Spectrum of Sunflower Hypergraphs

Prajnanaswaroopa S
sntrm4@rediffmail.com

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Abstract

Hypergraphs are generalization of graphs, which have several useful applications. Sunflower hypergraphs are interesting hypergraphs, which become linear in some cases. In this paper, we discuss the Siedel spectrum of these hypergraphs.

Introduction

Let $H = (V, E)$ be a hypergraph with $V = \{v_1, v_2, v_3, \dots, v_n\}$ and $E = \{e_1, e_2, e_3, \dots, e_m\}$. Each edge e_i is a subset of V. Let $A(H)$ be the adjacency matrix of H, which is defined by,

$$A(H) = \begin{cases} a_{ij} = |e_k : i \neq j; (v_i, v_j) \in e_k \subset E| \\ 0 \text{ otherwise.} \end{cases}$$

We denote by $O_{p,q}$, $J_{p,q}$ and I_p the zero matrix of order pq, the all ones matrix of order pq and identity matrices of order p respectively. In case $p = q$, we just write J_p or O_p. Then, the definition for Sediel matrix $S(H)$ of a hypergraph [10] is given by $S(H) = J_n - I_n - 2A(H)$. This paper has terminologies, method and proof technique heavily inspired from [5].

The characteristic polynomial of hypergraph H, with respect to the adjacency matrix is,

$$\phi_A(H, \lambda) = det(A - \lambda I).$$

A complete graph is a graph in which every pair of distinct vertices is connected by an edge. Then, the adjacency matrix of a complete graph on p vertices is denoted by K_p. Let $C_X(n)$ be the characteristic polynomial of X times the Siedel matrix of the complete graph on n vertices.

In [1], Banerjee and Das discuss the adjacency spectrum of the sunflower. Also, hyperstar [2] is a particular case of the sunflower hypergraph. Cardoso gives the adjacency spectrum of hyperstar. Rodríguez [6] emphasis on the Laplacian matrix of hypergraph. The properties of the adjacency and Laplacian spectrum of (k, r) regular hypergraph are presented in [4]. Here, we focus on the Seidel spectrum of the sunflower hypergraph.

Definitions

Definition 1. *A sunflower hypergraph $SH(n, p, h)$ is an h-uniform hypergraph of order $n = h + (k - 1)p$ and size $k(1 \leq p \leq h - 1$ and $h \geq 3)$ such that each edge consists of p distinct vertices and a common subset to all edges with $h - p$ vertices.*

Lemma 1. *The eigenvalues of the matrix $X = aI_n + bJ_n$ are given by $bn + a$ with multiplicity 1 and a with multiplicity $n - 1$.*

Lemma 2. [3] *Let A and D be square matrices of arbitrary orders. The determinant of the block matrix*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with invertible matrix D is given by $|M| = |D||A - BD^{-1}C| = \frac{||D|A - B \cdot adj(D)C|}{|D|^{n-1}}$.

For a square matrix A and vectors u, v, we have, the Sherman-Morrison formula [7], which is a special case of Woodbury formula [8],[9] is

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1 + v^T A^{-1} u}$$

.

Spectrum of sunflower hypergraph

The adjacency matrix of sunflower hypergraphs $SH(n, h, p)$, when $p \neq 1$ can be given as

$$A(SH(n,h,p)) = \begin{pmatrix} K_p & O & \dots & O & J_{p,h-p} \\ O_p & K_p & \dots & O_p & J_{p,h-p} \\ O_p & O_p & \ddots (k) - times & O_p & J_{p,h-p} \\ O_p & O_p & \dots & K_p & J_{p,h-p} \\ J_{h-p,p} & J_{h-p,p} & \dots & J_{h-p,p} & k(K_{h-p}) \end{pmatrix}$$

where $k = \frac{n-h}{p}$.

$$\begin{pmatrix} O_{n-h+1} & J_{n-h+1,h-1} \\ J_{h-1,n-h+1} & kK_{h-1} \end{pmatrix}.$$

Since the Siedel adjacency matrix of a hypergraph with adjacency matrix A of order n is defined as $J_n - I_n - 2A$, therefore, we get that, the matrices of which we wish to find the spectrum are

$$\begin{pmatrix} -K_p & J_p & \dots & J_p & -J_{p,h-p} \\ J_p & -K_p & \dots & J_p & -J_{p,h-p} \\ J_p & J_p & \ddots (k) - times & J_p & -J_{p,h-p} \\ J_p & J_p & \dots & -K_p & -J_{p,h-p} \\ -J_{h-p,p} & -J_{h-p,p} & \dots & -J_{h-p,p} & -X(K_{h-p}) \end{pmatrix},$$

where $X = 2k - 1$ and $p \neq 1$; and,

$$\begin{pmatrix} K_{n-h+1} & -J_{n-h+1,h-1} \\ -J_{h-1,n-h+1} & -XK_{h-1} \end{pmatrix},$$

when $p = 1$.

The generalization of the Lemma 1 is the following.

Theorem 1. *If A is a k order square matrix having constant row sum r and having the same eigenvectors as $aI + bJ$. Then, the eigenvalues of the matrix M defined by*

$$M = \begin{pmatrix} A & bJ_k & \dots & bJ_k \\ bJ_k & A & \dots & bJ_k \\ \vdots & \vdots & (n - times) & \vdots \\ bJ_k & \dots & \dots & A \end{pmatrix}$$

are given by $r + bk(n-1)$ with multiplicity 1, $r - bk$ with multiplicity $n - 1$ and d with multiplicity $k(n-1)$, where d is the eigenvalue of A with respect to the eigenvectors other than $(1\ 1\ 1 \dots\ 1)^T$.

Proof. From Lemma 1, we can construct eigenvectors for M as follows: Let $\vec{j}_i$ denote the all ones vector of order i. Let the eigenvectors of A except $\vec{j}_k$ be labelled as $e_1, e_2, \ldots, e_{k-1}$. Then, we have the eigenvectors of M to be $\vec{j}_{kn}, (\vec{j}_k\ 0\ \ldots\ -\vec{j}_k)^T, (0\ \vec{j}_k\ 0 \ldots -\vec{j}_k)^T, \ldots, (0\ 0\ \ldots 0\ \ -\vec{j}_k)^T, (e_1\ 0\ 0\ \ldots 0)^T, (e_2\ 0\ \ldots\ 0)^T, \ldots, (e_{k-1}\ 0\ \ldots\ 0)^T, (0\ e_1\ 0 \ldots\ 0)^T, \ldots, (0\ 0\ \ldots\ e_{k-1})^T$. The corresponding eigenvalues would then be $r + (bkn - bk) = r + bk(n-1)$ with multiplicity 1 (for eigenvector $\vec{j}_{kn}$), $r - bk$ with multiplicity $n-1$ (for eigenvectors $(\vec{j}_k\ 0\ \ldots\ -\vec{j}_k)^T, (0\ \vec{j}_k\ 0 \ldots -\vec{j}_k)^T, \ldots, (0\ 0\ \ldots 0\ \vec{j}_k\ \ -\vec{j}_k)^T$) and lastly the eigenvalue of A corresponding to the eigenvectors e_i with multiplicity $k(n-1)$. The vectors given above are actually eigenvectors can be easily verified by multiplication and by using the properties of A and bJ_k. □

Lemma 3. *If K_n is the adjacency matrix of the complete graph on n vertices, then adjugate of $M = -k(K_n) - \lambda I_n$ has the form*

$$\begin{pmatrix} C_{kp}(n-1) & (k-\lambda)^{n-2} & \ldots & k(k-\lambda)^{n-2} \\ (1-\lambda)^{n-2} & C_{kp}(n-1) & \ldots & k(k-\lambda)^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ k(k-\lambda)^{n-2} & k(k-\lambda)^{n-2} & \ldots & C_{kp}(n-1) \end{pmatrix}$$

, where $C_X(n) = (k-\lambda)^{n-1}(k - kn - \lambda)$ is the characteristic polynomial of M (or negative of k times the adjacency matrix of the complete graph of order n).

Proof. As $K_n = (k-\lambda)I_n - kJ_n$ is invertible, we again use the property that for any matrix A, $adj(A) = |A| \cdot A^{-1}$. To calculate the inverse of M, we use Sherman-Morrison formula [7]. By the formula, we have, for any square matrix A, $(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$, where u and v are vectors and $1 + v^TA^{-1}u \neq 0$. Here, we can take $u = (1\ 1\ \ldots (n - times)\ 1)^T$ and $v = (-k\ \ -k\ \ \ldots (n - times)\ \ -k)^T$ so that $uv^T = -kJ_n$ and $X = (k-\lambda)I_n$. Then, we get

$$\begin{aligned} M^{-1} &= \frac{1}{k-\lambda}I_n - \frac{\frac{1}{k-\lambda}I_n uv^T \frac{1}{k-\lambda}I_n}{1 + \frac{-kn}{k-\lambda}} \\ &= \frac{1}{k-\lambda}I_n - \frac{\frac{1}{(k-\lambda)^2}(-kJ_n)}{\frac{(k-\lambda)-kn}{k-\lambda}} \\ &= \frac{1}{(k-\lambda)(k-\lambda-kn)}(k-\lambda-n)I_n + kJ_n \end{aligned}$$

This implies that the adjugate then becomes $|M| \cdot M^{-1} = C_X(n)\frac{1}{(k-\lambda)(k-\lambda-kn)}((k-\lambda-kn)I_n + kJ_n)$ $= (k-\lambda)^{n-2}(k-\lambda-kn)I_n + (k-\lambda)^{n-2}J_n = (k-\lambda)^{n-2}((2k-\lambda-nk)-k)I_n + (k-\lambda)^{n-2}kJ_n$ $= (C_X(n-1) - (k-\lambda)^{n-2})I_n + k(k-\lambda)^{n-2}$. This matrix, when expanded, at once gives the desired result. □

Lemma 4. *If K_{h-p} denotes the adjacency matrix of the complete graph on h vertices, then we have $J_{n-h+p,h-p} \cdot adj(-XK_h - \lambda I_h) \cdot J_{h-p,n-h+p} = ((X-\lambda)^{(h-p-2)}(-\lambda + X(1-h+p))(h-p) + X(h-p))^2)J_{n-h+p}$.*

Proof. The proof is straight-forward multiplication, using Lemma 3 and noting that multiplication of $J_{m,n}$ with its transpose equals $nJ_{m,m}$. □

Theorem 2. *The spectrum of the Siedel matrix of $SH(n,h,p)$ with parameters n, h, p; $p \neq \{1, (h-1)\}$, and $h > 2$ is given by the roots of the characteristic polynomial $F(\lambda) = (X-\lambda)^{h-p-2}(1-2p-\lambda)^{k-1}(1-\lambda)^{n-h+p-k} \cdot A$, where $A = -2X^2h + (X^2+X)h^2 - (X^2-X)p^2 - ((X+1)h - (X-1)p - 2X - n - 1)\lambda^2 - \lambda^3 + X^2 + (X^2 - (X^2+X)h)n - (2Xh - (X^2+X)n)p - ((X+1)h^2 - (X-1)p^2 + X^2 - (X^2 + 3X)h - ((X+1)h - 2X)n + (X^2 + (X+1)n - X - 2h)p + 2X)\lambda$. In particular, it has an eigenvalue of 1 with multiplicity at least $n - h - k + p$, $X = 2k - 1$ with multiplicity at least $h - p - 2$, and $1 - 2p$ with multiplicity at least $k - 1$.*

Proof. The Siedel matrix of $SH(n,h,p)$ is given by:

$$S(SH) = \begin{pmatrix} -K_p & J_p & \dots & J_p & -J_{p,h-p} \\ J_p & -K_p & \dots & J_p & -J_{p,h-p} \\ J_p & J_p & \ddots (k)-times & J_p & -J_{p,h-p} \\ J_p & J_p & \dots & -K_p & -J_{p,h-p} \\ -J_{h-p,p} & -J_{h-p,p} & \dots & -J_{h-p,p} & -X(K_{h-p}) \end{pmatrix}$$

where $X = 2k-1$. We proceed to calculate the characteristic polynomial of the matrix $S(SH)$. This is nothing but determinant of the matrix $S(SH) - \lambda I_n$. In matrix form, this is

$$\begin{pmatrix} -K_p - \lambda I_p & J_p & \dots & J_p & -J_{p,h-p} \\ J_p & -K_p - \lambda I_p & \dots & J_p & -J_{p,h-p} \\ J_p & J_p & \ddots (k)-times & J_p & -J_{p,h-p} \\ J_p & J_p & \dots & -K_p - \lambda I_p & -J_{p,h-p} \\ -J_{h-p,p} & J_{h-p,p} & \dots & J_{h-p,p} & -X(K_{h-p}) - \lambda I_{h-p} \end{pmatrix}$$

By using Lemma 2, the determinant can be written as:

$$\frac{|-XK_{h-p} - \lambda I_{h-p}|M - J_{n-h+p,h-p} \cdot adj(-XK_{h-p} - \lambda I_{h-p})J_{h-p,n-h+p}|}{|-XK_{h-p} - \lambda I_h|^{n-h+p-1}},$$

where M is the block matrix of the first $n-h+p$ rows and columns given by

$$\begin{pmatrix} -K_p - \lambda I_p & J_p & \dots & J_p \\ J_p & -K_p - \lambda I_p & \dots & J_p \\ J_p & J_p & \ddots (k)-times & J_p \\ J_p & J_p & \dots & -K_p - \lambda I_p \end{pmatrix}.$$

Taking cognizance of the fact that $|-XK_{h-p} - \lambda I_{h-p}| = C_X(h-p)$ and, from Lemma 4, $J_{n-h+p,h-p} \cdot adj(-XK_{h-p} - \lambda I_h)J_{h-p,n-h+p} = ((X-\lambda)^{(h-p-2)}(-\lambda + X(1-h+p))(h-p) + X(h-p))^2)J_{n-h+p}$.

The determinant is given by:

$$\frac{|C_X(h-p)M - ((X-\lambda)^{(h-p-2)}(-\lambda + X(1-h+p))(h-p) + X(h-p))^2)J_{n-h+p}|}{(C_X(h-p))^{n-h+p-1}}.$$

We take $Y = (X-\lambda)^{(h-p-2)}((-\lambda + X(1-h+p))(h-p) + X(h-p)^2)$. Then, the above becomes, in block form:

$$\frac{1}{(C_X(h-p))^{n-h+p-1}} \begin{vmatrix} C_X(h-p)[-K_p - \lambda I_p] - YJ_p & J_p(C_X(h-p)-Y) & \dots & J_p(C_X(h-p)-Y) \\ J_p(C_X(h-p)-Y) & C_X(h-p)[-K_p - \lambda I_p] - YJ_p & \dots & J_p(C_X(h-p)-Y) \\ J_p(C_X(h-p)-Y) & J_p(C_X(h-p)-Y) & \ddots (k)-times & J_p(C_X(h-p)-Y) \\ J_p(C_X(h-p)-Y) & J_p(C_X(h-p)-Y) & \dots & C_X(h-p)[-K_p - \lambda I_p] - YJ_p \end{vmatrix}$$

Comparing the above matrix with Theorem 1, we get $A = C_X(h-p)[-K_p - \lambda I_p] - YJ_p$ and $b = C_X(h-p) - Y$. Therefore, as the eigenvalues of matrix in this case are A are $C_X(h-p)(n+1-p-h-\lambda) + Y(h-p-n)$ with multiplicity 1, $(C_X(h)(1-2p-\lambda))$ with multiplicity $k-1$ and $C_X(h-p)(1-\lambda)$ with multiplicity $n-h+p-k$, the determinant will be equal to $C_X(h-p)(n+1-p-h-\lambda) + Y(h-p-n)(C_X(h-p)(1-2p-\lambda))^{k-1}(C_X(h-p)(1-x))^{n-h+p-1}]$. Simplifying the expression using the form of $C_X(h-p) = (X-\lambda)^{h-p-1}(X(1-h+p) - \lambda)$, we get $F(\lambda) = (X-\lambda)^{h-p-2}(1-2p-\lambda)^{k-1}(1-\lambda)^{n-h+p-k} \cdot A$, where $A = -2X^2h + (X^2+X)h^2 - (X^2-X)p^2 - ((X+1)h - (X-1)p - 2X - n - 1)\lambda^2 - \lambda^3 + X^2 + (X^2 - (X^2+X)h)n - (2Xh - (X^2+X)n)p - ((X+1)h^2 - (X-1)p^2 + X^2 - (X^2+3X)h - ((X+$

$1)h - 2X)n + (X^2 + (X+1)n - X - 2h)p + 2X)\lambda$. The expression $F(\lambda)$ is therefore the characteristic polynomial of the desired Siedel matrix. Therefore, roots of the cubic polynomial A will fully determine the spectrum of G, as the eigenvalues $1, X = 2k-1$ and $2p-1$ are already known with their minimum multiplicities $(n-1-h+p), (h-p-2)$ and $k-1$ from the expression. □

Corollary 1. *The spectrum of the Siedel matrix of $SH(n, h, h-1)$ with $h > 2$ is given by the roots of the characteristic polynomial $H(\lambda) = (x^2 + (2p-1)x + 1 - n)(1 - 2p - \lambda)^{k-1}(1-\lambda)^{n-1-k}$. Therefore, the Siedel matrix has eigenvalues 1 with minimum multiplicity $n-1-k$, $1-2p$ with minimum multiplicity $k-1$.*

Proof. In this case, the matrix retains a similar structure, except that since $p = h-1 \implies h-p = 1$, the last block K_{h-p} is a zero matrix. In this case, the characteristic polynomial $C_X(h-p)$ reduces to just $-\lambda$, and corresponding the adjugate is 1. Therefore, in the discussion of the above theorem, we replace $C_X(h-p)$ by $-\lambda$ and Y by $(-1\ -1\ \ldots (n-1)-times\ -1) \cdot (-1\ -1\ \ldots (n-1)-times\ -1)^T = n-1$. Simplifying using these substitutions in the characteristic polynomial $F(\lambda)$ of the previous theorem, we get the desired characteristic polynomial as $(1-2p-\lambda)^{k-1}(1-\lambda)^{n-1-k} \cdot A$, where $A = -\lambda(n+2-2h-\lambda) - (n-1)^2$, which, on simplifying, is $H(\lambda) = (x^2 + (2p-1)x + 1 - n)(1-2p-\lambda)^{k-1}(1-\lambda)^{n-1-k}$. Therefore, the roots of the quadratic $(x^2 + (2p-1)x + 1 - n)$ fully determine the spectrum of the desired matrix in this case, as the other eigenvalues $1-2p$ and 1 have their minimum multiplicities already determined as $k-1$ and $n-1-k$ respectively. □

Theorem 3. *The spectrum of the Siedel matrix of $SH(n, h, 1)$ $h > 2$ is given by the roots of the polynomial $G(\lambda) = (-\lambda-1)^{n-h}(X-\lambda)^{h-3}A$, where $A = (X^2+X)h^2 - ((X+1)h - 3X - n)\lambda^2 - \lambda^3 - 2(X^2+X)h + (2X^2 - (X^2+X)h + X)n - ((X+1)h^2 + 2X^2 - (X^2+3X+2)h - ((X+1)h - 3X - 1)n + 1)\lambda + X$. In particular, it has eigenvalue of 1 with multiplicity at least $n-h$, and $X = 2k-1$ with multiplicity at least $h-3$.*

Proof. In this case, the Siedel matrix of $SH(n, h, 1)$ has the form

$$S(SH) = \begin{pmatrix} K_{n-h+1} & -J_{n-h+1,h-1} \\ -J_{h-1,n-h+1} & -XK_{h-1} \end{pmatrix}$$

, where $X = 2k-1 = 2(n-h+1) - 1 = 2n - 2h + 1$, and other symbols have their usual meaning.

We proceed to calculate the characteristic polynomial of the above matrix. This is nothing but determinant of the matrix $S(SH) - \lambda I_n$. In matrix form, this is

$$\begin{pmatrix} K_{n-h+1} - \lambda I_{n-h+1} & -J_{n-h+1,h-1} \\ -J_{h-1,n-h+1} & -XK_{h-1} - \lambda I_{h-1} \end{pmatrix}$$

By using Lemma 2, the determinant can be written as:

$$\frac{|-XK_{h-1} - \lambda I_{h-1}| K_{n-h+1} - J_{n-h+1,h-1} \cdot adj(-XK_{h-1} - \lambda I_{h-1}) J_{h-1,n-h+1}|}{|-XK_{h-1} - \lambda I_h|^{n-h}}$$

.

Taking cognizance of the fact that $|-XK_{h-1} - \lambda I_{h-1}| = C_X(h-1)$ and $J_{n-h+1,h-1} \cdot adj(-XK_{h-1} - \lambda I_h) J_{h-1,n-h+1} = ((X-\lambda)^{(h-1)}(-\lambda + X(1-h+1))(h-1) + X(h-1))^2) J_{n-h+1}$ from (2), we get the determinant as $\frac{|C_X(h-1)K_{n-h+1} - \lambda I_{n-h+1} - ((X-\lambda)^{(h-1)}(-\lambda + X(1-h+1))(h-1) + X(h-p))^2) J_{n-h+1}|}{(C_X(h-1))^{n-h}}$.

We take $Y = (X-\lambda)^{(h-3)}((-\lambda + X(2-h))(h-1) + X(h-1)^2)$. Then, the determinant becomes $\frac{|C_X(h-1)(K_{n-h+1} - \lambda I_{n-h+1}) - YJ_{n-h+1}|}{(C_X(h-1))^{n-h}} = \frac{|(C_X(h-1) - Y)J_{n-h+1} + C_X(h-1)(-\lambda-1)I_{n-h+1}|}{(C_X(h-1))^{n-h}}$. From Lemma 1, the eigenvalues of $(C_X(h-1) - Y)JY_{n-h+1} + (-\lambda - C_X(h-1))I_n$ are $(n-h+1)(C_X(h-1) - Y)$ with multiplicity 1 and $C_X(h-1)(-\lambda-1)$ with multiplicity $n-h$. Therefore, the determinant becomes $\frac{(C_X(h-1)(-\lambda-1))^{n-h}((C_X(h-1)(n-h-\lambda) + Y(h-1-n)}{(C_X(h-1))^{n-h}}$. Simplifying the expression using $C_X(h-p) = (X -$

$\lambda)^{h-2}(X(2-h)-\lambda)$, we get the determinant, which is the characteristic polynomial to be $G(\lambda) = (-\lambda - 1)^{n-h}(X-\lambda)^{h-3}A$, where $A = (X^2+X)h^2 - ((X+1)h - 3X - n)\lambda^2 - \lambda^3 - 2(X^2+X)h + (2X^2 - (X^2+X)h + X)n - ((X+1)h^2 + 2X^2 - (X^2+3X+2)h - ((X+1)h - 3X - 1)n + 1)\lambda + X$. □

Corollary 2. *The spectrum of the Siedel matrix of* $SH(n, 2, 1)$ *is given by* $\{n-1, -1, -1, \ldots\}$

Proof. In this case, the Siedel matrix is just the Siedel matrix of the star graph on n vertices. The spectrum of this graph is well known, which can be derived either from the above theorem, by taking the adjugate of $K_{h-p} - \lambda I_{h-p} = K_{-1} - \lambda I_1 = -\lambda$; or by the usual literature on bipartite spectral graph theory. □

To aid in verification of formulae, we provide here, as an appendix, the SageMath code that constructs the Siedel matrix of $SH(n, h, p)$ and computes its eigenvalues. The formulae verification is also performed for few cases.

Appendices

Appendix 1

Here is the code to generate the Siedel matrix $SH(12, 6, 3)$.

```
 def SiedelSunflowerMat(n,h,p):
    k=(n-h)/p+1
    q=graphs.CompleteGraph(p)
    P=q.am()
    A=graphs.CompleteGraph(h-p)
    H=k*A.am()
    x=n-h+p
    J=ones_matrix(x,h-p)
    N=block_matrix([[J],[H]])
    N1=J.transpose()
    Y1=[]
    for i in list(range(k)):
        Y1.append(P)
    Y=block_diagonal_matrix(Y1)
    Z=block_matrix([[Y],[N1]])
    S=block_matrix([[Z,N]])
    J2=ones_matrix(n)
    I2=identity_matrix(n)
    S1=J2-I2-2*S
    return S1
S=SiedelSunflowerMat(12,6,3)
S.eigenvalues()
```

The output was $[5, 5, -5, -5, 1, 1, 1, 1, 1, 1, 1, -11.717797887081347, 5.717797887081348]$. As a test, the roots of the cubic defined in Theorem 2 is found as under:

```
var('x')
n=12
h=6
p=3
k=((n-h)/p)+1
X=2*k-1
A=-2*X^2*h + (X^2 + X)*h^2 - (X^2 - X)*p^2 - ((X + 1)*h - (X - 1)*p - 2*X - n - 1)*x^2 - x^3 +
A.roots(ring=RR)
```

The output was $[(-11.7177978870813, 1), (5.00000000000000, 1), (5.71779788708135, 1)]$, which matches well with the expected result.

Appendix 2

Similarly, here is the code to generate the Siedel matrix of $SH(13, 4, 3)$ and then find its eigenvalues

```
 def SiedelSunflowerMat(n,h,p):
    k=(n-h)/p+1
    q=graphs.CompleteGraph(p)
    P=q.am()
    A=graphs.CompleteGraph(h-p)
    H=k*A.am()
    x=n-h+p
    J=ones_matrix(x,h-p)
    N=block_matrix([[J],[H]])
    N1=J.transpose()
    Y1=[]
    for i in list(range(k)):
        Y1.append(P)
    Y=block_diagonal_matrix(Y1)
    Z=block_matrix([[Y],[N1]])
    S=block_matrix([[Z,N]])
    J2=ones_matrix(n)
    I2=identity_matrix(n)
    S1=J2-I2-2*S
    return S1
S=SiedelSunflowerMat(13,4,3)
S.eigenvalues()
```

The output was found to be $[-5, -5, -5, 1, 1, 1, 1, 1, 1, 1, 1, -1.424428900898053, 8.42442890089806]$. As a verification, here is the code to verify the quadratic presented in Corollary 1.

```
var('x,n,p')
n=13
h=4
p=h-1
s=x^2+(2*p-n)*x+(1-n)
s.roots(ring=RR)
```

The output was found to be $[(-1.42442890089805, 1), (8.42442890089805, 1)]$, which is consonance with the prior output.

Appendix 3

Lastly, the code to generate the Siedel matrix of $SH(13, 6, 1)$and then find its eigenvalues is presented below:

```
 def SiedelSunflowerMat(n,h,p):
    k=(n-h)/p+1
    q=graphs.CompleteGraph(p)
    P=q.am()
    A=graphs.CompleteGraph(h-p)
    H=k*A.am()
```

```
    x=n-h+p
    J=ones_matrix(x,h-p)
    N=block_matrix([[J],[H]])
    N1=J.transpose()
    Y1=[]
    for i in list(range(k)):
        Y1.append(P)
    Y=block_diagonal_matrix(Y1)
    Z=block_matrix([[Y],[N1]])
    S=block_matrix([[Z,N]])
    J2=ones_matrix(n)
    I2=identity_matrix(n)
    S1=J2-I2-2*S
    return S1
S=SiedelSunflowerMat(13,6,1)
S.eigenvalues()
```

The output was found to be $[15, 15, 15, 15, -1, -1, -1, -1, -1, -1, -1, -60.59178786746157, 7.591787867461572]$. Correspondingly, the verification code is as under:

```
var('x')
n=13
h=6
k=n-h+1
X=2*k-1
s=(X-x)*(X*(2-h)-x)*(n-h-x)-((-x+X*(2-h))*(h-1)+X*(h-1)^2)*(n-h+1)
s.full_simplify()
s=(X^2 + X)*h^2 - ((X + 1)*h - 3*X - n)*x^2 - x^3 - 2*(X^2 + X)*h + (2*X^2 - (X^2 + X)*h + X)*r
s.roots(ring=RR)
```

The output was found to be $[(-60.5917878674616, 1), (7.59178786746157, 1), (15.0000000000000, 1)]$, which is agreement to the prior result.

Conclusion

We have used the block matrix technique, Sherman-Morrison formula and eigenvector reconstruction method to discuss the spectrum and characteristic polynomial of the Siedel matrix of sunflower hypergraphs in this work. The method can also be refined for more general matrices and hypergraphs, which can be used further for various applications.

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