Proof for specific type of continued fraction

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Abstract

I am going to prove the following result:

$$A_n = 2n+1+\cfrac{1}{2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\ddots}}}} = \frac{\displaystyle\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}$$

I am going to demonstrate and use two telescoping series.
Then I will combine both of them to create a new formula
and completing the proof by using an inductive proof.
(I am using Lambert's continued fraction for the base case)

Formula #1

$$\sum_{k=0}^{n-1}\left[2^{n-k}\,\frac{n!}{(2n)!}\,\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}-1\right]\right]=\ ?$$

we wanna find out what is the value of the expression above

$$\sum_{k=0}^{n-1}\left[2^{n-k}\,\frac{n!}{(2n)!}\,\frac{(n+k)!}{(n-k)!k!}\,\frac{2k}{(n+1-k)(n+k)}-2^{n-k}\,\frac{n!}{(2n)!}\,\frac{(n+k)!}{(n-k)!k!}\right]=$$

we will rewrite it in this way:

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\,\frac{n!}{(2n)!}\,\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\,\frac{n!}{(2n)!}\,\frac{(n+k)!(k+1)}{(n-k)!(k+1)!}\right]=$$

This is a telescoping series (I did not canceled the factorial terms in order to avoid the (-1)! case)

$$k=0::\left(2^{n-(0-1)}\,\frac{n!}{(2n)!}\,\frac{(n+(0-1))!0}{(n-(0-1))!0!}-2^{n-0}\,\frac{n!}{(2n)!}\,\frac{(n+0)!1}{(n-0)!(1)!}\right)$$

$$k=1::\left(2^{n-(1-1)}\,\frac{n!}{(2n)!}\,\frac{(n+(1-1))!1}{(n-(1-1))!1!}-2^{n-1}\,\frac{n!}{(2n)!}\,\frac{(n+1)!2}{(n-1)!2!}\right)$$

$$k=2::\left(2^{n-(2-1)}\,\frac{n!}{(2n)!}\,\frac{(n+(2-1))!2}{(n-(2-1))!2!}-2^{n-2}\,\frac{n!}{(2n)!}\,\frac{(n+2)!3}{(n-2)!3!}\right)$$

$$k=n-1::\left(2^{2}\,\frac{n!}{(2n)!}\,\frac{(2n-2)!(n-1)}{(2)!(n-1)!}-2^{1}\,\frac{n!}{(2n)!}\,\frac{(n+n-1)!n}{(n-n+1)!(n)!}\right)$$

we will write it a bit cleaner:

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\,\frac{n!}{(2n)!}\,\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\,\frac{n!}{(2n)!}\,\frac{(n+k)!}{(n-k)!k!}\right]=\left(0-2^{n-0}\,\frac{n!}{(2n)!}\,\frac{(n+0)!}{(n-0)!0!}\right)+$$

$$\left(2^{n-0}\,\frac{n!}{(2n)!}\,\frac{(n+0)!}{(n-0)!0!}-2^{n-1}\,\frac{n!}{(2n)!}\,\frac{(n+1)!}{(n-1)!1!}\right)+\left(2^{n-1}\,\frac{n!}{(2n)!}\,\frac{(n+1)!}{(n-1)!1!}-2^{n-2}\,\frac{n!}{(2n)!}\,\frac{(n+2)!}{(n-2)!2!}\right)+$$

$$\left(2^{n-2}\,\frac{n!}{(2n)!}\,\frac{(n+2)!}{(n-2)!2!}-2^{n-3}\,\frac{n!}{(2n)!}\,\frac{(n+3)!}{(n-3)!3!}\right)+...+\left(2^{2}\,\frac{n!}{(2n)!}\,\frac{(2n-2)!}{(2)!(n-2)!}-2^{1}\,\frac{n!}{(2n)!}\,\frac{(2n-1)!}{1!(n-1)!}\right)$$

now we will highlight the terms of the telescoping series, and we will get this:

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=0+$$

$$\left(-2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}+2^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!}{(n-0)!0!}\right)+\left(-2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}+2^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!}{(n-1)!1!}\right)+$$

$$\left(-2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}+2^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!}{(n-2)!2!}\right)+\left(-2^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!}{(n-3)!3!}+2^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!}{(n-3)!3!}\right)+$$

$$...+\left(-2^2\frac{n!}{(2n)!}\frac{(2n-2)!}{(2)!(n-2)!}+2^2\frac{n!}{(2n)!}\frac{(2n-2)!}{(2)!(n-2)!}\right)-2^1\frac{n!}{(2n)!}\frac{(2n-1)!}{1!(n-1)!}$$

only the first and last terms are left (the first term in this telescoping series is 0)

$$\sum_{k=0}^{n-1}\left[2^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}-2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\right]=0-2^1\frac{n!}{(2n)!}\frac{(2n-1)!}{1!(n-1)!}=-1$$

now we will go back to our original form that we wanted to find out that value for:

$$\sum_{k=0}^{n-1}\left[2^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}-1\right]\right]=-1$$

$$\frac{2^n n!}{(2n)!}\sum_{k=0}^{n-1}\left[\left(\frac{1}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[-1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+1=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{1}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[-1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{1}{2^n}\right)\frac{(2n)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{(-1)^n}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}\left[-1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0$$

$$\frac{(2n+1)(n+k)!}{(n-k)!k!}\cdot\frac{2k}{(n+1-k)(n+k)}=\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}$$

(The above result is a mathematical identity obtained by simple arithmetic)

$$\boxed{\sum_{k=0}^{n-1}\left[\left(\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{(-1)^n}{2^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0}$$

Formula #2

$$\sum_{k=0}^{n-1}\left[\frac{n!}{(2n)!}\left(\frac{(-2)^n}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}+1\right]\right]=?$$

we wanna find out what is the value of the expression above

$$\sum_{k=0}^{n-1}\left[\frac{n!}{(2n)!}\left(\frac{(-2)^n}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\frac{2k}{(n+1-k)(n+k)}+\frac{n!}{(2n)!}\left(\frac{(-2)^n}{(-2)^k}\right)\frac{(n+k)!}{(n-k)!k!}\right]=$$

we will rewrite it in this way:

$$\sum_{k=0}^{n-1}\left[-(-2)^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}+(-2)^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!(k+1)}{(n-k)!(k+1)!}\right]=$$

This is also a telescoping series! (I did not canceled the factorial terms in order to avoid the (-1)! case)

$$k=0::\left(-(-2)^{n-(0-1)}\frac{n!}{(2n)!}\frac{(n+(0-1))!0}{(n-(0-1))!(0)!}+(-2)^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!(0+1)}{(n-0)!(0+1)!}\right)$$

$$k=1::\left(-(-2)^{n-(1-1)}\frac{n!}{(2n)!}\frac{(n+(1-1))!1}{(n-(1-1))!(1)!}+(-2)^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!(1+1)}{(n-1)!(1+1)!}\right)$$

$$k=2::\left(-(-2)^{n-(2-1)}\frac{n!}{(2n)!}\frac{(n+(2-1))!2}{(n-(2-1))!(2)!}+(-2)^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!(2+1)}{(n-2)!(2+1)!}\right)$$

$$k=n-1::\left(-(-2)^{2}\frac{n!}{(2n)!}\frac{(n+(n-1-1))!(n-1)}{(n-(n-1-1))!(n-1)!}+(-2)^{1}\frac{n!}{(2n)!}\frac{(n+n-1)!(n-1+1)}{(n-n+1)!(n-1+1)!}\right)$$

we will write it a bit cleaner:

$$\sum_{k=0}^{n-1}\left[-(-2)^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}+(-2)^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!(k+1)}{(n-k)!(k+1)!}\right]=\left(0+(-2)^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!(1)}{(n-0)!(1)!}\right)+$$

$$\left(-(-2)^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!1}{(n-0)!(1)!}+(-2)^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!2}{(n-1)!(2)!}\right)+\left(-(-2)^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!2}{(n-1)!(2)!}+(-2)^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!3}{(n-2)!3!}\right)+$$

$$\left(-(-2)^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!3}{(n-2)!(3)!}+(-2)^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!4}{(n-3)!4!}\right)+...+\left(-(-2)^{2}\frac{n!}{(2n)!}\frac{(n+(n-2))!(n-1)}{(n-(n-2))!(n-1)!}+(-2)^{1}\frac{n!}{(2n)!}\frac{(n+(n-1))!n}{(n-(n-1))!(n)!}\right)$$

now we will highlight the terms of the telescoping series, and we will get this:

$$\sum_{k=0}^{n-1}\left[(-2)^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}+(-2)^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!(k+1)}{(n-k)!(k+1)!}\right]=0+$$

$$\left((-2)^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!1}{(n-0)!(1)!}-(-2)^{n-0}\frac{n!}{(2n)!}\frac{(n+0)!1}{(n-0)!(1)!}\right)+\left((-2)^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!2}{(n-1)!(2)!}+-(-2)^{n-1}\frac{n!}{(2n)!}\frac{(n+1)!2}{(n-1)!(2)!}\right)+$$

$$\left((-2)^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!3}{(n-2)!3!}-(-2)^{n-2}\frac{n!}{(2n)!}\frac{(n+2)!3}{(n-2)!3!}\right)+\left((-2)^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!4}{(n-3)!4!}-(-2)^{n-3}\frac{n!}{(2n)!}\frac{(n+3)!4}{(n-3)!4!}\right)+...+$$

$$\left((-2)^{2}\frac{n!}{(2n)!}\frac{(n+(n-2))!(n-1)}{(n-(n-2))!(n-1)!}-(-2)^{2}\frac{n!}{(2n)!}\frac{(n+(n-2))!(n-1)}{(n-(n-2))!(n-1)!}\right)+(-2)^{1}\frac{n!}{(2n)!}\frac{(n+(n-1))!n}{(n-(n-1))!(n)!}$$

only the first and last terms are left (the first term in this telescoping series is 0)

$$\sum_{k=0}^{n-1}\left[(-2)^{n-(k-1)}\frac{n!}{(2n)!}\frac{(n+(k-1))!k}{(n-(k-1))!(k)!}+(-2)^{n-k}\frac{n!}{(2n)!}\frac{(n+k)!(k+1)}{(n-k)!(k+1)!}\right]=0+(-2)^{1}\frac{n!}{(2n)!}\frac{(n+(n-1))!n}{(n-(n-1))!(n)!}=-1$$

now we will go back to our original form that we wanted to find out that value for:

$$\sum_{k=0}^{n-1}\left[\frac{n!}{(2n)!}\left(\frac{(-2)^{n}}{(-2)^{k}}\right)\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}+1\right]\right]=-1$$

$$\frac{(-2)^{n}n!}{(2n)!}\sum_{k=0}^{n-1}\left[\left(\frac{1}{(-2)^{k}}\right)\frac{(n+k)!}{(n-k)!k!}\left[\frac{2k}{(n+1-k)(n+k)}+1\right]\right]+1=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{1}{(-2)^{k}}\right)\frac{(n+k)!}{(n-k)!k!}\left[1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{1}{(-2)^{n}}\right)\frac{(2n)!}{n!}=0$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^{2}}{(-2)^{k}}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}\left[1+\frac{2k}{(n+1-k)(n+k)}\right]\right]+\left(\frac{e^{2}}{(-2)^{n}}\right)\frac{(2n+1)!}{n!}=0$$

$$\frac{(2n+1)(n+k)!}{(n-k)!k!}\cdot\frac{2k}{(n+1-k)(n+k)}=\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}$$

(The above result is a mathematical identity obtained by simple arithmetic)

$$\boxed{\sum_{k=0}^{n-1}\left[\left(\frac{e^{2}}{(-2)^{k}}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^{2}}{(-2)^{k}}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{e^{2}}{(-2)^{n}}\right)\frac{(2n+1)!}{n!}=0}$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{e^2}{(-2)^n}\right)\frac{(2n+1)!}{n!}=0 \quad \text{(Formula \#1)}$$

$$\sum_{k=0}^{n-1}\left[\left(\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{(-1)^n}{2^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0 \quad \text{(Formula \#2)}$$

Combining formula #1 and formula #2 will give us this:

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\left[\frac{(n+1+k)!}{(n+1-k)!k!}-\frac{(n-1+k)!}{(n-1-k)!k!}\right]\right]+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=0$$

$$\boxed{\begin{aligned}&\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=\\[2mm]&\left(\frac{e^2}{(-2)^n}-\frac{(-1)^n}{2^n}+\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}-\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}=\\[2mm]&\left(\frac{e^2}{(-2)^n}+\frac{(-1)^{n-1}}{2^n}\right)\frac{(2n+1)!}{n!}+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}+2\left(\frac{e^2}{(-2)^{n+1}}+\frac{(-1)^n}{2^{n+1}}\right)\frac{(2n+1)!}{n!}\end{aligned}}$$

(The above result is a mathematical identity obtained by simple arithmetic)

$$\sum_{k=0}^{n-1}\left[\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}\right]+$$

$$(2n+1)\left(\frac{e^2}{(-2)^n}+\frac{(-1)^{n-1}}{2^n}\right)\frac{(2n)!}{n!}+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(2n+1)!}{n!}+\left(\frac{e^2}{(-2)^{n+1}}+\frac{(-1)^n}{2^{n+1}}\right)\frac{(2n+2)!}{n+1!}=0$$

Let's restructure it a bit (by splitting the summation).

$$\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(2n+1)(n+k)!}{(n-k)!k!}+(2n+1)\left(\frac{e^2}{(-2)^n}+\frac{(-1)^{n-1}}{2^n}\right)\frac{(n+n)!}{(n-n)!n!}+$$

$$\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}+\left(\frac{e^2}{(-2)^n}+\frac{(-1)^n}{2^n}\right)\frac{(n+1+n)!}{(n+1-n)!n!}+\left(\frac{e^2}{(-2)^{n+1}}+\frac{(-1)^n}{2^{n+1}}\right)\frac{(n+1+n+1)!}{(n+1-n-1)!n+1!}=0$$

Let's restructure it a bit more:

$$(2n+1)\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}+\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}=\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}$$

we will divide by $\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}$ and we will get this:

$$2n+1+\frac{\displaystyle\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n+1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}=\frac{\displaystyle\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}$$

Let's make one more adjustment.

$$2n+1+\frac{1}{\dfrac{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n+1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}{\displaystyle\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}}}=\frac{\displaystyle\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}$$ ← Main result!

We will define a new series that we will call A_n as follows:

$$A_n=\frac{\displaystyle\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}\qquad\Rightarrow\qquad A_{n+1}=\frac{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n+1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}{\displaystyle\sum_{k=0}^{n+1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n+1+k)!}{(n+1-k)!k!}}$$

please notice that we got $2n+1+\dfrac{1}{A_{n+1}}=A_n$ from our main result!

this is what we wanted to get! (remember this because we will use this result later in the Induction step)

now lets find out what is the value of A_1

$$A_1=\frac{\displaystyle\sum_{k=0}^{0}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^1}{2^k}\right)\frac{(1-1+k)!}{(1-1-k)!k!}}{\displaystyle\sum_{k=0}^{1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{1-1}}{2^k}\right)\frac{(1+k)!}{(1-k)!k!}}=\frac{\left(\frac{e^2}{(-2)^0}-\frac{1}{2^0}\right)\frac{(0)!}{(0)!0!}}{\left(\frac{e^2}{(-2)^0}+\frac{1}{2^0}\right)\frac{(1)!}{(1)!0!}+\left(\frac{e^2}{(-2)^1}+\frac{1}{2^1}\right)\frac{(1+1)!}{(1-1)!1!}}=\frac{e^2-1}{\left(e^2+1\right)+\left(-e^2+1\right)}=\frac{e^2-1}{2}$$

Lambert's continued fraction

$$\tanh(x) = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{x^2}{5 + \cfrac{x^2}{7 + \cfrac{x^2}{9 + \cfrac{x^2}{11 + \ddots}}}}}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\tanh(1) = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{9 + \cfrac{1}{11 + \ddots}}}}}} = \frac{e^2 - 1}{e^2 + 1}$$

$$\frac{e^2 + 1}{e^2 - 1} = 1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{9 + \cfrac{1}{11 + \ddots}}}}}$$

$$\cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{9 + \cfrac{1}{11 + \ddots}}}}} = \frac{e^2 + 1}{e^2 - 1} - 1 = \frac{2}{e^2 - 1}$$

$$3 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{9 + \cfrac{1}{11 + \ddots}}}} = \frac{e^2 - 1}{2} = A_1 \qquad \text{(just like the value we got above for } A_1\text{)}$$

this is what we wanted!

(This is known result but I am showing how to get to it anyway)

A proof by induction

Proposition:

$$A_n = 2n+1 + \cfrac{1}{2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\ddots}}}} = \frac{\displaystyle\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}}$$

Base Case: $n=1$

$$A_1 = 3 + \cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\ddots}}}} = \frac{\displaystyle\sum_{k=0}^{0}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^1}{2^k}\right)\frac{(1-1+k)!}{(1-1-k)!k!}}{\displaystyle\sum_{k=0}^{1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{1-1}}{2^k}\right)\frac{(1+k)!}{(1-k)!k!}} = \frac{e^2-1}{2}$$ (I already showed that above!)

Induction step:

$$A_{n+1} = 2(n+1)+1 + \cfrac{1}{2(n+1)+3+\cfrac{1}{2(n+1)+5+\cfrac{1}{2(n+1)+7+\cfrac{1}{2(n+1)+9+\ddots}}}}$$

$$A_{n+1} = 2n+3 + \cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\cfrac{1}{2n+11+\ddots}}}}$$

$$A_n = 2n+1 + \cfrac{1}{2n+3+\cfrac{1}{2n+5+\cfrac{1}{2n+7+\cfrac{1}{2n+9+\ddots}}}} = \frac{\displaystyle\sum_{k=0}^{n-1}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^n}{2^k}\right)\frac{(n-1+k)!}{(n-1-k)!k!}}{\displaystyle\sum_{k=0}^{n}\left(\frac{e^2}{(-2)^k}+\frac{(-1)^{n-1}}{2^k}\right)\frac{(n+k)!}{(n-k)!k!}} = 2n+1+\frac{1}{A(n+1)}$$

As previously demonstrated in the "Main result" section on page 7.

QED