Weighted Riemann zeta limits on the real axis

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Abstract

It is investigated whether for real argument s the $(s-1)^{n+1}$ weighted Riemann zeta $\zeta^{(n)}(s)$ limits $s \downarrow 1$ do exist. Here, we will look into $n = 0, 1$. The answer to the question could very well be that assuming existence to be true gives a confusing outcome. That may support the possibility of incompleteness in concrete mathematics.

Keywords: Riemann zeta, limits, concrete mathematics, incompleteness

1 Introduction

The theory of prime number distribution has developed greatly [2]. Despite this, elementary approaches from 100 or so years ago [3], could still be valuable for obtaining results. Here, we will look into the Riemann zeta function with this relatively primitive approach.

The Riemann zeta function is a special case of the Dirichlet function [2], [3]. In the theory of the distribution of prime numbers, the Riemann zeta function holds a crucial position [3]. In addition, the Riemann zeta function is applied in physics as well [4]. If real arguments are employed in physics application of the zeta, then the result of this paper could be interesting to theoretical physics as well.

The zeta function is generally defined with a complex argument, $s = \sigma + i\tau$ and $(\sigma, \tau) \in \mathbb{R}^2$. The Riemann zeta, for real variable s, where obviously $\Im_m(s) = \tau = 0$, is a sub-case of the more general one with a complex co-domain.

Let us look at $s \in \mathbb{R}$, the zeta is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \exp\left[-s \log(n)\right] \tag{1}$$

Here, we will look at $s > 1$. From the "exp" format, the first derivative to s can be easily found.

$$\frac{d}{ds}\zeta(s) = \zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log(n)}{n^s} = -\frac{\log(2)}{2^s} - \sum_{n=3}^{\infty} \frac{\log(n)}{n^s}, \tag{2}$$

$\log(1) = 0$.

The weighthed $s \downarrow 1$ limits of zeta and of its first derivative are then given by

$$\lim_{s\downarrow 1}(s-1)\zeta(s) = 1 \qquad (3)$$
$$\lim_{s\downarrow 1}(s-1)^2\zeta'(s) = -1$$

The derivations of the limits in (3) are quite elementary, viz. [3, page 112; page 126]. A simple example will show how an elementary but crucial part of these derivations of weighted zeta limits works. Take e.g. s=2,

$$\int_1^\infty \frac{du}{u^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^\infty \frac{du}{u^2} \qquad (4)$$

Hence, one sets up an inequality relation. Here, e.g., $1 \leq \zeta(2) \leq 2$ while $\frac{d}{du}\left(\frac{1}{u^2}\right) < 0$ for all $u \geq 1$ enables (4)

There is, however, always the possibility that the two limits on (3), despite an inclusion based on something similar as in (4), do not exist. This possibility can be uncovered if assuming their correctness and existence leads to some form of confusion.

2 Motivation

The $\log(n)$ in (2) is the natural or Napier logarithm of the integer variable n. This entails that for $n > 2$, we find: $-\log(n) < -1$. This, in turn, enables the inequality

$$-\sum_{n=3}^{\infty} \frac{\log(n)}{n^s} < -\sum_{n=3}^{\infty} \frac{1}{n^s}$$

Because, from (1)

$$-\sum_{n=3}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} - \zeta(s) \qquad (5)$$

it follows looking at (2) that,

$$\zeta'(s) \leq c_0 - \zeta(s) \qquad (6)$$

where $c_0 = 1 + \frac{1}{2}(1 - \log(2))$. For $1 < s$, we note that the inequality $\frac{1}{2} > \frac{1}{2^s}$ holds.

2.1 Approximation limit

Subsequently, $s \downarrow 1$ is replaced with $0 < \epsilon_0 \to 0$ in $s = 1+\epsilon_0$. Basing ourselves upon (3) we can in approximation have:

$$\begin{aligned} 0 < \epsilon_0 \to 0 \ &\Rightarrow \epsilon_0\zeta(1+\epsilon_0) \approx 1 \Leftrightarrow \forall_{0<\epsilon_0}\exists_{0<\delta_0 \ll 1}|\epsilon_0\zeta(1+\epsilon_0) - 1| < \delta_0, \\ 0 < \epsilon_1 \to 0 \ &\Rightarrow \epsilon_1^2\zeta'(1+\epsilon_1) \approx -1 \Leftrightarrow \forall_{0<\epsilon_1}\exists_{0<\delta_1 \ll 1}|\epsilon_1^2\zeta'(1+\epsilon_1) + 1| < \delta_1 \end{aligned} \tag{7}$$

The $\epsilon_0, \epsilon_1, \delta_0$, and, δ_1 are real numbers.

The approximation definitions employed here are generally applicable. And so, the inequality of (6) then may read

$$\zeta'(1+\epsilon_0) \leq c_0 - \zeta(1+\epsilon_0) \tag{8}$$

Furthermore, let us for both x and $y \in \mathbb{R}$ introduce the following function, $g : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$,

$$g(x,y) = \cos^2(xy) + \frac{1}{2} \tag{9}$$

This function is positive for x and $y \in \mathbb{R}$. If we take, $x = \epsilon_0$ and have $y \in (y_0 - \delta, y_0 + \delta)$ and $\delta = O(\epsilon_0) > 0$ while

$$y_0 \equiv \frac{\pi}{4\epsilon_0} + k\frac{\pi}{\epsilon_0} \tag{10}$$

Here, $k \in \mathbb{N}$ considerably large but finite and tending to infinity to meet the required $0 < \epsilon_0 \to 0$, see (17) below.

Then, from the previous three expressions, we find

$$\zeta'(1+\epsilon_0) \leq c_0 - \zeta(1+\epsilon_0) + \frac{\cos^2(\epsilon_0 y) + \frac{1}{2}}{\epsilon_0} \tag{11}$$

And, $\left(\frac{\cos^2(\epsilon_0 y)+\frac{1}{2}}{\epsilon_0}\right) > 0$.

2.2 Contradiction in approximation

Subsequent left and right hand multiplication of (11) with ϵ_0 and observing (7) leads us, with $c_0\epsilon_0 \approx 0$, to the approximative inequaliy

$$\epsilon_0\zeta'(1+\epsilon_0) \lesssim -1 + \cos^2(\epsilon_0 y) + \frac{1}{2} = \cos^2(\epsilon_0 y) - \frac{1}{2} \tag{12}$$

Now, when $0 < \epsilon_0 \to 0$, then according to definition in (10) and, $y \in (y_0 - \delta, y_0 + \delta) \Leftrightarrow y_0 - \delta < y < y_0 + \delta$ with $\delta = O(\epsilon_0) > 0$, we can approximate $\zeta'(1+\epsilon_0)$ with

$$\zeta'(1+\epsilon_0) \lesssim -y\sin(2\epsilon_0 y) \tag{13}$$

This is so because of $0 < \epsilon_0 \to 0$ and, consequently $0 < \delta \to 0$, such that $y \to y_0$, but possibly still $|y - y_0| > 0$ looking at (10). Hence, $\epsilon_0 y \approx \frac{\pi}{4} + k\pi$.

The previous implies $\cos^2(\epsilon_0 y) \approx \frac{1}{2}$. Therefore, the limit, using the l'Hôpital rule in approximation, gives us

$$\frac{\cos^2(\epsilon_0 y) - \frac{1}{2}}{\epsilon_0} \approx -y \sin(2\epsilon_0 y) \tag{14}$$

with $y \in (y_0 - \delta, y_0 + \delta)$. Multiplication of (13) with ϵ_0^2 then leads in the approximation of (7) to

$$-1 \lesssim -\epsilon_0^2 y \sin(2\epsilon_0 y) \Leftrightarrow 1 \gtrsim \epsilon_0^2 y \sin(2\epsilon_0 y) \tag{15}$$

Now $2\epsilon_0 y$ is close to $\frac{\pi}{2} + 2k\pi$, with $k = 0, 1, 2 \ldots$. Therefore, it follows that $\sin(2\epsilon_0 y) \approx 1$. Suppose $\epsilon_0^2 y \approx x$ and $3 > x > 2$, for instance. Then

$$1 \gtrsim x > 2 \tag{16}$$

when,

$$\epsilon_0 = \frac{x}{\frac{\pi}{4} + k\pi} \tag{17}$$

Note, $y_0 = \frac{1}{x}\left(\frac{\pi}{4} + k\pi\right)^2$. This gives $0 < \epsilon_0 \to 0$ when k large and increasing. But it also gives confusion because 1 can not be $\gtrsim 2$.

3 Conclusion

In the paper, a confusing result in an approximation procedure is presented in the context of weighted zeta function limits. The approximation is derived from the correctness of the limits $s \downarrow 1$ of $(s-1)\zeta(s)$ and of $(s-1)^2\zeta'(s)$ in (3). The approximation procedure followed, based on equation (7), is, with good reason, claimed to be valid. This reason is the relation of our approximation procedure with the basic logical definition of a limit.

The result found fits the definition of concrete mathematical incompleteness, i.e., [1]: *...there are sentences (in the language of ZFC) that are neither provable nor refutable from the usual ZFC axioms for mathematics....* The basic limit definitions (7), as parts of concrete mathemaics, are properly grounded in ZFC. Then, the contradiction in the approximation of (12) based on (6), makes the limits in (3) neither provable nor refutable. For, we can state that the limits (3) exist because with valid concrete means a value can be obtained via inclusion inequality similar to (4). However, the limits (3) do not exist because with concrete mathematical rules, we can derive $1 > 2$ from their existence.

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