$$

in $\overline{H}$. Thus there is a sufficiently small $\hat{r} < r$ such that $\hat{\phi}$ is C^∞ up to the boundary in $B((0,0),\hat{r}) \cap \overline{H}$ and $\hat{\phi} = 0$ on $B((0,0),\hat{r}) \cap \partial H$. By direct calculation, we have

$$|\Delta_e\hat{\phi}| = |(\Delta_e\phi) \circ h||h'|^2 \leq C|\hat{\phi}|$$

in $B((0,0),\hat{r}) \cap \overline{H}$ for some constant C. Define $\tilde{\phi}$ in $B((0,0),\hat{r})$ by

$$\tilde{\phi}(x,y) = \begin{cases} \hat{\phi}(x,y) & \text{if } y \geq 0, \\ -\hat{\phi}(x,-y) & \text{if } y \leq 0. \end{cases} \tag{79}$$

Then it is easy to check that $\frac{\partial}{\partial x}\tilde{\phi}$, $\frac{\partial}{\partial y}\tilde{\phi}$, $\frac{\partial^2}{\partial x\partial y}\tilde{\phi}$, and $\frac{\partial^2}{\partial x^2}\tilde{\phi}$ are continuous in $B((0,0),\hat{r})$, and $\frac{\partial^2}{\partial x^2}\tilde{\phi} = 0$ in $B((0,0),\hat{r}) \cap \partial H$. From the inequality $|\Delta_e\hat{\phi}| \leq C|\hat{\phi}|$ in $B((0,0),\hat{r}) \cap \overline{H}$, it follows that $\frac{\partial^2}{\partial y^2}\tilde{\phi}$ is also continuous in $B((0,0),\hat{r})$. Since $\hat{\phi}$ is C^∞ up to the boundary in $B((0,0),\hat{r}) \cap \overline{H}$, we conclude that $\tilde{\phi}$ is in the Hölder space $C^{2,1}$ in $B((0,0),\hat{r})$, and moreover

$$|\Delta_e\tilde{\phi}| \leq C|\tilde{\phi}|$$

in $B((0,0),\hat{r})$. Thus by Aronszajn's unique continuation theorem [2], $\tilde{\phi}$ does not vanish of infinite order in L^1-sense at $(0,0)$. Let $\frac{\partial}{\partial\nu}_p = -\frac{\partial}{\partial y}_p$, for outer normal derivative $\frac{\partial}{\partial\nu}_p$ to Ω at

p. Since $\hat{\phi}$ is C^∞ function in a neighborhood in $\overline{H}$ around $(0,0)$ and vanishes of finite order at $(0,0)$, $\phi = \hat{\phi} \circ h^{-1}$ also vanishes of finite order at $p := (0,0)$. Then, since ϕ extends to be C^∞ up to the boundary of Ω, one has Taylor expansion such that for an integer $N \geq 1$

$$\phi = P_N + aP_{N+1} + 0\big(|(x,y)|^{N+1}\big)$$

holds in $\Pi := \overline{\Omega} \cap \overline{B(O,\tilde{r},\pi)}$, where $B(O,\tilde{r},\pi) = \{z : 0 < \arg(z) < \pi, |z| < \tilde{r}\}$, Π is convex for a sufficiently small number $\tilde{r}$, P_N is a non-zero polynomial of (minimal) degree N, and P_{N+1} is a polynomial of degree $N+1$ with $a \in \mathbb{R}$. In fact according to Taylor expansion $aP_{N+1}(x,y) + 0\big(|(x,y)|^{N+1}\big)$ is written by

$$+\frac{1}{(N+1)!}\,\frac{d^N}{dt^N}\Big|_{t=t_0}\Big\{x\frac{\partial}{\partial x}\phi(tx,ty) + y\frac{\partial}{\partial y}\phi(tx,ty)\Big\}$$

for all $(x,y) \in \Pi$, and for some $t_0 \in [0,1]$. We have

$$\Delta_e P_N = 0, \text{ in } \Pi,$$

since ϕ has no homogeneous polynomial whose degree is less than N. Thus the polynomial part P_N is homogeneous spherical harmonic. Therefore, ϕ has equi-angular N-system at $p = (0,0)$. □

Proposition 2.22. *Let $(0,0) \in \partial\Omega$, and $\frac{\partial}{\partial\nu}_{(0,0)} = -\frac{\partial}{\partial y}_{(0,0)}$. Suppose the nodal line of the second eigenfunction $\phi_2(0)$ of Ω has equi-angular three-system at $(0,0)$, that is, it vanishes of the third order at $(0,0)$, and $\phi_2(0) > 0$ in the middle open sector of the system. Let J_t be a simple deformation of Ω. Suppose $\frac{\partial}{\partial\nu}_{(0,0)}\frac{d}{dt}_{t=0}J_t^*\phi_2(t) \gneq 0$, where $\phi_2(t)$ is the second eigenfunction of $J_t(\Omega)$ converging to $\phi_2(0)$. Then, there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that for every $t \in (0,\varepsilon_1)$ the nodal line of $J_t^*\phi_2(t)$ is separated from a fixed neighborhood of boundary $B\big((0,0),\varepsilon_2\big) \cap \partial\Omega$.*

Proof. Let us map Ω by a conformal map $\mathfrak{h}$ to the upper-half plane $\mathbb{R}^{2+}$ with $(0,0) \mapsto (0,0)$. Let us remind ourselves that the equi-angular three-system at $(0,0)$ is representes by $-3x^2y + y^3$ as minimal degree, and the spherical harmonic two-system is represented by xy.

Assume that $\mathfrak{h}^{-1*}J_t^*\phi_2(t)$ has equi-angular three-system at $(x,0) = \big(c_1(t),0\big)$, for a differentiable real function $c_1(t)$ such that $c_1(0) = 0$. The function $c_1(t)$ is differentiable with respective to t-variable, since $\phi_2(t)$ is differentiable with respective to t-variable. Then, the system is represented by $-3\big(x - c_1(t)\big)^2 y + y^3 = -c_1^2(t)y + 2c_1(t)xy - x^2y + y^3$ near $(0,0)$. Then, $\frac{\partial}{\partial y}_{(x,y)=(0,0)}\frac{d}{dt}_{t=0}J_t^*\phi_2(t) = 0$.

Let us assume that $J_t^*\phi_2(t)$ has two equi-angular two-systems near $(0,0)$. We may consider the following expansion near $(0,0)$;

$$-3\big(x - c_2(t)\big)\big(x - c_3(t)\big)y + y^3,$$

where $c_2(t)$ and $c_3(t)$ are differentiable real functions such that $c_2(0) = 0 = c_3(0)$. Then, $\frac{\partial}{\partial y}_{(0,0)}\frac{d}{dt}_{t=0}\big\{-3\big(x - c_2(t)\big)\big(x - c_3(t)\big)y + y^3\big\} = 0$.

Consequently if $\frac{d}{dt}_{t=0}\frac{\partial}{\partial\nu}_{(0,0)}J_t^*\phi_2(t) \gneq 0$, then $J_t^*\phi_2(t)$ has neither equi-angular three-system nor equi-angular two systems for every $t \in (0,\varepsilon_1]$ on $B\big((0,0),\varepsilon_2\big) \cap \partial\Omega$. □

We describe a preliminary proposition which shows a general case of Hopf's boundary point lemma. Let $\{\phi_2,\phi_3\}$ be a basis of the second eigenspace of Ω.

Proposition 2.23. *The function*

$$\sum_{2\leq i,j\leq 3}\alpha_{i,j}\frac{\partial\phi_i}{\partial\nu}\frac{\partial\phi_j}{\partial\nu} \tag{80}$$

on the boundary $\partial\Omega$ has at most four zeros on $\partial\Omega$, where real coefficients $\alpha_{i,j}$ are not all zero.

Proof. Let $\alpha_{3,3} \neq 0$. Let us put (80) into the simple form $\frac{\partial\phi_3}{\partial\nu}\frac{\partial\phi_3}{\partial\nu} + \frac{\alpha_{2,3}+\alpha_{3,2}}{\alpha_{3,3}}\frac{\partial\phi_2}{\partial\nu}\frac{\partial\phi_3}{\partial\nu} + \frac{\alpha_{2,2}}{\alpha_{3,3}}\frac{\partial\phi_2}{\partial\nu}\frac{\partial\phi_2}{\partial\nu}$. Firstly, let us assume

$$\begin{aligned}&\Big(\frac{\alpha_{2,3}+\alpha_{3,2}}{\alpha_{3,3}}\frac{\partial\phi_2}{\partial\nu}\Big)^2 - 4\Big(\frac{\alpha_{2,2}}{\alpha_{3,3}}\frac{\partial\phi_2}{\partial\nu}\frac{\partial\phi_2}{\partial\nu}\Big)\\ =&\Big(\frac{\partial\phi_2}{\partial\nu}\Big)^2\Big\{\Big(\frac{\alpha_{2,3}+\alpha_{3,2}}{\alpha_{3,3}}\Big)^2 - 4\frac{\alpha_{2,2}}{\alpha_{3,3}}\Big\}\\ \geq&0.\end{aligned}$$

on a subset of $\partial\Omega$. Then from quadratic formula (79) is factorized into

$$\Big(\frac{\partial\phi_3}{\partial\nu} - e_1\frac{\partial\phi_2}{\partial\nu}\Big)\Big(\frac{\partial\phi_3}{\partial\nu} - e_2\frac{\partial\phi_2}{\partial\nu}\Big) \quad \text{on } \partial\Omega, \quad e_1, e_2 \in \mathbb{R}.$$

Each function of boundary $\frac{\partial\phi_3}{\partial\nu} - e_1\frac{\partial\phi_2}{\partial\nu}$ and $\frac{\partial\phi_3}{\partial\nu} - e_2\frac{\partial\phi_2}{\partial\nu}$ has at most two number of zeros on the subset of $\partial\Omega$. Secondly, on the complement of the above subset of $\partial\Omega$ we have

$$\Big(\frac{\partial\phi_2}{\partial\nu}\Big)^2\Big\{\Big(\frac{\alpha_{2,3}+\alpha_{3,2}}{\alpha_{3,3}}\Big)^2 - 4\frac{\alpha_{2,2}}{\alpha_{3,3}}\Big\} \lneq 0.$$

Then (79) has no zeros on this subset. Thus our proposition is valid. If $\alpha_{3,3} = 0$, then it is easily shown that (79) also factorized. Thus our proposition follows. □

Remark 2.10. Unlike the function (79), we have no knowledge for the number of zeros on $\partial\Omega$ of the following function

$$\alpha_{2,4}\frac{\partial\phi_2}{\partial\nu}\frac{\partial}{\partial\nu}\Big(\frac{\partial}{\partial\nu}{}_P K_{\lambda_2(0)}\Big) + \sum_{2\leq i,j\leq 3}\alpha_{i,j}\frac{\partial\phi_i}{\partial\nu}\frac{\partial\phi_j}{\partial\nu}. \tag{81}$$

Remind yourself that $\frac{\partial}{\partial\nu}\frac{\partial}{\partial\nu}{}_P K_{\lambda_2(0)}$ has also at most two zeros on boundary (Proposition 2.19). This formula can not be always factorized in linear factors like (80). For simplicity let us put (81) into the formula

$$\frac{\partial\phi_3}{\partial\nu}\frac{\partial\phi_3}{\partial\nu} + \frac{\alpha_{2,3}+\alpha_{3,2}}{\alpha_{3,3}}\frac{\partial\phi_2}{\partial\nu}\frac{\partial\phi_3}{\partial\nu} + \Big\{\frac{\alpha_{2,2}}{\alpha_{3,3}}\frac{\partial\phi_2}{\partial\nu} + \frac{\alpha_{2,4}}{\alpha_{3,3}}\frac{\partial}{\partial\nu}\Big(\frac{\partial}{\partial\nu}{}_P K_{\lambda_2(0)}\Big)\Big\}\frac{\partial\phi_2}{\partial\nu}.$$

When the following function

$$\Big(\frac{\alpha_{2,3}+\alpha_{3,2}}{\alpha_{3,3}}\frac{\partial\phi_2}{\partial\nu}\Big)^2 - 4\Big\{\frac{\alpha_{2,2}}{\alpha_{3,3}}\frac{\partial\phi_2}{\partial\nu} + \frac{\alpha_{2,4}}{\alpha_{3,3}}\frac{\partial}{\partial\nu}\Big(\frac{\partial}{\partial\nu}{}_P K_{\lambda_2(0)}\Big)\Big\}\frac{\partial\phi_2}{\partial\nu}$$

is not negative on an oen subset of $\partial\Omega$, considering the quadratic formula, we can not expect the factorization of (80) in linear factor consisting of $\frac{\partial\phi_2}{\partial\nu}$, $\frac{\partial\phi_3}{\partial\nu}$, and $\frac{\partial}{\partial\nu}\Big(\frac{\partial}{\partial\nu}{}_P K_{\lambda_2(0)}\Big)$ unlike (79). //

Corollary 2.24. *Let S_i, $i = 1, 2, 3, 4, 5$, be closed segments on $\partial\Omega$ which are disjoint one another. Let simple boundary functions $\mathfrak{G}_i$, $i = 1, 2, 3, 4, 5$, whose boundary supports are $\mathcal{S}_i$, respectively, be given and each $\mathfrak{G}_i$ do not vanish on S_i°. Then, there is at least one simple boundary function $\mathfrak{G}_k$, $1 \leq k \leq 5$, for which the value*

$$\int_{\partial\Omega}\mathfrak{G}_k\Big\{\sum_{2\leq i,j\leq 3}\alpha_{i,j}\frac{\partial\phi_i}{\partial\nu}\frac{\partial\phi_j}{\partial\nu}\Big\}dA$$

does not vanish, where constant coefficients $\alpha_{i,j}$ are not all zero.

We are to show our main proposition of section 2. Let us denote

$$\begin{cases} c^k_{ij} = -\int_\Omega\big\{\frac{d}{dt}{}_{t=0}\Delta_{J^{k}{}^*_t e}\phi_i(0)\big\}\phi_j(0)dxdy, & i,j \in \{2,3\}, \\ c^k_{24} = -\int_\Omega\big\{\frac{d}{dt}{}_{t=0}\Delta_{J^{k}{}^*_t e}\phi_2(0)\big\}\big\{\frac{\partial}{\partial\nu}{}_{(\zeta,\tau)=(P)}K_{\lambda_2(0)}\big\}dxdy. \end{cases} \tag{82}$$

To verify the proposition we apply significantly Hopf's boundary point lemma. It seems to be naturally true, but its proof is elementary and complicated. In the following Proposition 2.25 if a simple deformation J_t of Ω is given, the deformation denoted by $J_{\zeta t}$, $\zeta < 0$, can be defined from (22) in Definition 2.1.

Proposition 2.25. *Let ρ_2, ρ_3 and $\rho_4 > 0$ be real numbers such that $\rho_2 \neq \rho_3$, and let $\{\phi_2, \phi_3\}$ be an orthonormal basis of the second eigenspace of Δ_e in Ω associated with $\lambda_2(0)$, and assume that the nodal line of ϕ_2 be closed and meets $\partial\Omega$ at exactly one point P and ϕ_2 is positive in the nodal domain enclosed by the closed nodal line. Let eleven boundary functions $\mathfrak{G}_k$ of simple deformations J_t^k, $k = 1, 2, 3, ..., 11$, whose boundary supports $\mathcal{S}_k$ are disjointed each other be given, and let each $\mathfrak{G}_k$ does not vanish on the interior of its boundary support. Then, one can select four simple boundary functions $\mathfrak{G}_{k_j}$, and real numbers ζ_{k_j}, $j = 1, 2, 3, 4$, $1 \leq k_j \leq 11$, for which the followings are satisfied: Let us define a deformation $J : \Omega \times [0, \mathfrak{q}] \to \mathbb{R}^2$, $\mathfrak{q} \leq \min\{1, \frac{1}{\zeta_{k_1}}, \frac{1}{\zeta_{k_2}}, \frac{1}{\zeta_{k_3}}, \frac{1}{\zeta_{k_4}}\}$, with $\phi_3(\Omega_m)$-component eliminated which is represented by*

$$\uplus_{j=1}^4 J^{k_j}_{\zeta_{k_j} t} \uplus I_{c,t}(x, y), \quad 0 \leq t \leq \mathfrak{q},$$

where each $J_t^{k_j}$ is a simple deformation on $0 \leq t \leq 1$ with boundary function $\mathfrak{G}_{k_j}$, and $I_{c,t}$ is an inflation or deflation in c-rate. Then,

1) $\phi^*_{2,0}(t)$ and $\phi^*_{3,0}(t)$, $0 < t \leq 1$, converge to ϕ_2 and ϕ_3 as $t \to 0$, respectively,

2) $\frac{d}{dt}_{t=0}\lambda_2(t) = \rho_2$, $\frac{d}{dt}_{t=0}\lambda_3(t) = \rho_3$, and

3) $\frac{\partial}{\partial\nu}_P \frac{d}{dt}_{t=0}\phi^*_{2,0}(t) = \rho_4$ is satisfied.

Proof. From Proposition 2.14, Proposition 2.16, Proposition 2.17, Definition 2.5 and notation (82), it suffices to find J^{k_j}, $j = 1, 2, 3, 4$, for which the following system of linear equations has a solution $(\zeta_{k_1}, \cdots, \zeta_{k_4})$;

$$\begin{pmatrix} c^1_{22} & \cdots & c^4_{22} \\ c^1_{33} & \cdots & c^4_{33} \\ c^1_{23} & \cdots & c^4_{23} \\ c^1_{24} & \cdots & c^4_{24} \end{pmatrix} \begin{pmatrix} \zeta_{k_1} \\ \cdot \\ \cdot \\ \cdot \\ \zeta_{k_4} \end{pmatrix}$$

$$= \begin{pmatrix} -\int_\Omega \frac{d}{dt}_{t=0}\Big(\Delta_{(\uplus_{j=1}^4 J^{k_j}_{\zeta_{k_j} t})^* e}\phi_2\Big)\phi_2 \\ -\int_\Omega \frac{d}{dt}_{t=0}\Big(\Delta_{(\uplus_{j=1}^4 J^{k_j}_{\zeta_{k_j} t})^* e}\phi_3\Big)\phi_3 \\ -\int_\Omega \frac{d}{dt}_{t=0}\Big(\Delta_{(\uplus_{j=1}^4 J^{k_j}_{\zeta_{k_j} t})^* e}\phi_2\Big)\phi_3 \\ -\int_\Omega \frac{d}{dt}_{t=0}\Big(\Delta_{(\uplus_{j=1}^4 J^{k_j}_{\zeta_{k_j} t})^* e}\phi_2\Big)\frac{\partial}{\partial\nu}_P K_{\lambda_2(0)} \end{pmatrix} = \begin{pmatrix} \rho_2 + c\lambda_2(0) \\ \rho_3 + c\lambda_3(0) \\ 0 \\ \rho_4 - \tilde{\rho} \end{pmatrix}, \tag{83}$$

where $\tilde{\rho}$ denotes the increase of outer normal derivative at P of the $\phi_3(\Omega_m)$-component eliminated from $\uplus_{j=1}^4 J^{k_j}_{\zeta_{k_j} t}$. To find each J^{k_j} 4×4-matrix in (83) has to be transformed without change of rank into a regular matrix

$$\begin{pmatrix} d_1 & \cdot & \cdot & \cdot \\ 0 & d_2 & \cdot & \\ 0 & 0 & d_3 & \cdot \\ 0 & 0 & 0 & d_4 \end{pmatrix} \tag{84}$$

with $d_k \neq 0$ for $k = 1, 2, 3, 4$. Each d_i is calculated in

$$d_1 = c^1_{22},$$

$$d_2 = c^2_{33} - \frac{c^1_{33}}{c^1_{22}} c^2_{22},$$

$$d_3 = c^3_{23} - \frac{c^1_{23}}{c^1_{22}} c^3_{22} - \frac{c^2_{23} - \frac{c^1_{23}}{c^1_{22}} c^2_{22}}{d_2} \cdot \Big(c^3_{33} - \frac{c^1_{33}}{c^1_{22}} c^3_{22}\Big),$$

$$d_4 = \Big(c^4_{24} - \frac{c^1_{24}}{c^1_{22}} c^4_{22}\Big) - \Big\{ \frac{c^2_{24} - \frac{c^1_{24}}{c^1_{22}} c^2_{22}}{d_2} \Big\} \cdot \Big(c^4_{33} - \frac{c^1_{33}}{c^1_{22}} c^4_{22}\Big)$$
$$- \frac{\Big(c^3_{24} - \frac{c^1_{24}}{c^1_{22}} c^3_{22}\Big) - \Big\{ \frac{c^2_{24} - \frac{c^1_{24}}{c^1_{22}} c^2_{22}}{d_2} \Big\} \cdot \Big(c^3_{33} - \frac{c^1_{33}}{c^1_{22}} c^3_{22}\Big)}{d_3}$$
$$\cdot \Big\{ \Big(c^4_{23} - \frac{c^1_{23}}{c^1_{22}} c^4_{22}\Big) - \frac{c^2_{23} - \frac{c^1_{23}}{c^1_{22}} c^2_{22}}{d_2} \cdot \Big(c^4_{33} - \frac{c^1_{33}}{c^1_{22}} c^4_{22}\Big) \Big\}. \tag{85}$$

From Corollary 2.24 one can select by turn $\mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \in \{\mathfrak{G}_1, \cdots, \mathfrak{G}_{11}\}$ for which d_1, d_2, and d_3 do not vanish. For an example we obtain

$$d_2 := \int_\Omega \mathfrak{G}_{2_1} \big(\frac{\partial \phi_3}{\partial \nu} \frac{\partial \phi_3}{\partial \nu} - \frac{c^1_{33}}{c^1_{22}} \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_2}{\partial \nu} \big) \neq 0.$$

One can show that all $\mathfrak{G}_{1_1}$, $\mathfrak{G}_{2_1}$, and $\mathfrak{G}_{3_1}$ have to be distinct each other. Otherwise, for an example, while $1_1 = 2_1$, the entry $d_2 = 0$, since $c^1_{22} = c^2_{22}$ and $c^2_{33} = c^1_{33}$. When $\mathfrak{G}_{1_1}$, $\mathfrak{G}_{2_1}$, and $\mathfrak{G}_{3_1}$ are selected, the number of candidates for $\mathfrak{G}_{4_1}$ is eight. But, since function (81) can not be factorized in linear factors like (80), one can not select a simple boundary function $\mathfrak{G}_{4_j}$ for which d_4 does not vanish in the same way as we select $\mathfrak{G}_{1_1}$, $\mathfrak{G}_{2_1}$ and $\mathfrak{G}_{3_1}$. Accordingly, let us set

$$d_4 = c^4_{24} - \frac{c^1_{24}}{c^1_{22}} c^4_{22} - B_1 \big(c^4_{33} - \frac{c^1_{33}}{c^1_{22}} c^4_{22}\big)$$
$$- K_1 \Big\{ \Big(c^4_{23} - \frac{c^1_{23}}{c^1_{22}} c^4_{22}\Big) - B_2 \cdot \Big(c^4_{33} - \frac{c^1_{33}}{c^1_{22}} c^4_{22}\Big) \Big\}, \tag{86}$$

where

$$B_1 = \frac{c^2_{24} - \frac{c^1_{24}}{c^1_{22}} c^2_{22}}{d_2},$$
$$B_2 = \frac{c^2_{23} - \frac{c^1_{23}}{c^1_{22}} c^2_{22}}{d_2},$$
$$K_1 = - \frac{\big(c^3_{24} - \frac{c^1_{24}}{c^1_{22}} c^3_{22}\big) - \Big\{ \frac{c^2_{24} - \frac{c^1_{24}}{c^1_{22}} c^2_{22}}{d_2} \Big\} \cdot \big(c^3_{33} - \frac{c^1_{33}}{c^1_{22}} c^3_{22}\big)}{d_3}.$$

Let us denote by $\langle \mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l} \rangle$ triplet of boundary functions, $l = 1, 2, 3, ...$, such that each $\mathfrak{G}_{k_l}$, $k = 1, 2, 3$, is contained in $\{\mathfrak{G}_1, \cdots, \mathfrak{G}_{11}\}$, and makes d_k not vanish in order in a sense of Corollary 2.24. Let us define a function $\mathcal{A}\langle \mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l} \rangle$ on $\partial\Omega$ related to (86) by

$$\mathcal{A}\langle \mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l} \rangle$$
$$= \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_4}{\partial \nu} - \frac{c^1_{24}}{c^1_{22}} \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_2}{\partial \nu} - B_1 \big(\frac{\partial \phi_3}{\partial \nu} \frac{\partial \phi_3}{\partial \nu} - \frac{c^1_{33}}{c^1_{22}} \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_2}{\partial \nu} \big)$$
$$- K_1 \Big\{ \Big(\frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_3}{\partial \nu} - \frac{c^1_{23}}{c^1_{22}} \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_2}{\partial \nu} \Big) - B_2 \cdot \Big(\frac{\partial \phi_3}{\partial \nu} \frac{\partial \phi_3}{\partial \nu} - \frac{c^1_{33}}{c^1_{22}} \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_2}{\partial \nu} \Big) \Big\}, \tag{87}$$

where $\phi_4 := \frac{\partial}{\partial \nu}{}_P K_{\lambda_2(\Omega)}$, and c^k_{ij}, $k = 1, 2, 3$, is determined by $\mathfrak{G}_{k_l}$ in a sense of Corollary 2.24, (82), and Proposition 2.10.

Let us assume that (**assumption I.** although any triplet $\langle \mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l} \rangle$, $l = 1, 2, 3, ...$, be given, we have

$$\int_\Omega \mathfrak{G}_m \mathcal{A}\langle \mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l} \rangle = 0,$$

for any $\mathfrak{G}_m \in \{\mathfrak{G}_1, \cdots, \mathfrak{G}_{11}\} \setminus \{\mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l}\}$.) Let us assume that (**assumption II.** if we replace $\langle \mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \rangle$ with another triplet $\langle \mathfrak{G}_{1_2}, \mathfrak{G}_{2_2}, \mathfrak{G}_{3_2} \rangle$, then the coefficient K_1 defined by $\langle \mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \rangle$ turns into $K_1 + \gamma_{2,3}$, $0 \neq \gamma_{2,3} \in \mathbb{R}$.) From now on when we select a triplet $\langle \mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l} \rangle$ from $\{\mathfrak{G}_1, \cdots, \mathfrak{G}_{11}\}$, we will assume that if $l_1 \neq l_2$,

$$\mathfrak{G}_{1_{l_1}} \neq \mathfrak{G}_{1_{l_2}}.$$

Note that since the coefficient of function $\frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_4}{\partial \nu}$ in (87) is one, $\frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_4}{\partial \nu}$ is invariant under the choice $\langle \mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l} \rangle$. Coefficient of component $\frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_3}{\partial \nu}$ in (87) varies from K_1 to $K_1 + \gamma_{2,3}$ as $\langle \mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \rangle$ is replaced with $\langle \mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \rangle$, and therefore

$$\begin{aligned}&\mathcal{A}\langle \mathfrak{G}_{1_2}, \mathfrak{G}_{2_2}, \mathfrak{G}_{3_2} \rangle \\ =&\mathcal{A}\langle \mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \rangle + \gamma_{2,3} \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_3}{\partial \nu} + \gamma_{2,2} \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_2}{\partial \nu} + \gamma_{3,3} \frac{\partial \phi_3}{\partial \nu} \frac{\partial \phi_3}{\partial \nu}, \quad \gamma_{i,j} \in \mathbb{R}. \end{aligned} \tag{88}$$

From Proposition 2.23 function $\gamma_{2,3} \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_3}{\partial \nu} + \gamma_{2,2} \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_2}{\partial \nu} + \gamma_{3,3} \frac{\partial \phi_3}{\partial \nu} \frac{\partial \phi_3}{\partial \nu}$ has at most four zeros at boundary. Consequently, by Corollary 2.24 and by assumption I $\int_\Omega \mathfrak{G}_{l_1} \mathcal{A}\langle \mathfrak{G}_{1_2}, \mathfrak{G}_{2_2}, \mathfrak{G}_{3_2} \rangle \neq 0$ for a boundary function $\mathfrak{G}_{l_1} \in \{\mathfrak{G}_1, \cdots, \mathfrak{G}_{11}\} \setminus \cup_{l=1,2} \{1_l, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l}\}$. From assumption I it is impossible. Therefore, either assumption I or assumption II does not hold. If assumption I does not hold, our proof is done.

Then, we may assume that (**assumption III.** $\langle \mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \rangle$ replaced with another triplet $\langle \mathfrak{G}_{1_2}, \mathfrak{G}_{2_2}, \mathfrak{G}_{3_2} \rangle$, the coefficient K_1 is invariant.) Then, suppose that (**assumption IV.** the number $B_1 - K_1 B_2$ is variant when the triplet changes into $\langle \mathfrak{G}_{1_2}, \mathfrak{G}_{2_2}, \mathfrak{G}_{3_2} \rangle$.) Then, the coefficient of $\frac{\partial \phi_3}{\partial \nu} \frac{\partial \phi_3}{\partial \nu}$ in (87) equals to $B_1 - K_1 B_2$ and therefore it varies according to the changed triplet. By the same argument as preceding paragraph formula (88) holds with $\gamma_{3,3} \neq 0$. Then, Corollary 2.23 also implies that one can find a simple boundary function $\mathfrak{G}_{l_2}$ which makes $d_4 \neq 0$ among $\{\mathfrak{G}_1, \cdots, \mathfrak{G}_{11}\} \setminus \cup_{j=1}^2 \{\mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l}\}$. It is contradictory to either assumption I or assumption IV.

Thus we may assume that (**assumption V.** $\langle \mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \rangle$ changed into $\langle \mathfrak{G}_{1_2}, \mathfrak{G}_{2_2}, \mathfrak{G}_{3_2} \rangle$, $B_1 - K_1 B_2$ does not alter.) Then, under this assumption let us suppose that (**assumption VI.** $\langle \mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \rangle$ changed into $\langle \mathfrak{G}_{1_2}, \mathfrak{G}_{2_2}, \mathfrak{G}_{3_2} \rangle$, the coefficient $\frac{c_{24}^1}{c_{22}^1} - K_1 \frac{c_{23}^1}{c_{22}^1}$ in (87) alters.) Then, by assumption V the coefficient of $\frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_2}{\partial \nu}$ in (87) varies. By the same argument as the above Corollary 2.23 implies again that one can select a simple boundary function $\mathfrak{G}_{l_3}$ among $\{\mathfrak{G}_1, \cdots, \mathfrak{G}_{11}\} \setminus \cup_{l=1}^2 \{\mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l}\}$ for which $d_4 \neq 0$.

Therefore, let us suppose that (**assumption VII.** triplet $\langle \mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \rangle$ changed into $\langle \mathfrak{G}_{1_2},, \mathfrak{G}_{2_2}, \mathfrak{G}_{3_2} \rangle$, $\frac{c_{24}^1}{c_{22}^1} - K_1 \frac{c_{23}^1}{c_{22}^1} = K_2$ for a constant K_2.) Note that K_1 has been assumed to be invariant under this change of triplets. Thus, for two triplets $\langle \mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1} \rangle$ and $\langle \mathfrak{G}_{1_2},, \mathfrak{G}_{2_2}, \mathfrak{G}_{3_2} \rangle$ we have

$$c_{24}^1 - K_1 c_{23}^1 - K_2 c_{22}^1 = 0. \tag{89}$$

The procedure in the preceding paragraphs can be considered to progress in the same way for each pair of triplets $\{\mathfrak{G}_{1_1}, \mathfrak{G}_{2_1}, \mathfrak{G}_{3_1}\}$, $\{\mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathfrak{G}_{3_l}\}$, $j = 2, 3, 4, 5$, where $\mathfrak{G}_{1_{l_1}} \neq \mathfrak{G}_{1_{l_2}}$, if $l_1 \neq l_2$. Such pairs of triplets satisfy (89) for constants K_1 and K_2. Then, from Corollary 2.23 (89) does not hold, since the function

$$\begin{aligned}&\frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_4}{\partial \nu} - K_1 \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_3}{\partial \nu} - K_2 \frac{\partial \phi_2}{\partial \nu} \frac{\partial \phi_2}{\partial \nu} \\ =&\frac{\partial \phi_2}{\partial \nu} \Big(\frac{\partial \phi_4}{\partial \nu} - K_1 \frac{\partial \phi_3(\Omega)}{\partial \nu} - K_2 \frac{\partial \phi_2}{\partial \nu} \Big) \end{aligned} \tag{90}$$

does not vanish calculated with a boundary function $\mathfrak{G}_{l_4}$ which belongs to $\{\mathfrak{G}_{1_l} : l = 1, 2, 3, 4, 5\}$. For this, referring to Proposition 2.19, we observe that the right-hand side of (90) has at most three zeros on $\partial\Omega$. (If the nodal line of ϕ_2 is not closed, then (90) could have at most four zeros.) Thus, (89) fails for some triplet.

After all our assumption I is not true, that is, there is a quadruple $\{\mathfrak{G}_{1_l}, \mathfrak{G}_{2_l}, \mathcal{V}_{3_l}, \mathfrak{G}_{4_l}\}$ which makes d_4 not vanish. □

3 a Proof of Payne's Nodal Line Conjecture

In this section we assume the strict situation that the second eigenvalue of Ω is double and nodal line of a second eigenfunction in Ω is closed and touches the boundary. Then, Courant's nodal domain theorem [3] says that this nodal line meets boundary at exactly one point under the third order vanishing-system. Then, from Proposition 2.25, one can find a deformation $\mathcal{J}$ such that $\mathcal{J}_t(\Omega)$ has C^∞-boundary and has a unique normalized second eigenfunction whose nodal line is closed and separated from boundary for all t, $0 < t \le \mathfrak{q} \le 1$.

Let $\phi_{j,D}$ and $\lambda_j(D)$ denote the j-th normalized eigenfunction of a domain $D \subset \mathbb{R}^2$ and the j-th eigenvalue of D, respectively. The restiction of $\phi_{j,D}$ to D_0 is denoted by $\phi_{j,D}\big|_{D_0}$. Let us denote by

$$\Xi \subset \mathbb{R}^2$$

the rectangle $\{(x,y) : 0 < x < 2,\ 0 < y < 1\}$ whose corners are cut symmetrically and sufficiently slightly to be smoothen. According to [5] p.395, nodal line of $\phi_{2,\Xi}$ is $\{(x,y) : x = 1,\ 0 \le y \le 1\}$. Through inflation or deflation of $\Omega_1 := \Omega$, we set

$$\lambda_2(\Xi) = \lambda_2(\Omega_1). \tag{91}$$

By translation of Ω_1 we set

$$\mathfrak{F}_1 := \Omega_1 \cap \Xi \ne \emptyset, \quad \mathfrak{R}_1 := \Omega_1 \setminus \Xi \ne \emptyset.$$

We deform Ω_1 by a simple deformation $\mathcal{J}_t^1$ whose support lies in the closure of *remainder* $\mathfrak{R}_1$ so that $\mathcal{J}^1_{\mathfrak{q}_1}(\mathfrak{R}_1)$ may have thin and long i_2 bands $\mathfrak{S}_2^k$, $k = 1, 2, ..., i_2$ called *branch*, or have no more than i_2 discs $\mathfrak{B}_2^k$ with various radius called *blossom* attached to top of $\mathfrak{S}_2^k$. Otherwise, $\mathcal{J}_t^1$ tunnels into $\mathfrak{R}_1$. We denote this tunneled set by $\mathfrak{R}_1 \setminus \mathfrak{T}_2^k$. Thus we may write

$$\mathcal{J}^1(\Omega_1, \mathfrak{q}_1) := \mathcal{J}^1_{\mathfrak{q}_1}(\Omega_1) = \mathfrak{F}_1 \cup \Big\{\mathfrak{R}_1 \setminus \bigcup_k \mathfrak{T}_2^k\Big\} \cup \bigcup_k \mathfrak{S}_2^k \cup \bigcup_k \mathfrak{B}_2^k,$$

In fact this domain is regarded as the interior of the closed connected set $\overline{\mathfrak{F}_1} \cup \overline{\{\mathfrak{R}_1 \setminus \bigcup_k \mathfrak{T}_2^k\}} \cup \overline{\bigcup_k \mathfrak{S}_2^k} \cup \overline{\bigcup_k \mathfrak{B}_2^k}$.

According to proposition 2.25, a deformation $\mathcal{J}^1$ with $\phi_3(\Omega_m)$-component eliminated can be selected to satisfy the followings;
1) the nodal line of $\phi_{2,\mathrm{Im}(\mathcal{J}_t^1)}$ is closed and separated from boundary at all $t \in (0, \mathfrak{q}_1]$,
2) $\lambda_2\big(\mathrm{Im}(\mathcal{J}_t^1)\big)$ is simple at all $t \in (0, \mathfrak{q}_1]$, and
3) $\lambda_2(\Omega_1) = \lambda_2(\Xi) \lneq \lambda_2\big(\mathrm{Im}(\mathcal{J}^1_{\mathfrak{q}_1})\big)$.

Definition 3.1. Let us call such a deformation $\mathcal{J}_t^1$ a *splitting deformation*. Subsequently we fill up the set $\Xi \setminus \mathfrak{F}_1$ by a smooth deformation $\mathcal{F}^1 : \mathrm{Im}(\mathcal{J}^1_{\mathfrak{q}_1}) \times [0,1] \to \mathbb{R}^2$ called a *filling deformation* so that the support of $\mathcal{F}^1$ may lie in $\overline{\mathfrak{F}_1}$, $\mathcal{F}^1_s(\mathfrak{F}_1) \subset \Xi$ for all $s \in [0,1]$,

$$\mathcal{F}^1_{s_1}\big(\mathrm{Im}(\mathcal{J}^1_{\mathfrak{q}_1})\big) \subsetneq \mathcal{F}^1_{s_2}\big(\mathrm{Im}(\mathcal{J}^1_{\mathfrak{q}_1})\big),\ \text{if}\ 0 \le s_1 \lneq s_2 \le 1, \tag{92}$$

$$\lambda_2\big(\mathcal{F}^1_1\big(\mathrm{Im}(\mathcal{J}^1_{\mathfrak{q}_1})\big)\big) = \lambda_2(\Xi) \lneq \lambda_2\big(\mathrm{Im}(\mathcal{J}^1_{\mathfrak{q}_1})\big), \tag{93}$$

$\mathcal{F}^1_s\big(\mathrm{Im}(\mathcal{J}^1_{\mathfrak{q}_1})\big)$ may have the simple second eigenvalue at all $s \in [0,1)$, and nodal line of the second eigenfunction of $\mathcal{F}^1_s\big(\mathrm{Im}(\mathcal{J}^1_{\mathfrak{q}_1})\big)$ may not touch boundary at all $s \in [0,1)$. Let us denote

$$\Omega_2 := \mathcal{F}^1_1\big((\mathcal{J}^1_{\mathfrak{q}_1}(\Omega_1)\big),$$

and denote $\mathfrak{F}_2 := \Omega_2 \cap \Xi$, $\mathfrak{R}_2 := \Omega_2 \setminus \big(\mathfrak{F}_2 \cup \bigcup_k \mathfrak{S}_2^k \cup \bigcup_k \mathfrak{B}_2^k\big)$. We define Ω_m inductively by repeating above procedure;

$$\Omega_m := \mathcal{F}^{m-1}_1 \circ \mathcal{J}^{m-1}_{\mathfrak{q}_{m-1}}(\Omega_{m-1}), \quad m \ge 2. \qquad //$$

Definition 3.2. Let us select four simple deformations $\mathcal{J}_t^{m,k}$ with boundary supports $\mathcal{S}_m^k$, $k = 1, ..., 4$, and let they be $\uplus$-summands of a splitting deformation $\mathcal{J}_t^m := \uplus_{k=1}^4 \mathcal{J}_{\zeta_k t}^{m,k}$. We will call a simple deformation $\mathcal{J}^{m,j}$ *expansive*, if $\mathrm{Im}(\mathcal{J}_{t_1}^{m,j}) \subsetneq \mathrm{Im}(\mathcal{J}_{t_2}^{m,j})$, and *contractive,* if $\mathrm{Im}(\mathcal{J}_{t_1}^{m,j}) \supsetneq \mathrm{Im}(\mathcal{J}_{t_2}^{m,j})$ for $0 \le t_1 \lneq t_2 \le \mathfrak{q}_m$. //

Firstly, let us define an expansive deformation $\mathcal{J}_t^{m,j}$, $0 \le t \le \mathfrak{q}_m$, on its boundary support $\mathcal{S}_m^j$ as follows; denote the outer normal vector to $\mathcal{J}_t^{m,j}(\Omega_m)$ at each $(x,y) \in \mathcal{S}_m^j$, $0 \le t \le \mathfrak{q}_m$, by $\frac{\partial}{\partial \nu}_{\mathcal{J}_t^{m,j}(x,y)}$. Then, the followng has to be satisfied;

$$\lim_{\delta \to 0} \frac{\mathcal{J}_{0+\delta}^{m,j}(x,y) - \mathcal{J}_0^{m,j}(x,y)}{\delta} = \mathfrak{G}(x,y) \frac{\partial}{\partial \nu}_{\mathcal{J}_0^{m,j}(x,y)},$$

where $\mathfrak{G}$ is a non-negative smooth boundary function.

Let $\mathcal{J}_t^{m,j}$ be defined by a positive boundary function $\mathfrak{G}_m^j$ on $\mathcal{S}_m^j$, and let it turn out to correspond to an expansive simple deformation as the j-th $\uplus$-summand of a splitting deformation $\uplus_{k=1}^4 \mathcal{J}_{\zeta_k t}^{m,k}$. Then ζ_j is positive, and $\mathcal{J}_{\zeta_j t}^{m,j}$ deforms $\mathcal{S}_m^j$ to fill up a tunnel in $\mathfrak{R}_m$ from bottom, to form a *protrusion*, to grow branches, or to increase radii of blossoms. On the contrary, let $\mathcal{J}_t^{m,j}$ turn to be a contractive deformation. Then ζ_j is negative, and $\mathcal{J}_t^{m,j}$ tunnels into $\mathfrak{R}_m$, or diminishes branches and radii of blossoms. If it is assumed that ζ_j turns to be negative for the first time among the ordered set $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, the boundary function of $J_{\zeta_j t}^{m,j}$ does not need to be $\zeta_j \mathfrak{G}_m^j$, but we considering the linear system (83) and (84), $\mathfrak{G}_m^j$ can be redefined newly by an arbitrary smooth function on $\mathfrak{G}_m^j$ being negative on $\mathcal{S}_m^{j\circ}$ so that the coefficient ζ_j may alter only in positive sign. In this case for i, $i = j+1$, d_i may alter, and then $\mathfrak{G}_m^i$ may selected newly among given eleven boundary functions, and the column vector $\big(\rho_2 + c\lambda_2(0), \rho_3 + c\lambda_3(0), 0, \rho_4\big)^T$ in the system (83) and ζ_i may alter again. Provided ζ_i is also turned out to be negative, one may proceed in the same way as ζ_j. Consequently, when we designed to fill up the bottom of tunnel according to the given shape, if the deformation turns out to be contractive, then we can fill up the tunnel from bottom by designing at our disposal.

If expansive deformations is repeated, the branch reaches an appropriate length, and then we will inflate the portion around the top of branch. If expansiveness continues, the inflated portion becomes a blossom which forms a disc except for the negligible small portion of top of branch which is necessary to smooth the boundary.

If there are eleven blossoms $\mathfrak{B}_m^k$ in Ω_m, then we may deform only $\cup_k \mathfrak{B}_m^k$ by a splitting deformation $\mathcal{J}_t^m$ which inflates or deflates blossoms. Also we can deform Ω_m not to allow more than eleven blossoms.

Definition 3.3. For $f \in C^1(D)$ let us denote

$$\mathcal{E}[f] = \int_D |\nabla f|^2, \qquad \mathcal{E}^*[f] = \frac{\int_D |\nabla f|^2}{\int_D f^2}.$$

After translating $\mathfrak{S}_m^k$, we can set for each k without loss of generality

$$\mathfrak{S}_m^k = \Big\{(x,y) \in \mathbb{R}^2 : -\frac{l_m^k}{2} < x < \frac{l_m^k}{2},\ 0 < y < \omega_m^k\Big\}. \tag{94}$$

Although $\mathfrak{S}_m^k$ is bent, the validity of setting (94) in the succeeding arguments of proof is kept. (Refer to Corollary 3.11.) We denote for $-\frac{l_m^k}{2} \le \vartheta, x_0 < \frac{l_m^k}{2}$

$$\mathfrak{S}_{m,\vartheta}^k := \mathfrak{S}_m^k \cap \{(x,y) : -\vartheta \le x \le \vartheta\}, \qquad \mathfrak{s}_m^k(x_0) := \mathfrak{S}_m^k \cap \{x = x_0\}.$$

In what follows one side of the segment $\mathfrak{s}_m^k(0)$ connected with the blossom $\mathfrak{B}_m^k$ will be denoted by

$$\mathfrak{B}_m^{k+} := \mathfrak{S}_m^k \cap \Big\{(x,y) : 0 \le x \le l_m^k/2\Big\} \cup \mathfrak{B}_m^k, \tag{95}$$

and denote

$$\mathfrak{B}_m^{k-} := \Omega_m \setminus \mathfrak{B}_m^{k+}.$$

We denote by S_m the set of all indices k for which a blossom $\mathfrak{B}_m^k$ of Ω_m exists. We set for $k \in S_m$, $j \in \mathbb{N}$

$$\varrho_{j,m}^{k+} := \|\phi_{j,\Omega_m}|_{\mathfrak{B}_m^{k+}}\|_2, \quad \varrho_{j,m}^{k-} := \|\phi_{j,\Omega_m}|_{\mathfrak{B}_m^{k-}}\|_2 = \sqrt{1 - \big(\varrho_{j,m}^{k+}\big)^2}.$$

According to local estimation at boundary [6], Theorem 9.26 (It is stated in Theorem 3.7 later), one can notice $\big\{ \max_{\Omega_m} |\phi_{2,\Omega_m}| : \lambda_2(\Omega_m) = \lambda_2(\Xi),\ m = 1, 2, 3, ...\big\}$ has an upper bound. Then, by exponential decay theorem presented below in Lemma 3.1, there are positive constants C_1 and C_2 which depend only on the value $\lambda_2(\Omega_m)$ such that

$$\max_{\mathfrak{s}_m^k(\vartheta)} |\phi_{2,\Omega_m}| < C_1 \exp\Big(- C_2 \frac{\min\big\{|\frac{1}{2} l_m^k - \vartheta|,\ \ | - \frac{1}{2} l_m^k - \vartheta|\big\}}{\omega_m^k}\Big),$$

where ω_m denotes the width of $\mathfrak{S}_m^k$. We may assume ω_m^k are the same for all branches $\mathfrak{S}_m^k$ of Ω_m. Also l_m^k may be assumed to be constant for all $k \in S_m$. Let us denote

$$\Theta_{\omega_m,\vartheta} := C_1 \exp\Big(- C_2 \frac{\min\big\{|\frac{1}{2} l_m^k - \vartheta|,\ \ | - \frac{1}{2} l_m^k - \vartheta|\big\}}{\omega_m}\Big).$$

Fom now on the symbols C_1, C_2, C_3, etc. will denote positive absolute constants. //

Lemma 3.1. (Exponential Decay Theorem, extracted from [7]) *Let*

$$V = \big\{x + iy : 0 < y < 1/N, \quad 0 < x < x_m\big\}, \quad \frac{1}{N} << x_m,$$

and V_1 and V_2 be its vertical sides. Let W be a simply connected domain in $\mathbb{R}^2$ such that

$$\overline{W} \cap V_i \neq \emptyset, \quad W \subset V, \quad \text{and} \quad \partial W \setminus \{V_1 \cup V_2\} \subset V.$$

Let $\big(\Delta_e + \lambda\big)\mathfrak{u} = 0$ in W, $\sup_{V_i} |\mathfrak{u}| = e_i$, and $\mathfrak{u} = 0$ on $\partial W \setminus (V_1 \cup V_2)$. Let us denote $e = \max\{e_1, e_2\}$. Then,

$$|\mathfrak{u}(z)| \leq C_3 e \exp\big(- C_4 N \cdot \mathit{distance}(V_i, z)\big), \quad z \in W, \tag{96}$$

for constants C_3, C_4 which are independent of N and distance(V_i, z).

Proof. Let v be the solution of $\big(\Delta_e + \lambda\big)v = 0$ in V with $v = e = \max\{e_1, e_2\}$ on the vertical sides V_i, $i = 1, 2$, of V, and zero on the horizontal sides of V. Firstly, let $\mathfrak{u} \geq 0$ in an open subset $\mathcal{U}_1 \subset W$. Since $\Delta_e(v - \mathfrak{u}) + \lambda(v - \mathfrak{u}) = 0$ in W, from the weak maximum principle ([6] p 179, Theorem 8.1.) we have

$$\inf_W (v - \mathfrak{u}) \geq \inf_{\partial W} (v - \mathfrak{u})^- = 0,$$

where $(v - \mathfrak{u})^- = \min\{v - \mathfrak{u}, 0\}$. It implies that $\mathfrak{u} \leq v$ in $\mathcal{U}_1$. Secondly, if $\mathfrak{u} < 0$ in a sub-domain $\mathcal{U}_2 \subset W$, then

$$\inf_W (v + \mathfrak{u}) \geq \inf_{\partial W} (v + \mathfrak{u})^- = 0,$$

and it implies that $-\mathfrak{u} \leq v$ in $\mathcal{U}_2$. These two inequalities imply $v \geq |\mathfrak{u}|$ in W. Straightforward exponential decay estimation for v implies (96). □

Proposition 3.2. *There are C_i, $i = 5, 6, 7, 8, 9$, for which the followings are satisfied:*
(i) For each $k \in S_m$

$$\Big|\mathcal{E}\Big[\phi_{1,\Omega_m}|_{\mathfrak{B}_m^{k-}}\Big] - \mathcal{E}\Big[\big(\varrho_{1,m}^{k-}\big)\phi_{1,\mathfrak{B}_m^{k-}}\Big]\Big| < C_5 \frac{1}{\omega_m} \Theta_{\omega_m,\vartheta}^2.$$

(ii) Let us suppose that $\lambda_2(\mathfrak{B}_m^{k-}) - \lambda_1(\mathfrak{B}_m^{k-})$ is bigger than a constant $C_6 > 0$ for all $m \in \mathbb{N}$. Then,

$$\Big\|\phi_{1,\Omega_m}|_{\mathfrak{B}_m^{k-}} - \varrho_{1,m}^{k-}\phi_{1,\mathfrak{B}_m^{k-}}\Big\|_2^2 < C_7 \frac{1}{\omega_m} \Theta_{\omega_m,\vartheta}^2.$$

(iii)

$$\Big|\mathcal{E}\big[\phi_{1,\Omega_m}\big|_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\big]-\mathcal{E}\Big[\Big\{1-\sum_{k\in S_m}(\varrho_{1,m}^{k+})^2\Big\}^{\frac{1}{2}}\phi_{1,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\Big]\Big|<C_8\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta},\quad \textit{and}$$

(iv) Let us assume that $\lambda_2(\cap_{k\in S_m}\mathfrak{B}_m^{k-})-\lambda_1(\cap_{k\in S_m}\mathfrak{B}_m^{k-})$ is bigger than a constant $C_6>0$ for all $m\in\mathbb{N}$. Then,

$$\Big\|\phi_{1,\Omega_m}\big|_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}}-\Big\{1-\sum_{k\in S_m}(\varrho_{1,m}^{k+})^2\Big\}^{\frac{1}{2}}\phi_{1,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\Big\|_2^2<C_9\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}.$$

(v) Inequalities (i) and (ii) are also valid $\mathfrak{B}_m^{k-}$ replaced by $\mathfrak{B}_m^{k+}$.

Proof. (i) Let us define a piecewise $C^2(\mathfrak{B}_m^{k-})$-function h^- by

$$h^-(x,y)=\begin{cases}+\frac{\phi_{1,\Omega_m}(0,y)}{\vartheta}x+\phi_{1,\Omega_m}(0,y), & (x,y)\in\mathfrak{S}^k_{m,\vartheta}\cap\mathfrak{B}_m^{k-},\\ 0, & (x,y)\in\mathfrak{B}_m^{k-}\setminus\mathfrak{S}^k_{m,\vartheta}.\end{cases}\tag{97}$$

Then, since $\phi_{1,\mathfrak{B}_m^{k-}}$ has minimum energy among normalized functions in $C_0^2(\mathfrak{B}_m^{k-})$,

$$\big\|\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}-h^-\big\|_2^2\ \mathcal{E}\big[\phi_{1,\mathfrak{B}_m^{k-}}\big]\le\mathcal{E}\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}-h^-\big].\tag{98}$$

Also since $\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}$ has minimum energy among $C^2(\mathfrak{B}_m^{k-})$-functions which have the same boundary value and the same L_2-norm as $\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}$,

$$\big\|\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big\|_2^2\ \mathcal{E}\big[\phi_{1,\mathfrak{B}_m^{k-}}+h^-\big]\ge\big\|\phi_{1,\mathfrak{B}_m^{k-}}+h^-\big\|_2^2\ \mathcal{E}\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big].\tag{99}$$

Let us note the inequality

$$\begin{aligned}&\int_{\mathfrak{B}_m^{k-}}\Big[\Big\{\frac{\partial}{\partial x}\big(\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}-h^-\big)\Big\}^2+\Big\{\frac{\partial}{\partial y}\big(\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}-h^-\big)\Big\}^2\Big]\\ =&\mathcal{E}\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big]-\int_{\mathfrak{B}_m^{k-}}2\frac{\partial}{\partial x}\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\frac{\partial}{\partial x}h^--\int_{\mathfrak{B}_m^{k-}}2\frac{\partial}{\partial y}\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\frac{\partial}{\partial y}h^-\\ &+\int_{\mathfrak{B}_m^{k-}}\Big\{\Big(\frac{\partial}{\partial x}h^-\Big)^2+\Big(\frac{\partial}{\partial y}h^-\Big)^2\Big\}.\end{aligned}$$

From [6] Theorem 4.6 and Theorem 4.11 (Refer to Theorem 3.4 and Theorem 3.5 which will be stated later.) one can induce for $i,j=0,1,2,\ 0\le i+j\le 2$,

$$\begin{aligned}&\frac{\partial^{i+j}}{\partial x^i\partial y^j}\phi_{1,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}\\ \le&C_{10}\frac{1}{\omega_m^{|i+j|}}\sup_{\mathfrak{S}^k_{m,\vartheta}}|\phi_{1,\Omega_m}|\\ \le&C_{11}\frac{1}{\omega_m^{|i+j|}}\Theta_{\omega_m,\vartheta},\end{aligned}\tag{100}$$

where ω_m is the given width of branches of Ω_m. Note that Poisson's equation $\Delta u=w$ in [6] is satisfied by $u=\phi_{1,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}$ for $w=-\lambda_1(\Omega_m)\phi_{1,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}$. We have by (100)

$$\begin{aligned}&\Big|\int_{\mathfrak{B}_m^{k-}}2\frac{\partial}{\partial x}\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\frac{\partial}{\partial x}h^-\Big|\\ \le&2\Big\{\int_{\mathfrak{B}_m^{k-}\cap\mathfrak{S}^k_{m,\vartheta}}\Big|\frac{\partial}{\partial x}\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\Big|^2\cdot\int_{\mathfrak{B}_m^{k-}}\Big|\frac{\partial}{\partial x}h^-\Big|^2\Big\}^{\frac{1}{2}}\\ \le&2\Big\{C_{11}^2\frac{1}{\omega_m^2}\Theta^2_{\omega_m,\vartheta}\omega_m\vartheta\cdot\Theta^2_{\omega_m,\vartheta}\frac{1}{\vartheta^2}\omega_m\vartheta\Big\}^{\frac{1}{2}}=2C_{11}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}.\end{aligned}\tag{101}$$

Similarly, from (100)

$$\Big|\int_{\mathfrak{B}_m^{k-}} 2\frac{\partial}{\partial y}\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\frac{\partial}{\partial y}h^-\Big|$$
$$\leq 2\Big\{C_{11}^2\frac{1}{\omega_m^2}\Theta^2_{\omega_m,\vartheta}\omega_m\vartheta\cdot C_{11}^2\frac{1}{\omega_m^2}\Theta^2_{\omega_m,\vartheta}\omega_m\vartheta\Big\}^{\frac{1}{2}} = 2C_{11}^2\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}\vartheta.$$

From (100) $\mathcal{E}\big[h^-\big]$ is also bounded above to a constant times $\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}$. Thus, we can conclude

$$\mathcal{E}\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}} - h^-\big] \leq \mathcal{E}\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big] + C_{12}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}. \tag{102}$$

Similarly, one can show

$$\mathcal{E}\big[\phi_{1,\mathfrak{B}_m^{k-}} + h^-\big] \leq \mathcal{E}\big[\phi_{1,\mathfrak{B}_m^{k-}}\big] + C_{13}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}. \tag{103}$$

Thus, from (98)

$$\begin{aligned}
&\Big\{\big\|\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big\|_2^2 - \|\phi_{1,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}\cap\mathfrak{B}_m^{k-}}\|_2^2\Big\}\ \mathcal{E}\big[\phi_{1,\mathfrak{B}_m^{k-}}\big]\\
\leq&\big\|\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}} - h^-\big\|_2^2\ \mathcal{E}\big[\phi_{1,\mathfrak{B}_m^{k-}}\big]\\
\leq&\mathcal{E}\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}} - h^-\big]\\
\leq&\mathcal{E}\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big] + C_{12}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}.
\end{aligned} \tag{104}$$

Since $\big\|\phi_{1,\mathfrak{B}_m^{k-}} + h^-\big\|_2^2 \geq \big\|\phi_{1,\mathfrak{B}_m^{k-}}\big\|_2^2 - \big\|\phi_{1,\mathfrak{B}_m^{k-}}\big|_{\mathfrak{S}^k_{m,\vartheta}\cap\mathfrak{B}_m^{k-}}\big\|_2^2$, from (99)

$$\begin{aligned}
&\big\|\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big\|_2^2\Big\{\mathcal{E}\big[\phi_{1,\mathfrak{B}_m^{k-}}\big] + C_{13}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}\Big\}\\
\geq&\big\|\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big\|_2^2\ \mathcal{E}\big[\phi_{1,\mathfrak{B}_m^{k-}} + h^-\big]\\
\geq&\big\|\phi_{1,\mathfrak{B}_m^{k-}} + h^-\big\|_2^2\ \mathcal{E}\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big]\\
\geq&\Big\{\big\|\phi_{1,\mathfrak{B}_m^{k-}}\big\|_2^2 - \big\|\phi_{1,\mathfrak{B}_m^{k-}}\big|_{\mathfrak{S}^k_{m,\vartheta}\cap\mathfrak{B}_m^{k-}}\big\|_2^2\Big\}\mathcal{E}\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big].
\end{aligned} \tag{105}$$

From (104) and (105) (i) is verified.
(ii) For proof it suffices to show

$$\big\|\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}} - h^- - \varrho_{1,m}^{k-}\phi_{1,\mathfrak{B}_m^{k-}}\big\|_2^2 < C_{14}\Theta^2_{\omega_m,\vartheta}.$$

Let us set

$$\alpha_j = \int_{\mathfrak{B}_m^{k-}}\Big(\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}} - h^-\Big)\cdot\phi_{j,\mathfrak{B}_m^{k-}},\quad j\in\mathbb{N}.$$

Note the formula

$$\int_{\mathfrak{B}_m^{k-}}\nabla f\cdot\nabla g = -\int_{\mathfrak{B}_m^{k-}}(\Delta f)g,$$

for piecewise $C^2(\mathfrak{B}_m^{k-})$ functions $f,g\in C_0^0(\mathfrak{B}_m^{k-})$. Since $\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}} - h^-$ vanishes on $\partial\mathfrak{B}_m^{k-}$, from (i) and (102) we have

$$\begin{aligned}
C_{15}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta} \geq& \Big|\mathcal{E}\Big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}} - h^-\Big] - (\varrho_{1,m}^{k-})^2\mathcal{E}^*\Big[\phi_{1,\mathfrak{B}_m^{k-}}\Big]\Big|\\
=&\Big|\sum_{j=1}^{\infty}\alpha_j^2\lambda_j(\mathfrak{B}_m^{k-}) - (\varrho_{1,m}^{k-})^2\lambda_1(\mathfrak{B}_m^{k-})\Big|.
\end{aligned} \tag{106}$$

Since $\left(\varrho_{1,m}^{k-}\right)^2 - C_{16}\Theta_{\omega_m,\vartheta}^2 < \|\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}} - h^-\|_2^2 < \left(\varrho_{1,m}^{k-}\right)^2 + C_{16}\Theta_{\omega_m,\vartheta}^2$, we have

$$\left|\left(\varrho_{1,m}^{k-}\right)^2 - \sum_{j=1}^{\infty}\alpha_j^2\right| < C_{16}\Theta_{\omega_m,\vartheta}^2. \tag{107}$$

Then, from (106) and (107) we have

$$\begin{aligned} C_{17}\frac{1}{\omega_m}\Theta_{\omega_m,\vartheta}^2 \geq & \left|\sum_{j\in\mathbb{N}}\alpha_j^2\lambda_j(\mathfrak{B}_m^{k-}) - \sum_{j\in\mathbb{N}}\alpha_j^2\lambda_1(\mathfrak{B}_m^{k-})\right| \\ = & \sum_{j\in\mathbb{N}\setminus\{1\}}\alpha_j^2\Big\{\lambda_j(\mathfrak{B}_m^{k-}) - \lambda_1(\mathfrak{B}_m^{k-})\Big\} \\ > & \sum_{j\in\mathbb{N}\setminus\{1\}}\alpha_j^2\Big\{\lambda_2(\mathfrak{B}_m^{k-}) - \lambda_1(\mathfrak{B}_m^{k-})\Big\}. \end{aligned}$$

Thus, $\sum_{j\in\mathbb{N}\setminus\{1\}}\alpha_j^2 < \frac{C_{17}\frac{1}{\omega_m}\Theta_{\omega_m,\vartheta}^2}{\lambda_2(\mathfrak{B}_m^{k-})-\lambda_1(\mathfrak{B}_m^{k-})}$. Then, from (107) when $\left(\varrho_{1,m}^{k-}\right)^2 - \sum_{j=1}^{\infty}\alpha_j^2 \geq 0$,

$$0 \leq \left(\varrho_{1,m}^{k-}\right)^2 - \alpha_1^2 < C_{16}\Theta_{\omega_m,\vartheta}^2 + \frac{C_{17}\frac{1}{\omega_m}\Theta_{\omega_m,\vartheta}^2}{\lambda_2(\mathfrak{B}_m^{k-}) - \lambda_1(\mathfrak{B}_m^{k-})},$$

otherwise when $\left(\varrho_{1,m}^{k-}\right)^2 - \sum_{j=1}^{\infty}\alpha_j^2 < 0$,

$$-\frac{C_{17}\frac{1}{\omega_m}\Theta_{\omega_m,\vartheta}^2}{\lambda_2(\mathfrak{B}_m^{k-}) - \lambda_1(\mathfrak{B}_m^{k-})} < \alpha_1^2 - \sum_{j\in\mathbb{N}}\alpha_j^2 < \alpha_1^2 - \left(\varrho_{1,m}^{k-}\right)^2 \leq \sum_{j=1}^{\infty}\alpha_j^2 - \left(\varrho_{1,m}^{k-}\right)^2 < C_{16}\Theta_{\omega_m,\vartheta}^2.$$

Therefore, α_1^2 approximates to $\left(\varrho_{1,m}^{k-}\right)^2$, and (ii) follows.
(iii) We define h^- for each $k \in S_m$ by

$$h^-(x,y) = \begin{cases} +\frac{\phi_{1,\Omega_m}(0,y)}{\vartheta}x + \phi_{1,\Omega_m}(0,y), & (x,y) \in \mathfrak{S}_{m,\vartheta}^k \cap \mathfrak{B}_m^{k-}, \\ 0, & (x,y) \in \bigcap_{k\in S_m}\left(\mathfrak{B}_m^{k-} \setminus \mathfrak{S}_{m,\vartheta}^k\right). \end{cases}$$

The argument of proof of (iii) is similar to (i).
(iv) The way of proof of (iv) is the same as (ii).
(v) Proof of (v) follows those of (i) and (ii). □

Proposition 3.3. *Assume that for a $k \in S_m$*

$$\mathcal{E}^*[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}] = \mathcal{E}^*[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}] + \lambda_{1,m}^{k*}, \quad \lambda_{1,m}^{k*} > 0. \tag{108}$$

Then, we have

$$\left(\varrho_{1,m}^{k+}\right)^2 \cdot \lambda_{1,m}^{k*} < C_{18}\frac{1}{\omega_m}\Theta_{\omega_m,\vartheta}^2. \tag{109}$$

Proof. Let us define $\phi_{1,\Omega_m}^{\epsilon}$ by

$$\phi_{1,\Omega_m}^{\epsilon}(x,y) = \begin{cases} \big(1-\epsilon\big)\phi_{1,\Omega_m}(x,y), & (x,y) \in \overline{\mathfrak{B}_m^{k+}}, \\ f(\epsilon)\phi_{1,\Omega_m}(x,y), & (x,y) \in \mathfrak{B}_m^{k-}, \end{cases} \tag{110}$$

for $0 < \epsilon < \frac{1}{2}$, where f is a differentiable real function, and

$$f(\epsilon)^2\left(\varrho_{1,m}^{k-}\right)^2 + (1-\epsilon)^2\left(\varrho_{1,m}^{k+}\right)^2 = 1. \tag{111}$$

Then, $\|\phi_{1,\Omega_m}^{\epsilon}\|_2 = 1$. Let us define $\tilde{\phi}_{1,\Omega_m}^{\epsilon}$ by

$$\tilde{\phi}_{1,\Omega_m}^{\epsilon}(x,y) = \begin{cases} \phi_{1,\Omega_m}^{\epsilon}(x,y), & (x,y) \in \Omega_m \setminus \mathfrak{S}_{m,\vartheta}^k, \\ h_m^{k,\epsilon}(x,y)\phi_{1,\Omega_m}, & (x,y) \in \mathfrak{S}_{m,\vartheta}^k, \end{cases} \tag{112}$$

where $h_m^{k,\epsilon} : \mathfrak{S}_{m,\vartheta}^k \to \mathbb{R}$ is a smooth function defined by

$$h_m^{k,\epsilon}(x,y) = \frac{1-\epsilon-f(\epsilon)}{2\vartheta}x + \frac{1-\epsilon+f(\epsilon)}{2}. \tag{113}$$

We are to calculate an upper bound of

$$\begin{aligned}&\mathcal{E}\big[\tilde{\phi}^{\epsilon}_{1,\Omega_m}\big] - \mathcal{E}\big[\phi_{1,\Omega_m}\big]\\ =&\mathcal{E}\big[\phi^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big] + \mathcal{E}\big[\phi^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\big] - \mathcal{E}\big[\phi_{1,\Omega_m}\big]\\ &+\Big(\mathcal{E}[\tilde{\phi}^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}] - \mathcal{E}[\phi^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}]\Big)\end{aligned} \tag{114}$$

Direct calculation yields

$$\begin{aligned}&\mathcal{E}\big[\phi^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big] + \mathcal{E}\big[\phi^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\big]\\ =&f(\epsilon)^2\big(\varrho^{k-}_{1,m}\big)^2\Big\{\mathcal{E}^*[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}] - \lambda_1^{k*}\Big\} + (1-\epsilon)^2\big(\varrho^{k+}_{1,m}\big)^2\mathcal{E}^*[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}],\end{aligned} \tag{115}$$

From (115), (111), and from the equality $1-\big(\varrho^{k+}_{1,m}\big)^2 = \big(\varrho^{k-}_{1,m}\big)^2$, we have

$$\begin{aligned}&\mathcal{E}\big[\phi^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\big] + \mathcal{E}\big[\phi^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\big] - \mathcal{E}\big[\phi_{1,\Omega_m}\big]\\ =&\big\{1-\big(\varrho^{k+}_{1,m}\big)^2\big\}\big\{\mathcal{E}^*\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\big] - \lambda^{k*}_{1,m}\big\}\\ &-(-2\epsilon+\epsilon^2)\big(\varrho^{k+}_{1,m}\big)^2\big\{\mathcal{E}^*\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\big] - \lambda^{k*}_{1,m}\big\}\\ &+\big(\varrho^{k+}_{1,m}\big)^2\mathcal{E}^*\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\big] + (-2\epsilon+\epsilon^2)\big(\varrho^{k+}_{1,m}\big)^2\mathcal{E}^*\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\big]\\ &-\big\{1-\big(\varrho^{k+}_{1,m}\big)^2\big\}\big\{\mathcal{E}^*\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\big] - \lambda^{k*}_{1,m}\big\} - \big(\varrho^{k+}_{1,m}\big)^2\mathcal{E}^*\big[\phi_{1,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\big]\\ =&(-2\epsilon+\epsilon^2)\big(\varrho^{k+}_{1,m}\big)^2\lambda^{k*}_{1,m}.\end{aligned} \tag{116}$$

The other part in the right hand of (114) is calculated as follows;

$$\begin{aligned}&\mathcal{E}\big[\tilde{\phi}^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}\big] - \mathcal{E}\big[\phi^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}\big]\\ =&\int_{\mathfrak{S}^k_{m,\vartheta}}\Bigg[\Bigg\{\frac{1-\epsilon-f(\epsilon)}{2\vartheta}\phi_{1,\Omega_m} + \Bigg(\frac{1-\epsilon-f(\epsilon)}{2\vartheta}x + \frac{1-\epsilon+f(\epsilon)}{2}\Bigg)\frac{\partial\phi_{1,\Omega_m}}{\partial x}\Bigg\}^2\\ &\qquad+\Bigg(\frac{1-\epsilon-f(\epsilon)}{2\vartheta}x + \frac{1-\epsilon+f(\epsilon)}{2}\Bigg)^2\Bigg(\frac{\partial\phi_{1,\Omega_m}}{\partial y}\Bigg)^2\Bigg]dxdy\\ &-\int_{\mathfrak{S}^k_{m,\vartheta}\cap\mathfrak{B}_m^{k-}} f(\epsilon)^2\big|\nabla\phi_{1,\Omega_m}\big|^2 - \int_{\mathfrak{S}^k_{m,\vartheta}\cap\mathfrak{B}_m^{k+}}(1-\epsilon)^2\big|\nabla\phi_{1,\Omega_m}\big|^2.\end{aligned} \tag{117}$$

Note that $f(\epsilon)^2 = 1+(2\epsilon-\epsilon^2)\Big(\frac{\varrho^{k+}_{1,m}}{\varrho^{k-}_{1,m}}\Big)^2 = 1+a_1\epsilon+o(\epsilon) = 1+O(\epsilon)$, $\big(1-\epsilon-f(\epsilon)\big)^2 = a_2\epsilon+o(\epsilon)$, and $\big(1-\epsilon+f(\epsilon)\big)^2 = 2+a_3\epsilon+o(\epsilon)$ for real constants $a_i, i=1,2,3$. Then, from (100) and from (117) eliminating the second degree term ϵ^2 for a sufficiently small $\epsilon>0$, we can induce

$$\Big|\mathcal{E}[\tilde{\phi}^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}] - \mathcal{E}[\phi^{\epsilon}_{1,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}]\Big| < O(\epsilon)\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta} \le \epsilon C_{19}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}. \tag{118}$$

Consequently,

$$\begin{aligned}&\mathcal{E}\big[\tilde{\phi}^{\epsilon}_{1,\Omega_m}\big] - \mathcal{E}\big[\phi_{1,\Omega_m}\big]\\ \le&-(2\epsilon-\epsilon^2)\big(\varrho^{k+}_{1,m}\big)^2\lambda^{k*}_{1,m} + \epsilon C_{19}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}.\end{aligned} \tag{119}$$

If a constant κ_ϵ makes $\left\|\kappa_\epsilon \tilde{\phi}^\epsilon_{1,\Omega_m}\right\|_2 = 1$, then we have $\left|\kappa_\epsilon - 1\right| < \sqrt{C_{20}\epsilon\Theta^2_{\omega_m,\vartheta}}$, since $\left|\|\tilde{\phi}^\epsilon_{1,\Omega_m}\|_2^2 - \|\phi^\epsilon_{1,\Omega_m}\|_2^2\right| < C_{20}\epsilon\Theta^2_{\omega_m,\vartheta}$ for a constant C_{20}, and since $\|\phi^\epsilon_{1,\Omega_m}\|_2^2 = 1$. Therefore inequality (119) still holds when one replaces $\tilde{\phi}^\epsilon_{1,\Omega_m}$ and C_{19} by normalized function $\kappa_\epsilon\tilde{\phi}^\epsilon_{1,\Omega_m}$ and another constant C_{21}, respectively. Since eigenfunction ϕ_{1,Ω_m} has minimal energy,

$$0 \le \mathcal{E}\left[\kappa_\epsilon\tilde{\phi}^\epsilon_{1,\Omega_m}\right] - \mathcal{E}\left[\phi_{1,\Omega_m}\right] < -(2\epsilon - \epsilon^2)\left(\varrho^{k+}_{1,m}\right)^2\lambda^{k*}_{1,m} + \epsilon C_{21}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}.$$

and then setting ϵ small enough, we have (109). □

Remark 3.1. Let us denote the differentiable function $N^\epsilon_{1,m}$ of ϵ-variable by

$$\left|\mathcal{E}[\tilde{\phi}^\epsilon_{1,\Omega_m|\mathfrak{S}^k_{m,\vartheta}}] - \mathcal{E}[\phi^\epsilon_{1,\Omega_m|\mathfrak{S}^k_{m,\vartheta}}]\right| := N^\epsilon_{1,m} < O(\epsilon)\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta} \le \epsilon C_{19}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}.$$

$$0 \le \mathcal{E}\left[\kappa_\epsilon\tilde{\phi}^\epsilon_{1,\Omega_m}\right] - \mathcal{E}\left[\phi_{1,\Omega_m}\right] = -(2\epsilon - \epsilon^2)\left(\varrho^{k+}_{1,m}\right)^2\lambda^{k*}_{1,m} + N^\epsilon_{1,m}.$$

Since $\epsilon = 0$ is a minimum point of $\mathcal{E}\left[\kappa_\epsilon\tilde{\phi}^\epsilon_{1,\Omega_m}\right] - \mathcal{E}\left[\phi_{1,\Omega_m}\right]$,

$$0 = \frac{d}{d\epsilon}_{\epsilon\to 0}\left\{\mathcal{E}\left[\kappa_\epsilon\tilde{\phi}^\epsilon_{1,\Omega_m}\right] - \mathcal{E}\left[\phi_{1,\Omega_m}\right]\right\} = -2\left(\varrho^{k+}_{1,m}\right)^2\lambda^{k*}_{1,m} + \frac{d}{d\epsilon}_{\epsilon\to 0}N^\epsilon_{1,m}.$$

Therefore, (109) is described as

$$\left(\varrho^{k+}_{1,m}\right)^2 = \frac{\frac{d}{d\epsilon}_{\epsilon\to 0}N^\epsilon_{1,m}}{\lambda^{k*}_{1,m}} < \frac{C_{18}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}}{\lambda^{k*}_{1,m}}. \qquad //$$

We call f *uniformly Hölder continuous with exponent* α *in* Ω, iff the quantity

$$[f]_{\alpha;\Omega} = \sup_{\substack{x,y\in\Omega \\ x\ne y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}, \quad 0 < \alpha \le 1,$$

is finite. The Hölder spaces $C^{k,\alpha}(\Omega)$ are defined as the subspaces of $C^k(\Omega)$ consisting of functions whose k-th order partial derivatives are uniformly Hölder continuous with exponent α in Ω. For simplicity we write $C^{0,\alpha}(\Omega) = C^\alpha(\Omega)$. By setting $C^{k,0}(\Omega) = C^k(\Omega)$, we may include $C^k(\Omega)$ spaces among the $C^{k,\alpha}(\Omega)$ spaces for $0 \le \alpha \le 1$.

Let us set

$$[u]_{k,0;\Omega} = |D^k u|_{0;\Omega} = \sup_{|\beta|=k}\sup_{\Omega}|D^\beta u|, \quad \beta; \text{a multi-index},$$
$$[u]_{k,\alpha;\Omega} = [D^k u]_{\alpha;\Omega} = \sup_{|\beta|=k}[D^\beta u]_{\alpha;\Omega}.$$

With these semi-norms, we can define the related norm

$$\|u\|_{C^k(\overline{\Omega})} = |u|_{k;\Omega} = |u|_{k,0;\Omega} = \sum_{j=0}^k [u]_{j,0;\Omega} = \sum_{j=0}^k |D^j u|_{0;\Omega}, \tag{120}$$
$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = |u|_{k,\alpha;\Omega} = |u|_{k,0;\Omega} + [u]_{k,\alpha;\Omega} = |u|_{k;\Omega} + [u]_{k,\alpha;\Omega}$$
$$= \sum_{j=0}^k [u]_{j,0;\Omega} + [D^k u]_{\alpha;\Omega} = \sum_{j=0}^k |D^j u|_{0;\Omega} + \sup_{|\beta|=k}[D^\beta u]_{\alpha;\Omega}.$$

We set with $d = \text{diameter}(\Omega)$

$$\|u\|'_{C^k(\overline{\Omega})} = |u|'_{k;\Omega} = \sum_{j=0}^{k} d^j [u]_{j,0;\Omega} = \sum_{j=0}^{k} d^j |D^j u|_{0;\Omega} = \sum_{j=0}^{k} d^j \sup_{|\beta|=j} \sup_{\Omega} |D^\beta u|,$$

$$\begin{aligned}\|u\|'_{C^{k,\alpha}(\overline{\Omega})} &= |u|'_{k,\alpha;\Omega} = |u|'_{k;\Omega} + d^{k+\alpha}[u]_{k,\alpha;\Omega} = |u|'_{k;\Omega} + d^{k+\alpha}[D^k u]_{\alpha;\Omega} \\ &= \sum_{j=0}^{k} d^j |D^j u|_{0;\Omega} + d^{k+\alpha}[D^k u]_{\alpha;\Omega} \\ &= \sum_{j=0}^{k} d^j |D^j u|_{0;\Omega} + d^{k+\alpha} \sup_{|\beta|=k} [D^\beta]_{\alpha;\Omega}. \end{aligned} \tag{121}$$

Theorem 3.4. ([6] Theorem 4.6.) *Let $u \in C^2(\Omega)$, and $f \in C^\alpha(\Omega)$ satisfy Poisson's equation $\Delta u = f$ in Ω. Then $u \in C^{2,\alpha}(\Omega)$ and for any two concentric balls $B_1 = B_R(x_0)$, $B_2 = B_{2R}(x_0) \subset\subset \Omega$ we have*

$$|u|'_{2,\alpha;B_1} \leq C\big(|u|_{0;B_2} + R^2 |f|'_{0,\alpha;B_2}\big),$$

where $C = C(\alpha)$.

Let us denote by $\mathbb{R}^2_+$ the half-plane $y > 0$, and by T the hyperline, $y = 0$; $B_2 = B_{2R}(x_0)$, $B_1 = B_R(x_0)$ will be balls with center $x_0 \in \overline{\mathbb{R}}^2_+$ and we let $B_2^+ = B_2 \cap \mathbb{R}^2_+$, $B_1^+ = B_1 \cap \mathbb{R}^2_+$.

Theorem 3.5. ([6] Theorem 4.11.) *Let $u \in C^2(B_2^+) \cap C^0(\overline{B}_2^+)$, and $f \in C^\alpha(\overline{B}_2^+)$ satisfy $\Delta u = f$ in B_2^+, $u = 0$ on T. Then $u \in C^{2,\alpha}(\overline{B}_1^+)$ and we have*

$$|u|'_{2,\alpha;B_1^+} \leq C\big(|u|_{0;B_2^+} + R^2 |f|'_{0,\alpha;B_2^+}\big),$$

where $C = C(\alpha)$.

Theorem 3.6. ([6] Theorem 6.6.) *Let Ω be a $C^{2,\alpha}$ domain in $\mathbb{R}^n$ and let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of the second order linear differential equation $Lu = a^{ij}(x) D_{ij} u + b^i(x) D_i u + c(x) u = f$ in Ω, where $a^{ij} = a^{ji}$, $f \in C^\alpha(\overline{\Omega})$, and the coefficients of L satisfy, for positive constants λ, Λ,*

$$\sum_{i+j=2} a^{ij} \xi_i \xi_j \geq \lambda |\xi|^2 \quad \text{for all } x \in \Omega, \ \xi \in \mathbb{R}^n,$$

and

$$|a^{ij}|_{0,\alpha;\Omega}, \ |b^i|_{0,\alpha;\Omega}, \ |c|_{0,\alpha;\Omega} \ \leq \Lambda.$$

Let $\varphi(x) \in C^{2,\alpha}(\overline{\Omega})$ and suppose $u = \varphi$ on $\partial\Omega$. Then

$$|u|_{2,\alpha;\Omega} \leq C(|u|_{0;\Omega} + |\varphi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}).$$

where $C = C(n, \alpha, \lambda, \Lambda, \Omega)$.

Theorem 3.7. ([6] Theorem 9.26.) *Let an operator L represented by*

$$Lu = a^{ij}(x) D_{ij} u + b^i(x) D_i u + c(x) u$$

be strictly elliptic in Ω with fixed constants γ and ν such that

$$\frac{\Lambda}{\lambda} \leq \gamma, \quad \Big(\frac{|b|}{\lambda}\Big)^2, \ \frac{|c|}{\lambda} \leq \nu,$$

where λ, Λ denote, respectively, the minimum and maximum eigenvalues of the coefficient matrix $\big[a^{ij}\big]$. Let $u \in W^{2,n}(\Omega) \cap C^0(\overline{\Omega})$ satisfy $Lu \geq f$ in Ω, $u \leq 0$ on $B \cap \partial\Omega$ where $f \in L^n(\Omega)$ and $B = B_{2R}(y)$ is a ball in $\mathbb{R}^n$. Then, for any $p > 0$, we have

$$\sup_{\Omega \cap B_R(y)} u \leq C\Big\{\Big(\frac{1}{|B|}\int_{B\cap\Omega} (u^+)^p\Big)^{\frac{1}{p}} + \frac{R}{\lambda}\|f\|_{L^n(B\cap\Omega)}\Big\},$$

where $C = C(n, \gamma, \nu R^2, p)$.

Proposition 3.8. *There are positive constants C_{22}, C_{23}, C_{24}, and C_{25} which are independent of Ω_m under the boundedness assumption $\lambda_2(\Omega_m) = \lambda_2(\Xi)$, and for which the followings are satisfied: There exist $\mathfrak{z}_1$, $\mathfrak{z}_2 \in \mathbb{R}$ and $\phi_{2,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}$ such that*

(i)

$$\Big|\mathcal{E}\big[\phi_{2,\Omega_m}\big|_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\big] - \mathcal{E}\Big[\sum_{i=1,2}\mathfrak{z}_i\phi_{i,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\Big]\Big| < C_{22}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta},$$

(ii)

$$\Big\|\phi_{2,\Omega_m}\big|_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}} - \sum_{i=1,2}\mathfrak{z}_i\phi_{i,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\Big\|_2^2 < C_{23}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta},$$

where

$$\begin{cases} \int_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\Big\{\sum_{i=1,2}\mathfrak{z}_i\phi_{i,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\Big\}\cdot\phi_{1,\Omega_m} \\ \qquad = -\sum_{k\in S_m}\int_{\mathfrak{B}_m^{k+}}\phi_{2,\Omega_m}\cdot\phi_{1,\Omega_m}, \\ \mathfrak{z}_1^2+\mathfrak{z}_2^2 = 1-\sum_{k\in S_m}\big(\varrho_{2,m}^{k+}\big)^2. \end{cases} \tag{122}$$

(iii) To any $k \in S_m$ there correspond $\mathfrak{z}_3$, $\mathfrak{z}_4 \in \mathbb{R}$ such that

$$\big(\varrho_{2,m}^{k+}\big)^{-2}\Big|\mathcal{E}\big[\phi_{2,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\big] - \mathcal{E}\Big[\mathfrak{z}_3\phi_{1,\mathfrak{B}_m^{k+}} + \mathfrak{z}_4\phi_{2,\mathfrak{B}_m^{k+}}\Big]\Big| < C_{24}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}\big(\varrho_{2,m}^{k+}\big)^{-2}, \quad \textit{and}$$

(iv)

$$\big(\varrho_{2,m}^{k+}\big)^{-2}\Big\|\phi_{2,\Omega_m}\big|_{\mathfrak{B}_m^{k+}} - \Big(\mathfrak{z}_3\phi_{1,\mathfrak{B}_m^{k+}} + \mathfrak{z}_4\phi_{2,\mathfrak{B}_m^{k+}}\Big)\Big\|_2^2 < C_{25}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}\big(\varrho_{2,m}^{k+}\big)^{-2}, \tag{123}$$

where

$$\begin{cases} \int_{\mathfrak{B}_m^{k+}}\Big\{\mathfrak{z}_3\phi_{1,\mathfrak{B}_m^{k+}} + \mathfrak{z}_4\phi_{2,\mathfrak{B}_m^{k+}}\Big\}\cdot\phi_{1,\Omega_m} = -\int_{\mathfrak{B}_m^{k-}}\phi_{2,\Omega_m}\cdot\phi_{1,\Omega_m}, \\ \mathfrak{z}_3^2+\mathfrak{z}_4^2 = \big(\varrho_{2,m}^{k+}\big)^2. \end{cases}$$

(**Note.**) (ii) is valid $\cap_{k\in S_m}\mathfrak{B}_m^{k-}$ replaced by $\mathfrak{B}_m^{k-}$ with $\mathfrak{z}_1$ and $\mathfrak{z}_2$ limited by equations

$$\int_{\mathfrak{B}_m^{k-}}\Big\{\sum_{i=1,2}\mathfrak{z}_i\phi_{i,\mathfrak{B}_m^{k-}}\Big\}\cdot\phi_{1,\Omega_m} = -\int_{\mathfrak{B}_m^{k+}}\phi_{2,\Omega_m}\cdot\phi_{1,\Omega_m},$$

$$\mathfrak{z}_1^2+\mathfrak{z}_2^2 = 1-\big(\varrho_{2,m}^{k+}\big)^2.$$

Proof. (i) Considering the requirement of orthogonality of ϕ_{2,Ω_m} to ϕ_{1,Ω_m} which is the only difference between proof of Proposition 3.2 (i) and (iii), we define h^- by a function such that for each $k \in S_m$

$$h^-(x,y) = \begin{cases} +\frac{\phi_{2,\Omega_m}(0,y)}{\vartheta}x + \phi_{2,\Omega_m}(0,y), & (x,y)\in\mathfrak{S}^k_{m,\vartheta}\cap\mathfrak{B}_m^{k-}, \\ 0, & (x,y)\in\cap_{k\in S_m}\big(\mathfrak{B}_m^{k-}\setminus\mathfrak{S}^k_{m,\vartheta}\big). \end{cases} \tag{124}$$

Note that since $\sum_{i=1,2}\mathfrak{z}_i\phi_{i,\cap_k\mathfrak{B}_m^{k-}}$ is defined in order that it may have the minimal energy among functions in $C_0^2(\cap_{k\in S_m}\mathfrak{B}_m^{k-})$ which satisfy conditions (122), we have

$$\|\phi_{2,\Omega_m}\big|_{\cap_k\mathfrak{B}_m^{k-}} - h^-\|_2^2\mathcal{E}\Big[\sum_{i=1,2}\mathfrak{z}_i\phi_{i,\cap_k\mathfrak{B}_m^{k-}}\Big] \le \mathcal{E}\big[\phi_{2,\Omega_m}\big|_{\cap_k\mathfrak{B}_m^{k-}} - h^-\big].$$

The rest of proof follows the way of proof of Proposition 3.2 (i).
(ii) The way of proof of (ii) is similar to that of Proposition 3.2 (ii) using the result Proposition 3.2 (i). One of the differences between them is the following; let us denote

$$\alpha_j = \int_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\Big\{\phi_{2,\Omega_m}\big|_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}} - h^-\Big\}\cdot\phi_{j,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}.$$

From (i) we have

$$C_{26}\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta} \geq \left|\mathcal{E}\Big[\phi_{2,\Omega_m}\big|_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}} - h^-\Big] - \mathcal{E}\Big[\sum_{i=1,2}\mathfrak{z}_i\phi_{i,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\Big]\right|$$
$$= \left|\sum_{j=1}^{\infty}\alpha_j^2\lambda_j(\cap_{k\in S_m}\mathfrak{B}_m^{k-}) - \sum_{j=1,2}\mathfrak{z}_j\lambda_j(\cap_{k\in S_m}\mathfrak{B}_m^{k-})\right|$$
$$= \left|\sum_{j\in\mathbb{N}\setminus\{1,2\}}\alpha_j^2\lambda_j(\cap_{k\in S_m}\mathfrak{B}_m^{k-}) - \sum_{j=1,2}\{\mathfrak{z}_j^2-\alpha_j^2\}\lambda_j(\cap_{k\in S_m}\mathfrak{B}_m^{k-})\right|. \quad (125)$$

Considering the equation

$$\int_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\Big\{\sum_{j\in\mathbb{N}}\alpha_j\phi_{j,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}\Big\}\cdot\phi_{1,\Omega_m} = -\sum_{k\in S_m}\int_{\mathfrak{B}_m^{k+}}\phi_{2,\Omega_m}\cdot\phi_{1,\Omega_m}.$$

If $\alpha_j = \mathfrak{z}_j$ for $j = 1,2$, this equality holds only for $\alpha_j = 0$ for $3 \leq j$. But if α_1 varies, then this equality fails, since from Proposition 3.2 $\phi_{1,\Omega_m}\big|_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}}$ approximates to $\big\{1 - \sum_{k\in S_m}\big(\varrho_{1,m}^{k+}\big)^2\big\}^{\frac{1}{2}}\phi_{1,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}$, and $\phi_{j,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}$ for $j \geq 2$ is approximately orthogonal to $\phi_{1,\Omega_m}\big|_{\cap_{k\in S_m}\mathfrak{B}_m^{k-}}$. Therefore, since $\alpha_1^2+\alpha_2^2 = 1-\sum_{k\in S_m}\big(\varrho_{2,m}^{k+}\big)^2$, we may consider only the case that α_2 decreases. In this case $\sum_{j\geq 3}\alpha_j^2$ changes from zero to a positive number, and it implies energy of $\sum_{j\in\mathbb{N}}\alpha_j\phi_{j,\cap_{k\in S_m}\mathfrak{B}_m^{k-}}$ increases. It is absurd. Thus α_j^2 approximates to $\mathfrak{z}_j^2$, $j = 1,2$. The rest of proof follows Proposition 3.2 (ii).
(iii) The way of proof is similar to that of Proposition 3.2 (i).
(iv) The way of proof is similar to that of Proposition 3.2 (ii). □

Proposition 3.9. *For each $k \in S_m$ let us assume that*

$$\mathcal{E}^*\Big[\phi_{2,\Omega_m}\big|_{\mathfrak{B}_m^{k+}}\Big] = \mathcal{E}^*\Big[\phi_{2,\Omega_m}\big|_{\mathfrak{B}_m^{k-}}\Big] + \lambda_{2,m}^{k*}, \quad \lambda_{2,m}^{k*} > 0. \quad (126)$$

Then, there is a constant C_{27} such that for any $k \in S_m$

$$\big(\varrho_{2,m}^{k+}\big)^2\cdot\lambda_{2,m}^{k*} < C_{27}\Theta^2_{\omega_m,\vartheta}. \quad (127)$$

Thus, we have

$$\sum_{k\in S_m}\big(\varrho_{2,m}^{k+}\big)^2\cdot\lambda_{2,m}^{k*} < C_{27}|S_m|\Theta^2_{\omega_m,\vartheta}.$$

(**Note.**) If $\mathfrak{B}_m^{k+}$ and $\mathfrak{B}_m^{k-}$ are reversed in (126), then (127) also holds $\varrho_{2,m}^{k-}$ and $\varrho_{2,m}^{k+}$ reversed.

Proof. The proof is similar to proof of Proposition 3.3 except for the requirement of orthogonality of ϕ_{2,Ω_m} to ϕ_{1,Ω_m}.

Let us define $\phi^\epsilon_{2,\Omega_m}$ as follows; let Ω_m^+ and Ω_m^- the inner-nodal domain and the outer-nodal domain of ϕ_{2,Ω_m}, respectively. For C^1-real function f_j, $0 < f_j(\epsilon)$ for $\epsilon \geq 0$, $j = 2,3,4,5$,

$$\phi^\epsilon_{2,\Omega_m}(x,y) = \begin{cases} f_2(\epsilon)\phi_{2,\Omega_m}(x,y), & (x,y)\in\mathfrak{B}_m^{k+}\cap\Omega_m^+,\\ f_3(\epsilon)\phi_{2,\Omega_m}(x,y), & (x,y)\in\mathfrak{B}_m^{k+}\cap\Omega_m^+,\\ f_4(\epsilon)\phi_{2,\Omega_m}(x,y), & (x,y)\in\mathfrak{B}_m^{k-}\cap\Omega_m^+,\\ f_5(\epsilon)\phi_{2,\Omega_m}(x,y), & (x,y)\in\mathfrak{B}_m^{k-}\cap\Omega_m^-.\end{cases}$$

Let us define $\phi^\epsilon_{2,\Omega_m}$ in the same sense as (110) in order that

$$\Big\|\phi^\epsilon_{2,\Omega_m|\mathfrak{B}_m^{k-}}\Big\|_2^2\cdot\Big\|\phi_{2,\Omega_m|\mathfrak{B}_m^{k-}}\Big\|_2^{-2} = 1+b\epsilon+o(\epsilon)$$

for a positive constant b. Provided we define $\tilde{\phi}^{\epsilon}_{2,\Omega_m}$ in the same sense as definition (112), there exist f_j, $f_j(0)=1$, $j=2,3,4,5$, for which the following conditions are satisfied;

$$\|\tilde{\phi}^{\epsilon}_{2,\Omega_m}\|_2 = 1, \quad \int_{\Omega_m} \tilde{\phi}^{\epsilon}_{2,\Omega_m}\cdot\phi_{1,\Omega_m} = 0, \quad \text{and for sufficiently small all } \epsilon$$

$$0 \le \mathcal{E}\big[\tilde{\phi}^{\epsilon}_{2,\Omega_m}\big] - \mathcal{E}\big[\phi_{2,\Omega_m}\big] \le -C_{28}\epsilon\big(\varrho^{k+}_{2,m}\big)^2\lambda^{k*}_{2,m} + \epsilon C_{29}\Theta^2_{\omega_m,\vartheta}.$$

The last inequality implies (127). □

Remark 3.2. Considering Remark 3.1, we are to define two functions $N^{\epsilon}_{2,m}$ and $N^{\epsilon}_{2,m,t}$ of ϵ-variable and (ϵ,t)-variables, respectively. Let us denote

$$\Big|\mathcal{E}[\tilde{\phi}^{\epsilon}_{2,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}] - \mathcal{E}[\phi^{\epsilon}_{2,\Omega_m}\big|_{\mathfrak{S}^k_{m,\vartheta}}]\Big| := N^{\epsilon}_{2,m} < O(\epsilon)\frac{1}{\omega_m}\Theta^2_{\omega_m,\vartheta}.$$

Then, by the same argument as $N^{\epsilon}_{1,m}$

$$0 = \frac{d}{d\epsilon}_{\epsilon\to 0}\Big\{\mathcal{E}\big[\tilde{\phi}^{\epsilon}_{2,\Omega_m}\big] - \mathcal{E}\big[\phi_{2,\Omega_m}\big]\Big\} = -2\big(\varrho^{k+}_{2,m}\big)^2\lambda^{k*}_{2,m} + \frac{d}{d\epsilon}_{\epsilon\to 0}N^{\epsilon}_{2,m}. \qquad //$$

Remark 3.3. If ω_m is fixed, the larger $\lambda^{k*}_{2,m}$ is, the smaller $\varrho^{k+}_{2,m}$ becomes. As the contraposition the larger $\varrho^{k+}_{2,m}$ is, the smaller $\lambda^{k*}_{2,m}$ becomes. According to (127), no matter how small $\big(\varrho^{k+}_{2,m}\big)^2$ is, if we define $\Theta_{\omega_m,\vartheta}$ to be sufficiently small value, $\lambda^{k*}_{2,m}$ must be close to zero, that is, $\mathcal{E}^*\big[\phi_{2,\Omega_m}\big|_{\mathfrak{B}^{k+}_m}\big]$ approximates to $\mathcal{E}^*\big[\phi_{2,\Xi}\big] \approx \mathcal{E}^*\big[\phi_{2,\Omega_m}\big|_{\mathfrak{B}^{k-}_m}\big]$. //

Remark 3.4. Let us deform $\mathfrak{B}^{k+}_m$ by dilation and contraction of blossoms. Then, according to Proposition 3.9 (127), when $\big(\varrho^{k+}_{2,m}\big)^2$ and $\lambda^{k*}_{2,m}$ varies with fixed $\Theta_{\omega_{m_1},\vartheta}$, they correspond to each other one to one. That is, we infer that $\big(\varrho^{k+}_{2,m}\big)^2$ increases, if and only if $\lambda^{k*}_{2,m}$ decreases. Consequently, when $\big(\varrho^{k+}_{2,m}\big)^2$ assumed to be sufficiently small, a suitably large $\lambda^{k*}_{2,m}$ corresponds. Then, from Proposition 3.3 $\big(\varrho^{k-}_{1,m}\big)^2$ turns to be a large value, and then we can conclude that $|\mathfrak{z}_1|$ in Proposition 3.8 is determined to be sufficiently small and $|\mathfrak{z}_2|$ approximates to one. //

Repeating splitting deformations and filling deformations, one must attains to a situation such that remainder $\mathfrak{R}_m$ is simply connected and has arbitrarily small width. The *width* of $\mathfrak{R} \subset \mathfrak{R}_m$ is defined by the maximum among diameters of disks which are contained in $\mathfrak{R}$. Let us assume that the width of $\mathfrak{R}_m$ can not be arbitrarily small for any sufficiently large m. Then, we deform Ω_m repeatedly by splitting deformations whose supports lie in the remainder and blossoms and by filling deformations. Until there are eleven blossoms, we tunnel $\mathfrak{R}_m$, attach branches, or vary diameter of blossoms. When the number of blossoms reaches eleven, we deform only blossoms in order to vary their diameter by splitting deformation. But if splitting deformations which only attach branches and blossoms and deform blossoms can continue permanently, then we reach a situation such that $\Xi \subsetneq \Omega_\ell$ for a large $\ell > m$. It is impossible from the requirement 3) of Definition 3.1 of splitting deformation and requirement (93). Thus, $\mathcal{J}_m$ must tunnel the remainder for an large m. That is, we can make the $\mathfrak{R}_m$ be thinner as m becomes larger.

We illustrate a typical example of tunnelling. Let us set $\mathfrak{R}_m := \{(x,y)|\ 0<x<4,\ 0<y<1\}$. Consider the following unions of segments in $\mathfrak{R}_m$;

$$\begin{aligned}&\big\{(x,y)|\ x=1/3,\ 0<y<2/3\big\}\bigcup\big\{(x,y)|\ 1/3<x<11/3,\ y=2/3\big\}\\&\bigcup\big\{(x,y)|\ 1/3<x<5/3,\ y=1/3\big\}\bigcup\big\{(x,y)|\ x=2,\ 0<y<1/3\big\}\\&\bigcup\big\{(x,y)|\ 2<x<11/3,\ y=1/3\big\}.\end{aligned} \tag{128}$$

Then, we define tunnels in $\mathfrak{R}_m$ by some thin neighborhoods of above unions of segments which are disjoint each other. A simply connected band $\mathfrak{R}_m$ is divided into three ways around the positions $(5/3, 1/3)$ and $(11/3, 1/2)$.

From [6] Theorem 8.1 (the weak maximum principle) we obtain a corollary of Lemma 3.1. Under the hypotheses and notations of Lemma 3.1 we can obtain $\mathfrak{R} \subset \mathfrak{R}_m$ described by the following manner; let N in Lemma 3.1 be a sufficiently large integer, and let $\mathcal{B} : V \to \mathcal{B}(V) \subset \mathbb{R}^2$ be a bending diffeomorphism such that

$$\mathcal{B}^{-1}(\mathfrak{R}) = W, \tag{129}$$

$$\mathfrak{g} := \mathcal{B}^* \phi_{2,\Omega_m}\big|_{\mathfrak{R}},$$

$$\big|\mathcal{B}^{-1*} \frac{\partial}{\partial y}\big| = 1, \qquad \big\langle \mathcal{B}^* \frac{\partial}{\partial \xi}, \mathcal{B}^* \frac{\partial}{\partial \eta} \big\rangle = 0.$$

Thus $\mathcal{B}$ preserves $|\frac{\partial}{\partial y}|$, and therefore preserves $\frac{1}{N}$, the width of V. An elementary calculation shows

$$\Big(\mathcal{B}^* \frac{\partial}{\partial \xi}\Big)\mathfrak{g} = \frac{\partial}{\partial \xi}\mathfrak{g}\big(\mathcal{B}^{-1}(\xi, \eta)\big) = \frac{\partial}{\partial x}\mathfrak{g}\frac{\partial \mathcal{B}_x^{-1}}{\partial \xi} + \frac{\partial}{\partial y}\mathfrak{g}\frac{\partial \mathcal{B}_y^{-1}}{\partial \xi},$$

$$\mathcal{B}^* \frac{\partial}{\partial \xi} = \frac{\partial \mathcal{B}_x^{-1}}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial \mathcal{B}_y^{-1}}{\partial \xi}\frac{\partial}{\partial y}, \qquad \mathcal{B}^* \frac{\partial}{\partial \eta} = \frac{\partial \mathcal{B}_x^{-1}}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial \mathcal{B}_y^{-1}}{\partial \eta}\frac{\partial}{\partial y}, \tag{130}$$

where $\mathcal{B}^{-1} := \big(\mathcal{B}_x^{-1}, \mathcal{B}_y^{-1}\big)$.

Let $p \in \mathcal{B}\big(\partial V \setminus \{V1 \cup V_2\}\big)$, and let us denote the closed disk with maximal diameter contained in $\mathcal{B}(V)$ which touches p by $D(p)$, and denote the set of points $\mathcal{B}(\partial V) \cap \partial D(p) \setminus \{p\}$ by $\mathfrak{l}(p)$. Let us denote by $r(p)$ the radius of closed disk having maximum of the set of bounded radii of disks which meet p, but do not intersect with $\mathcal{B}(V)$. Then, let us assume that two points $p_1, p_2 \in \partial\mathcal{B}(V)$ lie at each side of p, and the distances from p_1 and p_2 to p are sufficiently small. Then, the distance from $\mathfrak{l}(p_1)$ to $\mathfrak{l}(p_2)$ is very large compared to the distance from p_1 to p_2. Set $\frac{\partial}{\partial \xi}$ is tangent to $\partial\mathcal{B}(V)$ at $q \in \mathfrak{l}(p)$, and $-\frac{\partial}{\partial \eta}$ is outer-normal derivative at q. Then, if we set $\big|\mathcal{B}^* \frac{\partial}{\partial \xi}_p\big| = 1$, then $\big|\mathcal{B}^* \frac{\partial}{\partial \xi}_q\big|$ is very small compared to one. The magnitude of $\big|\mathcal{B}^* \frac{\partial}{\partial \xi}_{\mathfrak{l}(p)}\big|$ is proportional to $\frac{r(p)}{r(p)+\text{diameter of } D(p)}$. One can tunnel the remainder in order that the width of the remainder may be much smaller than minimal value among all $r(p)$.

Corollary 3.10. (Corollary of Lemma 3.1.) Let $\mathfrak{R} \subset \mathfrak{R}_m$ be a simply connected open band with a sufficiently small width and a sufficiently large length, and let $\partial\mathfrak{R}$ consist of only a subset of $\partial\mathfrak{R}_m \cap \partial\Omega_m$ and more than two segments across $\mathfrak{R}_m$. Let us assume that
(i) either $\mathfrak{R}$ satisfies (129)
(ii) or $\mathfrak{R}$ contains a subset ϖ where $\mathfrak{R}$ is divided into several ways like the illustration (128), and boundary of ϖ consists of $\partial\mathfrak{R}_m$ and segments each of whom traverses a way of $\mathfrak{R}_m$ with an inequality diameter(ϖ) << diameter$(\mathfrak{R})$.
Then, the restriction $\phi_{2,\Omega_m}\big|_{\mathfrak{R}}$ decays exponentially.

Proof. (i) Consider the identities for a differential operator $\mathcal{G}$

$$\begin{cases} \Delta_{\mathcal{B}^* e}\mathfrak{g} = -\lambda_2(\Omega_m)\mathfrak{g} = (\Delta_e + \mathcal{G})\mathfrak{g}, \\ \Delta_e \mathfrak{g} = -\lambda_2(\Omega_m)\mathfrak{g} - \mathcal{G}\mathfrak{g}, \\ \Delta_e v = -\lambda_2(\Omega_m)v, \end{cases} \tag{131}$$

where v is a function given in the proof of Lemma 3.1. From (130), (131), and from the sentences following (130) there exists a constant C_{30} which is bounded below to

$$C_{31} \cdot \left(\frac{r(p)}{r(p) + \text{diameter of } D(p)}\right)^2,$$

for a constant C_{31}, and satisfies

$$C_{30}\lambda_2(\Omega_m)v > |\lambda_2(\Omega_m)\mathfrak{g} + \mathcal{G}\mathfrak{g}|, \quad \text{and} \quad C_{30}v - |\mathfrak{g}| \geq 0 \quad \text{on} \quad V_1 \cup V_2. \tag{132}$$

Then, considering the function $C_{30}v - \mathfrak{g}$, we have from (132)

$$\Delta_e(C_{30}v - \mathfrak{g}) = -\lambda_2(\Omega_m)(C_{30}v - \mathfrak{g}) + \mathcal{G}\mathfrak{g} < 0,$$

and then from weak maximum principle ([6] Theorem 8.1.)

$$\inf_{\mathcal{B}^{-1}(\mathfrak{R})}(C_{30}v - |\mathfrak{g}|) \geq \inf_{V_1\cup V_2}(C_{30}v - |\mathfrak{g}|)^- = 0,$$

where $(C_{30}v - |\mathfrak{g}|)^- = \min\{C_{30}v - |\mathfrak{g}|, 0\}$. Thus, $C_{30}v \geq |\mathfrak{g}|$, and from Lemma 3.1 $\mathfrak{g}$ decays exponentially for a sufficiently small width of $\mathfrak{R}$.
(ii) If $\phi_{2,\Omega_m}\big|_{\mathfrak{R}}$ does not decay exponentially in ϖ in which $\mathfrak{R}$ is divided into several ways, then one can redefine ϕ_{2,Ω_m} as follows in order that this redefined function which has unit norm and is orthogonal to ϕ_{1,Ω_m} may have energy less than that of ϕ_{2,Ω_m}; consider the existence of constant C_3 in inequality (96). Let us suppose that no matter how much $\Theta_{\omega_m,\vartheta}$ decreases, (96) fails in ϖ, that is, C_3 satisfying (96) does not exist. Then, we redefine $\phi_{2,\Omega}\big|_{\varpi}$ by multiplying a positive constant which is slightly smaller than one. Then, we can redefine $\phi_{2,\Omega_m}\big|_{\Omega_m\setminus\varpi}$ so that it may have a larger L_2-norm and a less energy than the original. We attain to a contradiction. □

Remark 3.5. One can tunnel $\mathfrak{R}_m \subset \Omega_m$ or fill up tunnels in $\mathfrak{R}_m$ so that $\mathfrak{R}_{m+j}$ may be sufficiently thin, and curvatures of the boundary $\partial\mathfrak{R}_{m+j}\cap\partial\Omega_{m+j}$ may have an upper bound for sufficiently large all $j \in \mathbb{N}$. Let $p \in \partial\mathfrak{R}_{m+j}\cap\partial\Omega_{m+j}$, and $\partial B(p, 2R)\cap\mathfrak{R}_{m+j}$ be connected. Let p be far sufficiently from $\mathfrak{F}_m$ and $\mathfrak{B}^k_m$ along $\mathfrak{R}_m$. Let

$$\mathfrak{H} : B(p, R) \cap \mathfrak{R}_{m+j} \to B(O, R) \cap \mathbb{R}^{2+},$$

$B(O, R)\cap\mathbb{R}^{2+} =: B(O, R, \pi)$, be an analytic function such that $\mathfrak{H}$ maps $B(p, R)\cap\partial\mathfrak{R}_{m+j}$ onto $\{(x, y) : -R < x < R,\ y = 0\}$, $\mathfrak{H}(p) = (0, 0)$. Referring to proof of Corollary 3.10, by a weak maximum principle $\mathfrak{H}^{-1*}\phi_{2,\Omega_{m+j}}\big|_{B(p,R)\cap\mathfrak{R}_{m+j}}$ decays exponentially in a rate $\Theta_{\omega'_m,\vartheta}$, where ω'_m denotes width of $\mathfrak{R}_m$. From Theorem 3.4 and Theorem 3.5 one can infer energy of ϕ_{2,Ω_m} restricted to the component of $\mathfrak{R}_m \cap \{(x, y) : -R < x < R\}$ which contains $B(p, R)\cap\mathfrak{R}_m$ also decays exponentially in a rate $\frac{1}{R}\Theta^2_{\omega'_m,\vartheta}$. //

Now let us assume that width of $\mathfrak{R}_m \cup \{\cup_k \mathfrak{S}^k_m\}$ is sufficiently small, and

$$\sum_{k\in S_m}(\varrho^{k+}_{2,m})^2 = \left\|\phi_{2,\Omega_m}\big|_{\bigcup_{k\in S_m}\mathfrak{B}^{k+}_m}\right\|_2^2 > \mathfrak{c}, \tag{133}$$

for a sufficiently small fixed constant $\mathfrak{c} > 0$, and for all m which are bigger than a sufficiently large $m_1 \in \mathbb{N}$. Assumption (133) implies from Proposition 3.9 that there exist a positive lower bound of maximal radii of blossoms $\mathfrak{B}^k_m$ over all $m > m_1$. Then, the value

$$\sup_{\mathfrak{B}^k_m}|\sum_{k=2}^{3}\alpha_k\phi_{k,\Omega_m} + \alpha_4 K_{\lambda_2(0)}|$$

has a positive lower bound over m and all α_k. Corollary 3.10 says $\sum_{k=2}^3\alpha_k\phi_{k,\Omega_m} + \alpha_4K_{\lambda_2(0)}$ decays exponentially in $\mathfrak{R}_m \cup \mathfrak{S}^k_m$, which implies $\sup_{\mathfrak{B}^k_m}|\sum_{k=2}^3\alpha_k\phi_{k,\Omega_m} + \alpha_4K_{\lambda_2(0)}|$ is much bigger than the supremum $\sup_{\mathfrak{V}_m}|\sum_{k=2}^3\alpha_k\phi_{k,\Omega_m} + \alpha_4K_{\lambda_2(0)}|$, where $\mathfrak{V}_m$ stands for a connected subset of $\mathfrak{R}_m\cup\mathfrak{S}^k_m$ whose boundary consists of a subset of boundary of $\mathfrak{R}_m\cup\mathfrak{S}^k_m$ and segments traversing $\mathfrak{R}_m\cup\mathfrak{S}^k_m$ and the length of $\mathfrak{R}_m\cup\mathfrak{S}^k_m\setminus\mathfrak{V}_m$ is a fixed constant much smaller than the length of $\mathfrak{R}_m\cup\mathfrak{S}^k_m$.

Consequently, if we denote by $\mathcal{C}_m : \Omega_m \times [0, 1] \to \mathbb{R}^2$ a deformation which collapses $\mathfrak{V}_m$, and if we define $\mathcal{J}^k_m$ by a splitting deformation which tunnels into, or attaches branches and blossoms to $\bigcup_k \mathfrak{B}^k_m$, then a sum of deformations $\mathcal{C}_m \uplus \mathcal{J}_m$ can split nodal line and eigenvalues like splitting deformations, since collapsing remainder causes a negligible changes in the second eigenfunctions and eigenvalues. Then, we define $\mathcal{F}_m\circ\mathcal{C}_m\uplus\mathcal{J}_m(\Omega_m) : \Omega_{m+1}$, and deform Ω_{m+1}

repeatedly by deformations mentioned in Definition 3.1. When remainder has a sufficiently small width, we perform again a sume of collapsing deformation and splitting deformation stated in this paragraph. If we repeat this procedure, then we attain to the case such that $\mathrm{vol}(\Xi) \lneqq \mathrm{vol}(\mathfrak{F}_\ell)$ for a large ℓ, and attain to an inequality $\lambda_2(\mathfrak{F}_\ell) \lneqq \lambda_2(\Xi)$. It is absurd, since $\lambda_2(\Omega_\ell) = \lambda_2(\Xi)$.

Consequently, we may assume that

$$\left\| \phi_{2,\Omega_{m_1}} \big|_{\bigcup_{k\in S_{m_1}} \mathfrak{B}^{k+}_{m_1}} \right\|_2^2 \le \mathfrak{c}, \tag{134}$$

for some large m_1 and for the sufficiently small constant $\mathfrak{c}$ given in (133). Let us consider the identity

$$\begin{aligned}\lambda_2(\Xi) =& \lambda_2(\Omega_{m_1}) \\ =& \big\{1 - \sum_k \big(\varrho^{k+}_{2,m_1}\big)^2\big\}\mathcal{E}^*[\phi_{2,\Omega_{m_1}}\big|_{\cap_k \mathfrak{B}^{k-}_{m_1}}] + \sum_k \big(\varrho^{k+}_{2,m_1}\big)^2 \mathcal{E}^*[\phi_{2,\Omega_{m_1}}\big|_{\mathfrak{B}^{k+}_{m_1}}].\end{aligned}$$

Proposition 3.9 implies

$$\sum_{k\in S_{m_1}} \big(\varrho^{k+}_{2,m_1}\big)^2 \cdot \mathcal{E}^*[\phi_{2,\Omega_{m_1}}\big|_{\mathfrak{B}^{k+}_{m_1}}] = \sum_{k\in S_{m_1}} \big(\varrho^{k+}_{2,m_1}\big)^2 \cdot \Big(\mathcal{E}^*[\phi_{2,\Omega_{m_1}}\big|_{\mathfrak{B}^{k-}_{m_1}}] + \lambda^{k*}_{2,m_1}\Big)$$

is bounded above to $\mathfrak{c}2\lambda_2(\Xi) + C_{27}|S_{m_1}|\Theta^2_{\omega_{m_1},\vartheta}$. Thus, we have

$$0 < \lambda_2(\Xi) - \big\{1 - \sum_k \big(\varrho^{k+}_{2,m_1}\big)^2\big\}\mathcal{E}^*[\phi_{2,\Omega_{m_1}}\big|_{\cap_k \mathfrak{B}^{k-}_{m_1}}] < \mathfrak{c}2\lambda_2(\Xi) + C_{27}|S_{m_1}||\Theta^2_{\omega_{m_1},\vartheta}. \tag{135}$$

Considering Proposition 3.8 (i) and (ii), we define $\overline{\phi}_{2,\mathfrak{F}_{m_1}}$ by a function such that for each k

$$\overline{\phi}_{2,\mathfrak{F}_{m_1}} = \begin{cases} \sum_{j=1,2} \mathfrak{z}_j \phi_{j,\bigcap_{k\in S_{m_1}} \mathfrak{B}^{k-}_{m_1}} & \text{in } \mathfrak{F}_{m_1} \setminus V_k, \\ 0 & \text{in } \mathfrak{B}^{k-}_{m_1} \setminus \mathfrak{F}_{m_1}, \ \text{ and} \end{cases}$$

$\overline{\phi}_{2,\mathfrak{F}_{m_1}}$ is defined in V_k in order that $\overline{\phi}_{2,\mathfrak{F}_{m_1}}\big|_{\mathfrak{F}_{m_1}}$ may be of $C^2(\mathfrak{F}_{m_1})$, where V_k is a small neighborhood of $\overline{\mathfrak{F}_{m_1}} \cap \overline{\mathfrak{R}^k_{m_1}}$. Furthermore, we may define $\overline{\phi}_{2,\mathfrak{F}_{m_1}}$ in V_k so that

$$\mathcal{E}[\phi_{2,\Omega_{m_1}}\big|_{\cap_k \mathfrak{B}^{k-}_{m_1}}] \approx \mathcal{E}[\overline{\phi}_{2,\mathfrak{F}_{m_1}}].$$

Then, from (135) for sufficiently small values $\mathfrak{c}$ and ω_{m_1}

$$\mathcal{E}[\overline{\phi}_{2,\mathfrak{F}_{m_1}}] \approx \mathcal{E}[\phi_{2,\Xi}].$$

Then, by using the method of proof of Proposition 3.8 (ii) one can infer in L_2-norm

$$\overline{\phi}_{2,\mathfrak{F}_{m_1}} \approx \phi_{2,\Xi},$$

where $\overline{\phi}_{2,\mathfrak{F}_{m_1}} := 0$ in $\Xi \setminus \mathfrak{F}_{m_1}$. Therefore, we have in L_2-norm

$$\phi_{2,\Xi} \approx \sum_{j=1,2} \mathfrak{z}_j \phi_{j,\bigcap_{k\in S_{m_1}} \mathfrak{B}^{k-}_{m_1}}\big|_{\mathfrak{F}_{m_1}}.$$

Then, according to Remark 3.4,

$$\phi_{2,\Xi} \approx \mathfrak{z}_2 \phi_{2,\bigcap_{k\in S_{m_1}} \mathfrak{B}^{k-}_{m_1}}\big|_{\mathfrak{F}_{m_1}}. \tag{136}$$

From Proposition 3.8 (ii) when $|\mathfrak{z}_2|$ approximates to one, the nodal line of $\phi_{2,\Omega_{m_1}}$ lies in $\mathfrak{F}_{m_1}$. Then, (135) implies a contradiction.

To show this contradiction firstly note that if (136) is true, since $\phi_{2,\Omega_{m_1}}$ has a closed nodal line, we may say without loss of generality the nodal line of Ω_{m_1} is sufficiently close to the segments $\{(x,y) : x = 0,1,\ 0 < y < 1\}$ and $\{(x,y) : x = 0,1,\ 0 < y < 1\}$. Let us denote by $\Omega^-_{m_1}$ and $\Omega^+_{m_2}$ the outer and inner nodal domain of $\phi_{2,\Omega_{m_1}}$, respectively. Thus $\Omega^-_{m_1}$ must contain a sufficiently narrow and long simply connected band

$$W = \Omega^-_{m_1} \cap \{(x, y) : 0 < x < 1/2,\ 1/4 < y < 3/4\}.$$

Then, by Lemma 3.1 $\left|\phi_{2,\Omega_{m_1}}|_{\mathfrak{F}_{m_1}}\right|$ decays exponentially in W. Let W have length $2\vartheta'$ and width smaller than ω'_{m_1} which denotes width of the smallest rectangle containing W.

We are to follow the arguments in [7] of David Jerison. Lemma 2 [7] with the roles of $\Omega_1 := \Omega^+_{m_1}$ and $\Omega_2 := \Omega^-_{m_1}$ reversed says that there is $\zeta \in \partial W \cap \partial\Omega^+_{m_1}$ such that

$$\left|\nabla\phi_{2,\Omega_{m_1}}(\zeta)\right| \le \frac{C_{32}}{r} \max_{\partial B(z,s)\cap\overline{\Omega}^-_{m_1}} \left|\phi_{2,\Omega_{m_1}}\right| \le \frac{C_{32}}{r}\Theta_{\omega'_{m_1},\vartheta'}, \tag{137}$$

where $B(z, r) \subset \Omega^+_{m_1}$, $\zeta \in \partial W \cap \partial B(z, r)$, $2r < s < 2\omega'_{m_1}$, and $\Theta_{\omega'_{m_1},\vartheta'}$ stands for the absolute constant defined in Definition 3.3. Regardless of shape of $\partial W \cap \partial\Omega^+_{m_1}$, the radius r and ζ could be selected in order that $\frac{C_{32}}{r}\Theta_{\omega'_{m_1},\vartheta'}$ may decay exponentially as ω'_{m_2} decreases. Since $|\phi_{2,\Omega_{m_1}}|$ is superharmonic in $\Omega^+_{m_1}$, comparison with a harmonic function (Hopf lemma) implies

$$\min_{\overline{B}(z,R/2)} \left|\phi_{2,\Omega_{m_1}}\right| \le R\left|\nabla\phi_{2,\Omega_{m_1}}(\zeta)\right|, \tag{138}$$

where $B(z, R) \subset \Omega^+_{m_1}$ with $\zeta \in \partial B(z, R) \cap \partial\Omega^+_{m_1}$. For this refer to the paragraph succeeding Lemma 3 in [7]. Thus,

$$\min_{\overline{B}(z,R/2)} \left|\phi_{2,\Omega_{m_1}}\right| \le R\left|\nabla\phi_{2,\Omega_{m_1}}(\zeta)\right| \le \frac{C_{32}R}{r}\Theta_{\omega'_{m_1},\vartheta'}. \tag{139}$$

Consequently, (139) implies that $\phi_{2,\Omega_{m_1}}$ can not approximate to $\phi_{2,\Xi}$ in L_2-norm in $\Omega^+_{m_1}$, and then we attain a contradiction. □

4 About another Proof of Payne's conjecture

Given Ω described in preceding sections, we deform Ω_m only by splitting deformations which strictly decreases $\lambda_2(\Omega_m)$. Then, diameter of a blossom of Ω_m must keep increasing as m grows larger. But since the closed nodal line of $\phi_2(\Omega_m)$ can not lie in a blossom if $\frac{\text{length of } \mathfrak{S}_m}{\text{width of } \mathfrak{S}_m}$ is bigger than a suitably large constant, it means a contradiction.

REFERENCES

[1] Giovanni Alessandrini, *Nodal lines of eigenfunctions of the fixed membrane problem in general convex domains,* Comment. Math. Helv., **69**(1994), 142-154.
[2] N. Aronszajn, *A unique continuation theorem for solution of elliptic partial differential equations or inequalities of second order,* J. Math. Pures Appl., **36**(1957), 235-249.
[3] Issac Chavel, *Eigenvalues in Riemannian geometry,* Academic Press, Inc., Orlando, Florida, 1984.
[4] S. Y. Cheng, *Eigenfunctions and nodal sets*, Comment. Math. Helv., **51**(1976), 43-55.
[5] R. Courant and D. Hilbert, *Methods of mathematical physics,* **1**, Interscience Publishers, London, 1963. German edition, 1937.
[6] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order, second edition,* Springer-Verlag, New York Tokyo, 1983
[7] David Jerison, *The first nodal line of a convex planar domain,* International Mathematics Research notices, (1991 No.1), 1-5.

[8] Chang-Shou Lin, *On the second eigenfunctions of the laplacian in* $\mathbb{R}^2$, Commun. Math. Phys., **111**(1987), 161-166.
[9] D. Kellogg, *Harmonic functions and Green's integral,* Trans. Amer. Math. Soc., **13** (1912), 109-132.
[10] Zhang Liqun, *On the multiplicity of the second eigenvalue of laplacian in* $\mathbb{R}^2$, Comm. in Analysis and Geometry, **3**(1995), 273-296.
[11] Antonios D. Melas, *On the nodal line of the second eigenfunction of the laplacian in* $\mathbb{R}^2$, J. Differential Geometry, **35**(1992), 255-263.
[12] M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and N. Nadirashvili, *The nodal line of the second eigenfunction of the Laplacian in* $\mathbb{R}^2$ *can be closed,* Duke Math. Journal **90**(1997), 631-640.
[13] L. E. Payne *Isoperimetric inequalities and their applications,* SIAM Rev., **9**(1967), 453-488.
[14] F. Treves, *Basic linear partial differential equations,* Academic Press, (1975).
[15] K. Uhlenbeck, *Generic properties of eigenfunctions,* Amer. J. Math., **98**(1975), 1059-1078.

Department of Mathematics, Seoul National University,
Seoul 151-742 Korea