Generalization for specific type of continued fraction

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Abstract

I came across "The Ramanujan Machine" on the Internet and, using my intuition on those kind of stuff, I found some interesting results. The link to the website mentioned above is https://www.ramanujanmachine.com/results/ and the results I found is listed below:

Formula #1

$$n+\cfrac{1}{n+1+\cfrac{2}{n+2+\cfrac{3}{n+3+\cfrac{4}{\dots}}}}=\frac{\frac{1}{n!}}{e-\sum_{k=0}^{n}\frac{1}{k!}}$$

Formula #2

$$n+2-\cfrac{1}{n+3-\cfrac{2}{n+4-\cfrac{3}{n+5-\cfrac{4}{\dots}}}}=\frac{\frac{(-1)^n}{n!}}{\sum_{k=0}^{n}\frac{(-1)^k}{k!}-\frac{1}{e}}$$

Formula #3

$$1+\cfrac{n+1}{2+\cfrac{n+2}{3+\cfrac{n+3}{4+\cfrac{n+4}{\dots}}}}=\frac{\frac{d^n}{dx^n}\left(xe^{\frac{x}{1-x}}\right)}{\frac{d^{n-1}}{dx^{n-1}}\left(\frac{x}{1-x}e^{\frac{x}{1-x}}\right)}$$

at point $x=0$ and $n>1$

Formula #4

$$1-\cfrac{n+1}{2-\cfrac{n+2}{3-\cfrac{n+3}{4-\cfrac{n+4}{\dots}}}}=\frac{n\frac{d^n}{dx^n}\left(e^{\frac{x}{x-1}}\right)}{\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{n!}{k!}}=\frac{n\frac{d^n}{dx^n}\left(e^{\frac{x}{x-1}}\right)}{e^x\frac{d^n}{dx^n}\left(\frac{x^n}{e^x}\right)}$$

at point $x=0$

Now here I am using formula #1 to get formula #5
(By the way you can do the same on all the other formulas as well)

Formula #5

$$n+\cfrac{1+m}{n+1+\cfrac{2+m}{n+2+\cfrac{3+m}{n+3+\cfrac{4+m}{n+4+\cfrac{5+m}{\dots}}}}}=\cfrac{m}{\cfrac{m-1}{\cfrac{m-2}{\cfrac{\ddots}{\cfrac{3}{\cfrac{2}{\cfrac{1}{\cfrac{\frac{1}{(n-m)!}}{e-\sum_{k=0}^{n-m}\frac{1}{k!}}-(n-m)}-(n-m+1)}-(n-m+2)}-(n-m+3)}-\ddots}-(n-m+m-2)}-(n-m+m-1)}$$

$$2-\cfrac{1}{3-\cfrac{2}{4-\cfrac{3}{5-\cfrac{4}{\cdots}}}}=+\frac{e}{1e-0!}=\frac{e}{e-1}$$

1.58197670686932642438…

$$3-\cfrac{1}{4-\cfrac{2}{5-\cfrac{3}{6-\cfrac{4}{\cdots}}}}=-\frac{e}{0e-1!}=e$$

(The Ramanujan Machine)

$$4-\cfrac{1}{5-\cfrac{2}{6-\cfrac{3}{7-\cfrac{4}{\cdots}}}}=+\frac{e}{1e-2!}=\frac{e}{e-2}$$

3.78442238235466562875...

$$5-\cfrac{1}{6-\cfrac{2}{7-\cfrac{3}{8-\cfrac{4}{\cdots}}}}=-\frac{e}{2e-3!}=-\frac{e}{2e-6}$$

4.82447016745576732333…

$$6-\cfrac{1}{7-\cfrac{2}{8-\cfrac{3}{9-\cfrac{4}{\cdots}}}}=+\frac{e}{9e-4!}=\frac{e}{9e-24}$$

5.85160064959487971142…

$$7-\cfrac{1}{8-\cfrac{2}{9-\cfrac{3}{10-\cfrac{4}{\cdots}}}}=-\frac{e}{44e-5!}=-\frac{e}{44e-120}$$

6.87129660173296177453...

$$8-\cfrac{1}{9-\cfrac{2}{10-\cfrac{3}{11-\cfrac{4}{\cdots}}}}=+\frac{e}{265e-6!}=\frac{e}{265e-720}$$

7.88628876557801878356...

$$9-\cfrac{1}{10-\cfrac{2}{11-\cfrac{3}{12-\cfrac{4}{\cdots}}}}=-\frac{e}{1854e-7!}=-\frac{e}{1854e-5040}$$

8.89810304707489785707...

$$n+2-\cfrac{1}{n+3-\cfrac{2}{n+4-\cfrac{3}{n+5-\cfrac{4}{\dots}}}}=\frac{(-1)^n e}{d_n e-n!}$$

$$d_n=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}\approx\frac{n!}{e}$$

1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570, 176214841, 2290792932, 32071101049, 481066515734, 7697064251745, 130850092279664, 2355301661033953, …

https://oeis.org/A000166

or you can use this formula if you want:

$$n+2-\cfrac{1}{n+3-\cfrac{2}{n+4-\cfrac{3}{n+5-\cfrac{4}{\dots}}}}=\frac{\frac{(-1)^n}{n!}}{\sum_{k=0}^{n}\frac{(-1)^k}{k!}-\frac{1}{e}}$$

$$0+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{\cdots}}}}=\frac{1}{e-1}$$ 0.581976706869326424385002005…

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{\cdots}}}}=\frac{1}{e-2}$$ 1.392211191177332814376552878…

$$2+\cfrac{1}{3+\cfrac{2}{4+\cfrac{3}{5+\cfrac{4}{\cdots}}}}=\frac{1}{2e-5}$$ 2.290616692785362422107533414…

$$3+\cfrac{1}{4+\cfrac{2}{5+\cfrac{3}{6+\cfrac{4}{\cdots}}}}=\frac{1}{6e-16}$$ 3.229025365397119837675231238…

$$4+\cfrac{1}{5+\cfrac{2}{6+\cfrac{3}{7+\cfrac{4}{\cdots}}}}=\frac{1}{24e-65}$$ 4.188238134527411898322660568…

$$5+\cfrac{1}{6+\cfrac{2}{7+\cfrac{3}{8+\cfrac{4}{\cdots}}}}=\frac{1}{120e-326}$$ 5.159441842083972585306850648l…

$$6+\cfrac{1}{7+\cfrac{2}{8+\cfrac{3}{9+\cfrac{4}{\cdots}}}}=\frac{1}{720e-1957}$$ 6.1381140537326221409501709982…

$$7+\cfrac{1}{8+\cfrac{2}{9+\cfrac{3}{10+\cfrac{4}{\cdots}}}}=\frac{1}{5040e-13700}$$ 7.1217242609829384029720880055…

$$n+\cfrac{1}{n+1+\cfrac{2}{n+2+\cfrac{3}{n+3+\cfrac{4}{\cdots}}}}=\frac{1}{n!e-d_n}$$

$$d_n=n!\sum_{k=0}^{n}\frac{1}{k!}$$

$1\cdot 1+1=2$
$2\cdot 2+1=5$
$3\cdot 3+1=16$
$16\cdot 4+1=65$
$65\cdot 5+1=326$
$326\cdot 6+1=1957$

1, 2, 5, 16, 65, 326, 1957, 13700, 109601, 986410, 9864101, 108505112, 1302061345, 16926797486, 236975164805, 3554627472076, 56874039553217, 966858672404690, …

https://oeis.org/A000522

or you can use this formula if you want:

$$n+\cfrac{1}{n+1+\cfrac{2}{n+2+\cfrac{3}{n+3+\cfrac{4}{\cdots}}}}=\frac{\frac{1}{n!}}{e-\sum_{k=0}^{n}\frac{1}{k!}}$$

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{\cdots}}}}=\frac{1}{e-2}$$

$$1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{\cdots}}}}=e-1$$

$$1+\cfrac{3}{2+\cfrac{4}{3+\cfrac{5}{4+\cfrac{6}{\cdots}}}}=\frac{2}{1}$$

$$1+\cfrac{4}{2+\cfrac{5}{3+\cfrac{6}{4+\cfrac{7}{\cdots}}}}=\frac{9}{4}$$

$$1+\cfrac{5}{2+\cfrac{6}{3+\cfrac{7}{4+\cfrac{8}{\cdots}}}}=\frac{52}{21}$$

$$1+\cfrac{6}{2+\cfrac{7}{3+\cfrac{8}{4+\cfrac{9}{\cdots}}}}=\frac{365}{136}$$

$$1+\cfrac{7}{2+\cfrac{8}{3+\cfrac{9}{4+\cfrac{10}{\cdots}}}}=\frac{3006}{1045}$$

$$1+\cfrac{8}{2+\cfrac{9}{3+\cfrac{10}{4+\cfrac{11}{\cdots}}}}=\frac{28357}{9276}$$

$$f(x)=xe^{\frac{x}{1-x}}$$ $$f(0)=0e^{\frac{0}{1}}=0$$

$$f'(x)=e^{\frac{x}{1-x}}\cdot\frac{(x^2+1-x)}{(1-x)^2}$$ $$f'(0)=e^{\frac{0}{1}}\cdot\frac{(1)}{(1)^2}=1$$

$$f''(x)=e^{\frac{x}{1-x}}\cdot\frac{(2-x)}{(1-x)^4}$$ $$f''(0)=e^{\frac{0}{1}}\cdot\frac{(2)}{(1)^4}=2$$

$$f'''(x)=e^{\frac{x}{1-x}}\cdot\frac{(3x^2-11x+9)}{(1-x)^6}$$ $$f'''(0)=e^{\frac{0}{1}}\cdot\frac{(9)}{(1)^6}=9$$

$$f''''(x)=e^{\frac{x}{1-x}}\cdot\frac{(-12x^3+64x^2-103x+52)}{(1-x)^8}$$ $$f''''(0)=e^{\frac{0}{1}}\cdot\frac{(52)}{(1)^8}=52$$

https://oeis.org/A006152 Exponential generating function: $e^{\frac{x}{1-x}}$ $A_n=\frac{d^n}{dx^n}\left(xe^{\frac{x}{1-x}}\right)$

0,1, 2, 9, 52, 365, 3006, 28357, 301064, 3549177, 45965530, 648352001, 9888877692, 162112109029, 2841669616982, 53025262866045, 1049180850990736, 21937381717388657,

$$f(x)=\frac{x}{1-x}\cdot e^{\frac{x}{1-x}}$$ $$f(0)=\frac{0}{1}\cdot e^{\frac{0}{1}}=0$$

$$f'(x)=e^{\frac{x}{1-x}}\cdot\frac{(1)}{(1-x)^3}$$ $$f'(0)=e^{\frac{0}{1}}\cdot\frac{(1)}{(1)^5}=1$$

$$f''(x)=e^{\frac{x}{1-x}}\cdot\frac{(4-3x)}{(1-x)^5}$$ $$f''(0)=e^{\frac{0}{1}}\cdot\frac{(4)}{(1)^5}=4$$

$$f'''(x)=e^{\frac{x}{1-x}}\cdot\frac{(12x^2-32x+21)}{(1-x)^7}$$ $$f'''(0)=e^{\frac{0}{1}}\cdot\frac{(21)}{(1)^7}=21$$

$$f''''(x)=e^{\frac{x}{1-x}}\cdot\frac{(-60x^3+240x^2-315x+136)}{(1-x)^9}$$ $$f''''(0)=e^{\frac{0}{1}}\cdot\frac{(136)}{(1)^9}=136$$

https://oeis.org/A052852 Expansion of e.g.f: $\frac{x}{1-x}\cdot e^{\frac{x}{1-x}}$ $B_n=\frac{d^n}{dx^n}\left(\frac{x}{1-x}e^{\frac{x}{1-x}}\right)$

0, 1, 4, 21, 136, 1045, 9276, 93289, 1047376, 12975561, 175721140

$$1+\cfrac{n+1}{2+\cfrac{n+2}{3+\cfrac{n+3}{4+\cfrac{n+4}{\dots}}}}=\frac{\frac{d^n}{dx^n}\left(xe^{\frac{x}{1-x}}\right)}{\frac{d^{n-1}}{dx^{n-1}}\left(\frac{x}{1-x}e^{\frac{x}{1-x}}\right)}$$

at point $x=0$ and $n>1$

$$1-\cfrac{1}{2-\cfrac{2}{3-\cfrac{3}{4-\cfrac{4}{\cdots}}}}=0$$

0

$$1-\cfrac{2}{2-\cfrac{3}{3-\cfrac{4}{4-\cfrac{5}{\cdots}}}}=-\infty$$

$-\infty$

$$1-\cfrac{3}{2-\cfrac{4}{3-\cfrac{5}{4-\cfrac{6}{\cdots}}}}=\frac{-2}{-1}$$

2

$$1-\cfrac{4}{2-\cfrac{5}{3-\cfrac{6}{4-\cfrac{7}{\cdots}}}}=\frac{-3}{-4}$$

0.75

$$1-\cfrac{5}{2-\cfrac{6}{3-\cfrac{7}{4-\cfrac{8}{\cdots}}}}=\frac{4}{-15}$$

-0.26666666666666666666666666666666…

$$1-\cfrac{6}{2-\cfrac{7}{3-\cfrac{8}{4-\cfrac{9}{\cdots}}}}=\frac{95}{-56}$$

-1.69642857142857142857142857142886…

$$1-\cfrac{7}{2-\cfrac{8}{3-\cfrac{9}{4-\cfrac{10}{\cdots}}}}=\frac{906}{-185}$$

-4.89729729729729729729729729729729…

$$1-\cfrac{8}{2-\cfrac{9}{3-\cfrac{10}{4-\cfrac{11}{\cdots}}}}=\frac{7637}{-204}$$

-37.43623221298753386949154637967 5…

$$1-\cfrac{9}{2-\cfrac{10}{3-\cfrac{11}{4-\cfrac{12}{\dots}}}}=\frac{62728}{6209}$$

10.102754066677431484160113796532…

$$1-\cfrac{10}{2-\cfrac{11}{3-\cfrac{12}{4-\cfrac{13}{\dots}}}}=\frac{508671}{112400}$$

4.525542704626334519572953736654 8…

$$f(x)=e^{\frac{x}{x-1}}$$

$$f(0)=e^{\frac{0}{-1}}=1$$

$$f'(x)=e^{\frac{x}{x-1}}\cdot\frac{-1}{(x-1)^2}$$

$$f'(0)=e^{\frac{0}{-1}}\cdot\frac{-1}{(-1)^2}=-1$$

$$f''(x)=e^{\frac{x}{x-1}}\cdot\frac{(2x-1)}{(x-1)^4}$$

$$f''(0)=e^{\frac{0}{-1}}\cdot\frac{(-1)}{(-1)^4}=-1$$

$$f'''(x)=e^{\frac{x}{x-1}}\cdot\frac{(-6x^2+6x-1)}{(x-1)^6}$$

$$f'''(0)=e^{\frac{0}{-1}}\cdot\frac{(-1)}{(-1)^6}=-1$$

$$f''''(x)=e^{\frac{x}{x-1}}\cdot\frac{(24x^3-36x^2+12x+1)}{(x-1)^8}$$

$$f''''(0)=e^{\frac{0}{-1}}\cdot\frac{(1)}{(-1)^8}=1$$

$$f'''''(x)=e^{\frac{x}{x-1}}\cdot\frac{(-120x^4+240x^3-120x^2-20x+19)}{(x-1)^{10}}$$

$$f'''''(0)=e^{\frac{0}{0-1}}\cdot\frac{(19)}{(0-1)^{10}}=19$$

https://oeis.org/A293116 Expansion of e.g.f: $e^{\frac{x}{x-1}}$ $n\frac{d^n}{dx^n}\left(e^{\frac{x}{x-1}}\right)$

1, -1, -1, -1, 1, 19, 151, 1091, 7841, 56519, 396271, 2442439, 7701409, -145269541, -4833158329, -104056218421, -2002667085119, -37109187217649, -679877731030049, …

[No oeis] $A_n=n\frac{d^n}{dx^n}\left(e^{\frac{x}{x-1}}\right)$

0,-1,-2,-3,4, 95, 906, 7637, 62728, 508671

$$L_n(x)=\frac{e^x}{n!}\frac{d^n}{dx^n}\left(\frac{x^n}{e^x}\right)=\frac{1}{n!}\left(\frac{d}{dx}-1\right)^n x^n=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{k!}x^k$$

Laguerre polynomials

https://oeis.org/A009940

1, 0, -1, -4, -15, -56, -185, -204, 6209, 112400, 1520271, 19165420, 237686449, 2944654296,

$$B_n(x) = e^x \frac{d^n}{dx^n}\left(\frac{x^n}{e^x}\right) = \left(\frac{d}{dx} - 1\right)^n x^n \text{ at point x=1}$$

$$B_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{n!}{k!}$$

$$1 - \cfrac{n+1}{2 - \cfrac{n+2}{3 - \cfrac{n+3}{4 - \cfrac{n+4}{\dots}}}} = \frac{n \dfrac{d^n}{dx^n}\left(e^{\frac{x}{x-1}}\right)}{\sum_{k=0}^{n} (-1)^k \binom{n}{k} \dfrac{n!}{k!}} = \frac{n \dfrac{d^n}{dx^n}\left(e^{\frac{x}{x-1}}\right)}{e^x \dfrac{d^n}{dx^n}\left(\dfrac{x^n}{e^x}\right)}$$

at point $x = 0$

Using formula #1 to get formula #5

Proof:

Lets start with formula #1

$$\boxed{n+\cfrac{1}{n+1+\cfrac{2}{n+2+\cfrac{3}{n+3+\cfrac{4}{\dots}}}}=\frac{\frac{1}{n!}}{e-\sum_{k=0}^{n}\frac{1}{k!}}}$$

Rearrange $\Rightarrow$ $$n+1+\cfrac{2}{n+2+\cfrac{3}{n+3+\cfrac{4}{\dots}}}=\frac{1}{\frac{\frac{1}{n!}}{e-\sum_{k=0}^{n}\frac{1}{k!}}-n}$$

after we rearranged we will set $n\leftarrow n-1$ and we will get:

$$n+\cfrac{2}{n+1+\cfrac{3}{n+2+\cfrac{4}{n+3+\cfrac{5}{\dots}}}}=\frac{1}{\frac{\frac{1}{(n-1)!}}{e-\sum_{k=0}^{n-1}\frac{1}{k!}}-(n-1)}$$

Rearrange $\Rightarrow$ $$n+1+\cfrac{3}{n+2+\cfrac{4}{n+3+\cfrac{5}{\dots}}}=\frac{2}{\frac{1}{\frac{\frac{1}{(n-1)!}}{e-\sum_{k=0}^{n-1}\frac{1}{k!}}-(n-1)}-n}$$

after we rearranged we will set $n\leftarrow n-1$ and we will get:

$$n+\cfrac{3}{n+1+\cfrac{4}{n+2+\cfrac{5}{n+3+\cfrac{6}{\dots}}}}=\frac{2}{\frac{1}{\frac{\frac{1}{(n-2)!}}{e-\sum_{k=0}^{n-2}\frac{1}{k!}}-(n-2)}-(n-1)}$$

Rearrange $\Rightarrow$

$$n+1+\cfrac{4}{n+2+\cfrac{5}{n+3+\cfrac{6}{\cdots}}}=\cfrac{3}{\cfrac{2}{\cfrac{1}{\cfrac{\frac{1}{(n-2)!}}{e-\sum_{k=0}^{n-2}\frac{1}{k!}}-(n-2)}-(n-1)}-n}$$

after we rearranged we will set $n \leftarrow n-1$ and we will get:

$$n+\cfrac{4}{n+1+\cfrac{5}{n+2+\cfrac{6}{n+3+\cfrac{7}{\cdots}}}}=\cfrac{3}{\cfrac{2}{\cfrac{1}{\cfrac{\frac{1}{(n-3)!}}{e-\sum_{k=0}^{n-3}\frac{1}{k!}}-(n-3)}-(n-2)}-(n-1)}$$

Now lets repeat m times from the base form and we will get this:

$$n+\cfrac{1+m}{n+1+\cfrac{2+m}{n+2+\cfrac{3+m}{n+3+\cfrac{4+m}{n+4+\cfrac{5+m}{\cdots}}}}}=\cfrac{m}{\cfrac{m-1}{\cfrac{m-2}{\cfrac{\ddots}{\cfrac{3}{\cfrac{2}{\cfrac{1}{\cfrac{\frac{1}{(n-m)!}}{e-\sum_{k=0}^{n-m}\frac{1}{k!}}-(n-m)}-(n-m+1)}-(n-m+2)}-(n-m+3)}-\ddots}-(n-m+m-2)}-(n-m+m-1)}$$

LHS : Infinite continued fraction
RHS : Finite continued fraction with $m+1$ steps

Examples:

M=3

$$n+\cfrac{1+3}{n+1+\cfrac{2+3}{n+2+\cfrac{3+3}{n+3+\cfrac{4+3}{n+4+\cfrac{5+3}{\cdots}}}}}=\cfrac{3}{\cfrac{2}{\cfrac{1}{\cfrac{\frac{1}{(n-3)!}}{e-\sum_{k=0}^{n-3}\frac{1}{k!}}-(n-3)}-(n-3+1)}-(n-3+2)}$$

n=10

$$10+\cfrac{1+3}{10+1+\cfrac{2+3}{10+2+\cfrac{3+3}{10+3+\cfrac{4+3}{10+4+\cfrac{5+3}{\dots}}}}}=\cfrac{3}{\cfrac{2}{\cfrac{1}{\cfrac{\frac{1}{(10-3)!}}{e-\sum_{k=0}^{10-3}\frac{1}{k!}}-(10-3)}-(10-3+1)}-(10-3+2)}$$

$$10+\cfrac{4}{11+\cfrac{5}{12+\cfrac{6}{13+\cfrac{7}{14+\cfrac{8}{\dots}}}}}=\cfrac{3}{\cfrac{2}{\cfrac{1}{\cfrac{\frac{1}{7!}}{e-\sum_{k=0}^{7}\frac{1}{k!}}-7}-8}-9}=\cfrac{3}{\cfrac{2}{\cfrac{1}{\cfrac{1}{7!e-7!\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}\right)}-7}-8}-9}$$

$$10+\cfrac{4}{11+\cfrac{5}{12+\cfrac{6}{13+\cfrac{7}{14+\cfrac{8}{\dots}}}}}=\cfrac{3}{\cfrac{2}{\cfrac{1}{\cfrac{1}{5040e-13700}-7}-8}-9}=\frac{390454-143640e}{442680e-1203329}=10.3508228202276233420756\dots$$

10.000000000000000000000000000000000000000000000000000000000000000...
10.363636363636363636363636363636363636363636363636363636363636363...
10.350364963503649635036496350364963503649635036496350364963503649...
10.350839198700595560368164591229020032485110990795885219274499187...
10.350822239624119028974158183241973375097885669537979639780736100...
10.350822840508259577825149689111840606573612033637787118943196069...
10.350822819532217148508739487946583733604827065213602437658097537...
10.350822820250972051499656840351403959630729997679731514395339585...
10.350822820226856973184564771845796159685249682659562353995928806...
10.350822820227647906256300427660780329853708189477641265328010867...
10.350822820227622573699048815891408290144985229438822744288293982...

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10.350822820227623342075602230137969081015999065582118803635657661...