Generalization for specific type of continued fraction

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Abstract

I came across "The Ramanujan Machine" on the Internet and, using my intuition on those kind of stuff, I found some interesting results. The link to the website mentioned above is https://www.ramanujanmachine.com/results/ and the results I found is listed below:

$$2-\cfrac{1}{3-\cfrac{2}{4-\cfrac{3}{5-\cfrac{4}{\cdots}}}}=+\frac{e}{1e-0!}=\frac{e}{e-1}$$

1.58197670686932642438…

$$3-\cfrac{1}{4-\cfrac{2}{5-\cfrac{3}{6-\cfrac{4}{\cdots}}}}=-\frac{e}{0e-1!}=e$$

(The Ramanujan Machine)

$$4-\cfrac{1}{5-\cfrac{2}{6-\cfrac{3}{7-\cfrac{4}{\cdots}}}}=+\frac{e}{1e-2!}=\frac{e}{e-2}$$

3.78442238235466562875...

$$5-\cfrac{1}{6-\cfrac{2}{7-\cfrac{3}{8-\cfrac{4}{\cdots}}}}=-\frac{e}{2e-3!}=-\frac{e}{2e-6}$$

4.82447016745576732333…

$$6-\cfrac{1}{7-\cfrac{2}{8-\cfrac{3}{9-\cfrac{4}{\cdots}}}}=+\frac{e}{9e-4!}=\frac{e}{9e-24}$$

5.85160064959487971142…

$$7-\cfrac{1}{8-\cfrac{2}{9-\cfrac{3}{10-\cfrac{4}{\cdots}}}}=-\frac{e}{44e-5!}=-\frac{e}{44e-120}$$

6.87129660173296177453...

$$8-\cfrac{1}{9-\cfrac{2}{10-\cfrac{3}{11-\cfrac{4}{\cdots}}}}=+\frac{e}{265e-6!}=\frac{e}{265e-720}$$

7.88628876557801878356...

$$9-\cfrac{1}{10-\cfrac{2}{11-\cfrac{3}{12-\cfrac{4}{\cdots}}}}=-\frac{e}{1854e-7!}=-\frac{e}{1854e-5040}$$

8.89810304707489785707...

$$n+2-\cfrac{1}{n+3-\cfrac{2}{n+4-\cfrac{3}{n+5-\cfrac{4}{\cdots}}}}=\frac{(-1)^n e}{d_n e-n!}$$

$$d_n=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}\approx\frac{n!}{e}$$

1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570, 176214841, 2290792932, 32071101049, 481066515734, 7697064251745, 130850092279664, 2355301661033953, …

https://oeis.org/A000166

or you can use this formula if you want:

$$n+2-\cfrac{1}{n+3-\cfrac{2}{n+4-\cfrac{3}{n+5-\cfrac{4}{\cdots}}}}=\frac{\dfrac{(-1)^n}{n!}}{\displaystyle\sum_{k=0}^{n}\frac{(-1)^k}{k!}-\frac{1}{e}}$$

$$0+\cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{\cdots}}}}=\frac{1}{e-1}$$ 0.581976706869326424385002005…

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{\cdots}}}}=\frac{1}{e-2}$$ 1.392211191177332814376552878…

$$2+\cfrac{1}{3+\cfrac{2}{4+\cfrac{3}{5+\cfrac{4}{\cdots}}}}=\frac{1}{2e-5}$$ 2.290616692785362422107533414…

$$3+\cfrac{1}{4+\cfrac{2}{5+\cfrac{3}{6+\cfrac{4}{\cdots}}}}=\frac{1}{6e-16}$$ 3.229025365397119837675231238…

$$4+\cfrac{1}{5+\cfrac{2}{6+\cfrac{3}{7+\cfrac{4}{\cdots}}}}=\frac{1}{24e-65}$$ 4.188238134527411898322660568…

$$5+\cfrac{1}{6+\cfrac{2}{7+\cfrac{3}{8+\cfrac{4}{\cdots}}}}=\frac{1}{120e-326}$$ 5.1594418420839725853068506481…

$$6+\cfrac{1}{7+\cfrac{2}{8+\cfrac{3}{9+\cfrac{4}{\cdots}}}}=\frac{1}{720e-1957}$$ 6.1381140537326221409501709982…

$$7+\cfrac{1}{8+\cfrac{2}{9+\cfrac{3}{10+\cfrac{4}{\cdots}}}}=\frac{1}{5040e-13700}$$ 7.1217242609829384029720880055…

$$n+\cfrac{1}{n+1+\cfrac{2}{n+2+\cfrac{3}{n+3+\cfrac{4}{\dots}}}}=\frac{1}{n!e-d_n}$$

$$d_n=n!\sum_{k=0}^{n}\frac{1}{k!}$$

$1\cdot 1+1=2$
$2\cdot 2+1=5$
$3\cdot 3+1=16$
$16\cdot 4+1=65$
$65\cdot 5+1=326$
$326\cdot 6+1=1957$

1, 2, 5, 16, 65, 326, 1957, 13700, 109601, 986410, 9864101, 108505112, 1302061345, 16926797486, 236975164805, 3554627472076, 56874039553217, 966858672404690, …

https://oeis.org/A000522

or you can use this formula if you want:

$$n+\cfrac{1}{n+1+\cfrac{2}{n+2+\cfrac{3}{n+3+\cfrac{4}{\dots}}}}=\frac{\frac{1}{n!}}{e-\sum_{k=0}^{n}\frac{1}{k!}}$$

$$1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{\cdots}}}}=\frac{1}{e-2}$$

$$1+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{\cdots}}}}=e-1$$

$$1+\cfrac{3}{2+\cfrac{4}{3+\cfrac{5}{4+\cfrac{6}{\cdots}}}}=\frac{2}{1}$$

$$1+\cfrac{4}{2+\cfrac{5}{3+\cfrac{6}{4+\cfrac{7}{\cdots}}}}=\frac{9}{4}$$

$$1+\cfrac{5}{2+\cfrac{6}{3+\cfrac{7}{4+\cfrac{8}{\cdots}}}}=\frac{52}{21}$$

$$1+\cfrac{6}{2+\cfrac{7}{3+\cfrac{8}{4+\cfrac{9}{\cdots}}}}=\frac{365}{136}$$

$$1+\cfrac{7}{2+\cfrac{8}{3+\cfrac{9}{4+\cfrac{10}{\cdots}}}}=\frac{3006}{1045}$$

$$1+\cfrac{8}{2+\cfrac{9}{3+\cfrac{10}{4+\cfrac{11}{\cdots}}}}=\frac{28357}{9276}$$

$$f(x) = xe^{\frac{x}{1-x}} \qquad f(0) = 0e^{\frac{0}{1}} = 0$$

$$f'(x) = e^{\frac{x}{1-x}} \cdot \frac{(x^2+1-x)}{(1-x)^2} \qquad f'(0) = e^{\frac{0}{1}} \cdot \frac{(1)}{(1)^2} = 1$$

$$f''(x) = e^{\frac{x}{1-x}} \cdot \frac{(2-x)}{(1-x)^4} \qquad f''(0) = e^{\frac{0}{1}} \cdot \frac{(2)}{(1)^4} = 2$$

$$f'''(x) = e^{\frac{x}{1-x}} \cdot \frac{(3x^2-11x+9)}{(1-x)^6} \qquad f'''(0) = e^{\frac{0}{1}} \cdot \frac{(9)}{(1)^6} = 9$$

$$f''''(x) = e^{\frac{x}{1-x}} \cdot \frac{(-12x^3+64x^2-103x+52)}{(1-x)^8} \qquad f''''(0) = e^{\frac{0}{1}} \cdot \frac{(52)}{(1)^8} = 52$$

https://oeis.org/A006152 Exponential generating function: $e^{\frac{x}{1-x}}$ $A_n = \frac{d^n}{dx^n}\left(xe^{\frac{x}{1-x}}\right)$

0,1, 2, 9, 52, 365, 3006, 28357, 301064, 3549177, 45965530, 648352001, 9888877692, 162112109029, 2841669616982, 53025262866045, 1049180850990736, 21937381717388657,

$$f(x) = \frac{x}{1-x} \cdot e^{\frac{x}{1-x}} \qquad f(0) = \frac{0}{1} \cdot e^{\frac{0}{1}} = 0$$

$$f'(x) = e^{\frac{x}{1-x}} \cdot \frac{(1)}{(1-x)^3} \qquad f'(0) = e^{\frac{0}{1}} \cdot \frac{(1)}{(1)^5} = 1$$

$$f''(x) = e^{\frac{x}{1-x}} \cdot \frac{(4-3x)}{(1-x)^5} \qquad f''(0) = e^{\frac{0}{1}} \cdot \frac{(4)}{(1)^5} = 4$$

$$f'''(x) = e^{\frac{x}{1-x}} \cdot \frac{(12x^2-32x+21)}{(1-x)^7} \qquad f'''(0) = e^{\frac{0}{1}} \cdot \frac{(21)}{(1)^7} = 21$$

$$f''''(x) = e^{\frac{x}{1-x}} \cdot \frac{(-60x^3+240x^2-315x+136)}{(1-x)^9} \qquad f''''(0) = e^{\frac{0}{1}} \cdot \frac{(136)}{(1)^9} = 136$$

https://oeis.org/A052852 Expansion of e.g.f: $\frac{x}{1-x} \cdot e^{\frac{x}{1-x}}$ $B_n = \frac{d^n}{dx^n}\left(\frac{x}{1-x} e^{\frac{x}{1-x}}\right)$

0, 1, 4, 21, 136, 1045, 9276, 93289, 1047376, 12975561, 175721140

$$1 + \cfrac{n+1}{2 + \cfrac{n+2}{3 + \cfrac{n+3}{4 + \cfrac{n+4}{\dots}}}} = \frac{\frac{d^n}{dx^n}\left(xe^{\frac{x}{1-x}}\right)}{\frac{d^n}{dx^n}\left(\frac{x}{1-x}e^{\frac{x}{1-x}}\right)}$$

at point $x = 0$ and $n > 1$