Quasi-diagonalization and Quasi-Jordanization of Real Matrices in Real Number Field

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Abstract

A real matrix may not be similar to a diagonal matrix or a Jordan canonical matrix in the real number field. However, it is valuable to discuss the quasi-diagonalization and quasi-Jordanization of matrices in the field of real numbers. Because the characteristic polynomial of a real matrix is a real coefficient polynomial, the complex eigenvalues and eigenvector chains occur in complex conjugate pairs. So we can re-select the base vectors to quasi-diagonalize it or quasi-Jordanize it into blocks whose dimensions are no larger than 2.

In this paper, we prove these conclusions and give the method of finding transition matrix from the Jordan canonical form matrix to the quasi-diagonalized matrix.

Keywords: Real Matrix, Diagonal Matrix, Jordan Canonical Matrix, Eigenvalues, Eigenvector Chain, Base Vector, Diagonalization, Quasi-diagonalization, Jordanization, Quasi-Jordanization

1. Introduction

Even though in the complex number field some real or complex matrices cannot still be not diagonalized, because there are not enough linearly independent eigenvectors in the space, any matrix can be reduced to the equivalent standard form [1].

Suppose that the linear transformation matrix $\mathcal{A}$ of the linear transformation in the n-dimensional space under the basis vectors $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$, is A, i.e

$$\mathcal{A}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)A \tag{1}$$

If we select a new set of basis vectors $\varepsilon'_1, \varepsilon'_2, \cdots, \varepsilon'_n$ whose relationship with the old set of basis vectors is

$$(\varepsilon'_1, \varepsilon'_2, \cdots, \varepsilon'_n) = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)S \tag{2}$$

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we have

$$\begin{aligned}
&\mathcal{A}(\varepsilon_1', \varepsilon_2', \cdots, \varepsilon_n') = \mathcal{A}(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)S = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)AS \\
&= (\varepsilon_1', \varepsilon_2', \cdots, \varepsilon_n')S^{-1}AS
\end{aligned} \tag{3}$$

So the corresponding matrix to $\mathcal{A}$ with the new basis is going to be

$$A' = S^{-1}AS \tag{4}$$

Even in the field of complex numbers, some matrices cannot be diagonalized. Let's take the simplest example

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{5}$$

Both of the eigenvalues are 0, but the eigenvectors are only $(k, 0)^T$, because for any $(x, y)^T$ we have

$$A(x, y)^T = (y, 0)^T. \tag{6}$$

which means that as long as $y \neq 0$, $(x, y)^T$ can never be an eigenvector.

According to Jordan canonical form theory, any matrix A can be similarly reduced to Jordan canonical form, which means that the transformation matrix of $\mathcal{A}$ with a new set of base vectors is

$$A' = \begin{pmatrix} J_{n_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{n_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & J_{n_k}(\lambda_k) \end{pmatrix} \tag{7}$$

where

$$J_{n_i}(\lambda_i) = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_i \end{pmatrix}, \tag{8}$$

$$\sum_{i=1}^{s} n_i = n \tag{9}$$

That is to say, there is a new basis vector group of $\boldsymbol{\alpha}_{11}, \boldsymbol{\alpha}_{12}, \cdots, \boldsymbol{\alpha}_{1n_1}, \boldsymbol{\alpha}_{21}, \boldsymbol{\alpha}_{22}, \cdots, \boldsymbol{\alpha}_{2n_2}, \cdots\cdots \boldsymbol{\alpha}_{k1}, \boldsymbol{\alpha}_{k2}, \cdots, \boldsymbol{\alpha}_{kn_k}$. denoting the matrix whose columns are the coordinates of these vectors $Z_{11}, Z_{12}, \cdots, Z_{1n_1}, Z_{21}, Z_{22}, \cdots, Z_{2n_2}, \cdots\cdots Z_{k1}, Z_{k2}, \cdots, Z_{kn_k}$ under the old basis vector group as S, then

$$A' = S^{-1}AS \tag{10}$$

is a Jordan canonical matrix. That tells us that although there are not enough linearly independent eigenvectors, we can find enough cyclic vectors that are column vectors of S to make

$$\begin{cases} AZ_{i\ n_i} = \lambda_k Z_{i\ n_i} + Z_{i\ n_i-1} \\ AZ_{i\ n_i-1} = \lambda_i Z_{i\ n_i-1} + Z_{i\ n_i-2} \\ \vdots \\ AZ_{i\ 2} = \lambda_i Z_{i\ 2} + Z_{i\ 1} \\ AZ_{i\ 1} = \lambda_i Z_{i\ 1} \end{cases} \tag{11}$$

is satisfied $\forall\, i$. In other words, the linear subspace $L(\boldsymbol{\alpha}_{i1}, \boldsymbol{\alpha}_{i2}, \cdots, \boldsymbol{\alpha}_{in_i})$ spanned by $\boldsymbol{\alpha}_{i1}, \boldsymbol{\alpha}_{i2}, \cdots, \boldsymbol{\alpha}_{in_i}$ is an invariant subspace of $\mathcal{A}$ and an nilpotent subspace of $\lambda_i \mathcal{E} - \mathcal{A}$, and this transformation is geometrically called "shear", where $\mathcal{A}$ denotes the identical transformation.

The conclusion that any matrix can be reduced to the equivalent standard form will not be proved here and the method of finding the transition matrix will not be discussed here because even the most intuitive and concise method would take a long time to discuss thoroughly. The focus of this paper is to discuss the quasi-diagonalization and quasi-Jordanization of real matrices.

2. The Quasi-diagonalization of the Real Matrices That Can Be diagonalized in The Complex Number Field

Let us first discuss the case where a real matrix can be diagonalized. Since the matrix elements of a real matrix are all real numbers, the real and imaginary parts of the eigenvectors corresponding to its real eigenvalues are both eigenvectors. Therefore, we do not need the imaginary numbers to participate in the search for the eigenvectors of the real eigenvalues of a real matrix.

Since the coefficients of the characteristic polynomial of a real matrix are all real numbers, the complex conjugate of any complex root is also its root, so its complex root is complex conjugate in pairs. So if λ is an eigenvalue, $\bar{\lambda}$ is also an eigenvalue. So when we select the eigenvectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$ of the eigenvalues $\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \cdots, \lambda_m, \bar{\lambda}_m, \lambda_{m+1}, \lambda_{m+2}, \cdots, \lambda_{n-m}$ as the new base vectors, the transformation matrix is

$$A' = diag(\lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2, \cdots, \lambda_m, \bar{\lambda}_m, \lambda_{m+1}, \lambda_{m+2}, \cdots, \lambda_{n-m}) \tag{12}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_m$ is not a real number, and $\lambda_{m+1}, \lambda_{m+2}, \cdots, \lambda_{n-m}$ is a real number.

For complex root λ_j, let coordinate of the corresponding eigenvector be $\boldsymbol{\alpha}_j$ be

$$Z_j = X_j + iY_j \tag{13}$$

where all components of.

$$X_j = (x_{1j}, x_{2j}, \cdots, x_{nj})^T \tag{14}$$

$$Y_j = (y_{1j}, y_{2j}, \cdots, y_{nj})^T \tag{15}$$

are real numbers. By

$$AZ_j = \lambda_j(X_j + iY_j) \Longleftrightarrow A\bar{Z}_j = \bar{\lambda}_j \bar{Z}_j = \bar{\lambda}_j(X_j - iY_j) \tag{16}$$

we know the eigenvector of $\bar{\lambda}_j$ is

$$\bar{Z}_j = X_j - iY_j \tag{17}$$

Therefore, we have

$$\mathcal{A}X_j = \frac{\mathcal{A}(Z_j + \bar{Z}_j)}{2} = \frac{\mathcal{A}Z_j + \mathcal{A}\bar{Z}_j}{2} = Re(\lambda_j)X_j - Im(\lambda_j)Y_j \tag{18}$$

$$\mathcal{A}Y_j = \frac{\mathcal{A}(Z_j - \bar{Z}_j)}{2} = \frac{\mathcal{A}Z_j - \mathcal{A}\bar{Z}_j}{2} = Im(\lambda_j)X_j + Re(\lambda_j)Y_j \tag{19}$$

Of course, this conclusion can only be directly derived by saying that the real part of $\mathcal{A}Z_j$ is equal to the real part of $\lambda_j Z_j$, and the imaginary part of $\mathcal{A}Z_j$ is equal to the imaginary part of $\lambda_j Z_j$. So with setting

$$\lambda_j = |\lambda_j| e^{-i\theta_j} = |\lambda_j|(\cos\theta_j - i\sin\theta_j) \tag{20}$$

we have

$$\mathcal{A}(X_j, Y_j) = |\lambda_j|(X_j, Y_j) \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix} \tag{21}$$

That tells us the subtransformation matrix is

$$B_j = |\lambda_j| \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix} \tag{22}$$

when we select X_j and Y_j as the new basis vectors. This matrix is $|\lambda_j|$ times the orthogonal matrix, and if the two old base vectors itself is orthogonal and their magnitudes are equal, the matrix denotes the composite of a rotation transformation with $-theta$ and a multiplication transformation with $|\lambda|$ in the vector of the 2-dimensional plane. Obviously, if we choose the vectors $X_1, Y_1, X_2, Y_2, \cdots, X_m, Y_m, Z_{m+1}, Z_{m+2}, \cdots, Z_{n-m}$ as the new base vectors, then the matrix corresponding to $\mathcal{A}$ is

$$A' = \begin{pmatrix} B_1 & O & \cdots & O & O & O & \cdots & O \\ O & B_2 & \cdots & O & O & O & \cdots & O \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ O & O & \cdots & B_m & O & O & \cdots & O \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ O & O & \cdots & O & \lambda_{m+1} & 0 & \cdots & 0 \\ O & O & \cdots & O & 0 & \lambda_{m+2} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ O & O & \cdots & O & 0 & 0 & \cdots & \lambda_{n-m} \end{pmatrix} \tag{23}$$

3. The Quasi-Jordanization of the Real Matrices That Cannot Be diagonalized in The Complex Number Field

A real matrix may not be diagonalized even in a complex number field. For a maximal set of cyclic eigenvectors $Z_{i1}, Z_{i2}, \cdots, Z_{in_i}$, we have (11) and

$$\begin{cases} A\bar{Z}_{i\ n_i} = \bar{\lambda}_k \bar{Z}_{i\ n_i} + \bar{Z}_{i\ n_i-1} \\ A\bar{Z}_{i\ n_i-1} = \bar{\lambda}_i \bar{Z}_{i\ n_i-1} + \bar{Z}_{i\ n_i-2} \\ \vdots \\ A\bar{Z}_{i\ 2} = \bar{\lambda}_i \bar{Z}_{i\ 2} + Z_{i\ 1} \\ A\bar{Z}_{i\ 1} = \bar{\lambda}_i \bar{Z}_{i\ 1} \end{cases} \tag{24}$$

and $\bar{Z}_{i1}, \bar{Z}_{i2}, \cdots, \bar{Z}_{in_i}$ is also a maximal set of cyclic eigenvectors. So we can find a new set of vectors $\boldsymbol{\alpha}_{11}, \boldsymbol{\alpha}_{12}, \cdots, \boldsymbol{\alpha}_{1n_1}, \bar{\boldsymbol{\alpha}}_{11}, \bar{\boldsymbol{\alpha}}_{12}, \cdots, \bar{\boldsymbol{\alpha}}_{1n_1}, \boldsymbol{\alpha}_{21}, \boldsymbol{\alpha}_{22}, \cdots, \boldsymbol{\alpha}_{2n_2}$, $\bar{\boldsymbol{\alpha}}_{21}, \bar{\boldsymbol{\alpha}}_{22}, \cdots, \bar{\boldsymbol{\alpha}}_{2n_2}, \cdots\cdots, \boldsymbol{\alpha}_{m1}, \boldsymbol{\alpha}_{m2}, \cdots, \boldsymbol{\alpha}_{mn_m}, \bar{\boldsymbol{\alpha}}_{m1}, \bar{\boldsymbol{\alpha}}_{m2}, \cdots, \bar{\boldsymbol{\alpha}}_{mn_m}, \boldsymbol{\alpha}_{m+1\ 1}$, $\boldsymbol{\alpha}_{m+1\ 2}, \cdots, \boldsymbol{\alpha}_{m+1\ n_{m+1}}, \boldsymbol{\alpha}_{m+2\ 1}, \boldsymbol{\alpha}_{m+2\ 2}, \cdots, \boldsymbol{\alpha}_{m+2\ n_{m+2}}, \cdots\cdots, \boldsymbol{\alpha}_{k1}, \boldsymbol{\alpha}_{k2}, \cdots$, $\boldsymbol{\alpha}_{kn_k}$ and select their coordinates under the original base vectors $Z_{11}, Z_{12}, \cdots, Z_{1n_1}$, $\bar{Z}_{11}, \bar{Z}_{12}, \cdots, \bar{Z}_{1n_1}, Z_{21}, Z_{22}, \cdots, Z_{2n_2}, \bar{Z}_{21}, \bar{Z}_{22}, \cdots, \bar{Z}_{2n_2}, \cdots\cdots, Z_{m1}, Z_{m2}, \cdots$, $Z_{mn_m}, \bar{Z}_{m1}, \bar{Z}_{m2}, \cdots, \bar{Z}_{mn_m}, Z_{m+1\ 1}, Z_{m+1\ 2}, \cdots, Z_{m+1\ n_{m+1}}, Z_{m+2\ 1}, Z_{m+2\ 2}$, $\cdots, Z_{m+2\ n_{m+2}}, \cdots\cdots, Z_{k1}, Z_{k2}, \cdots, Z_{kn_k}$ as new basis vectors, then the matrix of $\mathcal{A}$ becomes

$$\begin{aligned} A' = diag(&J_{n_1}(\lambda_1), J_{n_1}(\bar{\lambda}_1), J_{n_2}(\lambda_2), J_{n_2}(\bar{\lambda}_2), \cdots, J_{n_m}(\lambda_m), J_{n_m}(\bar{\lambda}_m), \\ &J_{n_{m+1}}(\lambda_{m+1}), J_{n_{m+2}}(\lambda_{m+2}), \cdots, J_{n_k}(\lambda_k)) \end{aligned} \tag{25}$$

Since the matrix elements of a real matrix are all real numbers, the real and imaginary parts of the eigenvectors corresponding to its real eigenvalues are both eigenvectors. Therefore, we do not need to involve imaginary numbers when looking for the cyclic eigenvectors $Z_{m+1\ 1}, Z_{m+1\ 2}, \cdots, Z_{m+1\ n_{m+1}}$, $Z_{m+2\ 1}, Z_{m+2\ 2}, \cdots, Z_{m+2\ n_{m+2}}, \cdots\cdots, Z_{k1}, Z_{k2}, \cdots, Z_{kn_k}$ corresponding to the Jordan blocks the real eigenvalues $\lambda_{m+1}, \lambda_{m+2}, \cdots, \lambda_k$.

For complex root λ_j, let the coordinate of the corresponding vector $\boldsymbol{\alpha}_{jl}$ be

$$Z_{jl} = X_{jl} + iY_{jl} \tag{26}$$

where

$$X_{j\ l} = (x_{1\ j\ l}, x_{2\ j\ l}, \cdots, x_{n\ j\ l})^T \tag{27}$$

$$Y_{j\ l} = (y_{1\ j\ l}, y_{2\ j\ l}, \cdots, y_{n\ j\ l})^T \tag{28}$$

are real numbers. By

$$AZ_{jl} = \lambda_j(X_{jl} + iY_{jl}) \Longleftrightarrow A\bar{Z}_{jl} = \bar{\lambda}_{jl}\bar{Z}_{jl} = \bar{\lambda}_j(X_{jl} - iY_{jl}) \tag{29}$$

we know the vector of $\bar{\lambda}_j$ is

$$\bar{Z}_j = X_j - iY_j \tag{30}$$

Therefore, we have

$$\mathcal{A}X_{j\ l} = \frac{\mathcal{A}(Z_{j\ l} + \bar{Z}_{j\ l})}{2} = \frac{\mathcal{A}Z_{j\ l} + \mathcal{A}\bar{Z}_{j\ l}}{2} = Re(\lambda_j)X_{j\ l} - Im(\lambda_j)Y_{j\ l} + X_{j\ l-1} \quad (31)$$

$$\mathcal{A}Y_{j\ l} = \frac{\mathcal{A}(Z_{j\ l} - \bar{Z}_{j\ l})}{2} = \frac{\mathcal{A}Z_{j\ l} - \mathcal{A}\bar{Z}_{j\ l}}{2} = Im(\lambda_j)X_{j\ l} + Re(\lambda_j)Y_{j\ l} + Y_{j\ l-1} \quad (32)$$

Of course, this conclusion can only be directly derived by saying that the real part of $\mathcal{A}Z_{j\ l}$ is equal to the real part of $\lambda_j Z_j$, and the imaginary part of $\mathcal{A}Z_j$ is equal to the imaginary part of $\lambda_j Z_j$. So with setting

$$\lambda_j = |\lambda_j|e^{-i\theta_j} = |\lambda_j|(\cos\theta_j - i\sin\theta_j) \quad (33)$$

we have

$$\mathcal{A}(X_{j\ l}, Y_{j\ l}) = |\lambda_j|(X_{j\ l}, Y_{j\ l}) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} + (X_{j\ l-1}, Y_{j\ l-1}) \quad (34)$$

Obviously, if we choose the vectors $X_{11}, Y_{11}, X_{12}, Y_{12}, \cdots, X_{1n_1}, Y_{1n_1}, X_{21}, Y_{21}$, $X_{22}, Y_{22}, \cdots, X_{2n_2}, Y_{2n_2}, \cdots\cdots, X_{m1}, Y_{m1}, X_{m2}, Y_{m2}, \cdots, X_{mn_m}, Y_{mn_m}, Z_{m+1\ 1}$, $Z_{m+1\ 2}, \cdots, Z_{m+1\ n_{m+1}}, Z_{m+2\ 1}, Z_{m+2\ 2}, \cdots, Z_{m+2\ n_{m+2}}, \cdots\cdots, Z_{k1}, Z_{k2}, \cdots$, Z_{kn_k} as the new base vectors, then the matrix corresponding to $\mathcal{A}$ is

$$A' = diag(J_{n_1}(\lambda_1, \bar{\lambda}_1), J_{n_2}(\lambda_2, \bar{\lambda}_2), \cdots, J_{n_m}(\lambda_m, \bar{\lambda}_m), J_{n_{m+1}}(\lambda_{m+1}), J_{n_{m+2}}(\lambda_{m+2}), \cdots, J_{n_k}(\lambda_k)) \quad (35)$$

where

$$B_i = |\lambda_i| \begin{pmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (36)$$

4. The Method of Finding Transition Matrix from the Jordan Canonical Form Matrix to the Quasi-diagonalized Matrix

The basis vectors for the canonical form of Jordan is $Z_{jl}, \bar{Z}_{jl}$, and if we want to transform the matrix to the quasi-Jordan canonical form in the field of real numbers, we need to select X_{jl}, Y_{jl} as the base vectors.

According to

$$A' = diag(J_{n_1}(\lambda_1, \bar{\lambda}_1), J_{n_2}(\lambda_2, \bar{\lambda}_2), \cdots, J_{n_m}(\lambda_m, \bar{\lambda}_m), J_{n_{m+1}}(\lambda_{m+1}), J_{n_{m+2}}(\lambda_{m+2}), \cdots, J_{n_k}(\lambda_k)) \quad (37)$$

where

$$\begin{cases} Z_{jl} = X_{jl} + iY_{jl}, \\ \bar{Z}_{jl} = X_{jl} - iY_{jl} \end{cases} \quad (38)$$

we have

$$\begin{cases} X_{jl} = \frac{Z_{jl} + \bar{Z}_{jl}}{2}, \\ Y_{jl} = \frac{Z_{jl} - \bar{Z}_{jl}}{2} \end{cases} \quad (39)$$

Thus we find the base vector for quasi-Jordanization of the matrix.

5. Giving an Example

Assuming matrix

$$A = \begin{pmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{pmatrix} \tag{40}$$

This matrix is already a Jordan canonical form one, but we observe that the complex roots of this matrix appear in complex conjugate pairs, and the structure of the Jordan block corresponds to the same, so we must be able to transform it back to the real matrix by similarity transformation. Obviously, because of

$$\begin{cases} Z_1 = (1,0,0,0)^T, \\ Z_2 = (0,1,0,0)^T, \\ \bar{Z}_1 = (0,0,1,0)^T, \\ \bar{Z}_2 = (0,0,0,1)^T, \end{cases} \tag{41}$$

we have

$$\begin{cases} X_1 = \frac{Z_1+\bar{Z}_1}{2} = (\frac{1}{2},0,\frac{1}{2},0)^T, \\ X_2 = \frac{Z_2+\bar{Z}_2}{2} = (0,\frac{1}{2},0,\frac{1}{2})^T, \\ Y_1 = \frac{Z_1-\bar{Z}_1}{2i} = (-\frac{i}{2},0,\frac{i}{2},0)^T, \\ Y_2 = \frac{Z_2-\bar{Z}_2}{2i} = (0,-\frac{i}{2},0,\frac{i}{2})^T, \end{cases} \tag{42}$$

The reason why we think of $(1,0,0,0)^T$ and $(0,0,1,0)^T$ as complex conjugate vectors is that they are two vectors of equal status corresponding to the complex conjugate eigenvalues. There is no matter between they are not complex conjugate numerically. So if we select X_1, X_2, Y_1, Y_2 as new basis vectors, and we can transform the matrix to the Jordan canonical form

$$A' = (X_1, X_1, Y_1, Y_2)^{-1} A (X_1, X_1, Y_1, Y_2) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{43}$$

which is peculiar case for $\theta = \frac{\pi}{2}$ and what we have expected.

6. Conclusions

We have proved that a real matrix can definitely be quasi-Jordanized to a 2-dimensional blocks in the range of the real number field, derived the transition matrix from the canonical form to quasi-Jordan form in the real number field and give a specific example.

If a real matrix can be diagonalized, the complex eigenvalues must be in complex conjugate pairs. We can quasi-diagonalize such matrix into 2-dimensional matrix blocks in the real number field. In a similar way, if a real matrix cannot be diagonalized, the eigenvalues must also be in complex conjugate pairs, we can quasi-Jordanize such matrix similarity into 2-dimensional matrix blocks in the real number field.

[1] E. Browne, On the reduction of a matrix to a canonical form, The American Mathematical Monthly 47 (7) (1940) 437–450.