Finding Rational Points of Circles, Spheres, Hyper-Spheres via Stereographic Projection and Quantum Mechanics

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November 2023

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Abstract

One of the consequences of Fermat's last theorem is the existence of a countable infinite number of rational points on the unit circle, which allows in turn, to find the rational points on the unit sphere via the inverse stereographic projection of the homothecies of the rational points on the unit circle. We proceed to iterate this process and obtain the rational points on the unit S^3 via the inverse stereographic projection of the homothecies of the rational points on the previous unit S^2. One may continue this iteration/recursion process ad infinitum in order to find the rational points on unit hyper-spheres of arbitrary dimension $S^4, S^5, \cdots, S^N$. As an example, it is shown how to obtain the rational points of the unit S^{24} that is associated with the Leech lattice. The physical applications of our construction follow and one finds a direct relation among the $N+1$ quantum states of a spin-$\frac{N}{2}$ particle and the rational points of a unit S^N hyper-sphere embedded in a flat Euclidean R^{N+1} space.

Keywords : Fermat Last Theorem; Rational Points; Surfaces; Leech Lattice; Quantum Mechanics.

1 Rational Points in Circles, Spheres and Hyper-Spheres

Fermat's Last Theorem (FLT) states that no three positive integers a, b, and c satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2. The cases $n = 2$ has infinitely many solutions and is directly related to the generation of Pythagorean triples. The first proof of FLT was carried by Wiles [1]. After Wiles' proof there have been other attempts like the one based on Euler's double equations [2].

We begin by recalling the geometrical procedure how to generate Pythagorean triples, namely triples of non-zero integers (positive, negative) a, b, c obeying $a^2 + b^2 = c^2$, like $3^2 + 4^2 = 5^2$. Let us find how to generate an infinite family of points $\mathbf{P}$ lying on the circle of radius unity such that their coordinates $\mathbf{P} = (x, y)$ are *rational* numbers obeying

$$x^2 \ + \ y^2 \ = \ 1, \ \ x \ \equiv \ \frac{a}{c}, \ \ y \ \equiv \ \frac{b}{c} \tag{1}$$

One draws a unit circle centered at the origin and chooses any point $\mathbf{X} = (\frac{p}{q}; 0)$ lying on the *real* line whose coordinates $\frac{p}{q}$ (with p, q integers) are *rational* numbers. Take the north pole of the unit circle $\mathbf{N} = (0, 1)$ and draw the *straight* line connecting the north pole $\mathbf{N}$ with the point $\mathbf{X}$ on the real line. Choose $\frac{p}{q} > 1$ so the point $\mathbf{X}$ lies outside the unit circle (the construction also works for a point inside the unit circle). The *intersection* of the straight line $\mathbf{N}\ \mathbf{X}$ with the unit circle will generate the desired rational points on the unit circle whose coordinates are given by the following rational numbers[1]

$$x \ = \ \frac{2pq}{p^2 + q^2}, \ \ y \ = \ \frac{p^2 - q^2}{p^2 + q^2}, \ \ x^2 \ + \ y^2 \ = \ 1 \tag{2}$$

The substitution $p = m + n, \ q = m - n$ allows to recast eq-(2) in the equivalent form

$$x \ = \ \frac{m^2 - n^2}{m^2 + n^2}, \ \ y \ = \ \frac{2mn}{m^2 + n^2}, \ \ x^2 \ + \ y^2 \ = \ 1 \tag{3}$$

And vice versa, given any point $\mathbf{P} = (x, y)$ with *rational* coordinates on the unit circle, the intersection of the line joining $\mathbf{P}$ and $\mathbf{N}$ with the real line $\mathbf{R}$ occurs at a point $\mathbf{X}$ with *rational* values. Consequently, one has established a *birational* map (bijective mapping of rational points to rational points and vice versa) between the real line and the unit circle.

By setting $m = 2, n = 1$ one automatically recovers the point in the unit circle with rational coordinates $(\frac{3}{5}, \frac{4}{5})$ and corresponding to the Pythagorean triple $(3, 4, 5)$. Setting $m = 3, n = 2$ yields the point in the unit circle $(\frac{5}{13}, \frac{12}{13})$ and corresponding to the Pythagorean triple $5, 12, 13$. And so forth. Due to the symmetry of the unit circle with respect the x and y axis the most general form of all the rational points in the unit circle are given by

$$x \ = \ \pm \ \frac{m^2 - n^2}{m^2 + n^2}, \ \ y \ = \ \pm \ \frac{2mn}{m^2 + n^2}, \ \ x^2 \ + \ y^2 \ = \ 1 \tag{4}$$

. with m, n integers. Due to the identities

$$cos^2\alpha \ + \ sin^2\alpha \ = \ 1; \ \ sin(2\alpha) \ = \ 2 \ sin(\alpha) \ cos(\alpha), \ \ cos(2\alpha) \ = \ cos^2(\alpha) \ - \ sin^2(\alpha) \tag{5}$$

[1]The fraction $\frac{2u}{u^2+1} \leq 1$ with $u = \frac{p}{q}$ for all values of u

one can recognize that eq.(4) can be recast in terms of trigonometric functions after setting

$$cos(\alpha) \ = \ \frac{m}{\sqrt{m^2+n^2}}, \ \ sin(\alpha) \ = \ \frac{n}{\sqrt{m^2+n^2}} \Rightarrow cos(2\alpha) \ = \ \frac{m^2-n^2}{m^2+n^2}, \ \ sin(2\alpha) \ = \ \frac{2mn}{m^2+n^2} \tag{6}$$

and obeying $cos^2(2\alpha) \ + \ sin^2(2\alpha) = 1$.

This geometrical construction is just a very special case of the *stereographic* projection of a sphere onto the equatorial plane. Let us find now the points on the sphere of unit radius centered at the origin and whose coordinates (x, y, z) are given by rational numbers. The *stereographic* projection of a sphere onto the equatorial plane is given by selecting any point on the sphere $\mathbf{P} = (x, y, z)$ and drawing the straight line joining that point $\mathbf{P}$ with the north pole $\mathbf{N} = (0, 0, 1)$. The *intersection* of the line $\mathbf{N}\ \mathbf{P}$ with the equatorial plane is given by a point $\mathbf{Q}$ whose coordinates are given by X, Y. The relationship among the x, y, z coordinates and X, Y is given by

$$x \ = \ \frac{2X}{X^2+Y^2+1}, \ \ y \ = \ \frac{2Y}{X^2+Y^2+1}, \ \ z \ = \ \frac{X^2+Y^2-1}{X^2+Y^2+1}, \ \ x^2+y^2+z^2 = 1 \tag{7}$$

Inverting these relations yields

$$X \ = \ \frac{x}{1-z}, \ \ Y \ = \ \frac{y}{1-z}, \ \ X^2 \ + \ Y^2 \ = \ \frac{1+z}{1-z} \tag{8}$$

It is clear from eq-(7) that if X, Y are rational numbers, or integers, then x, y, z are automatically rational-valued.

The intersection of the xy-plane with the unit sphere S^2 is an equatorial great circle S^1 of unit radius. Hence we can use the rational points on the unit circle S^1 found in eqs.(3,4), and denoted now by $\tilde{x}, \tilde{y}$ to avoid confusion, and scale them by a factor of $\lambda_1 \geq 1$ and set

$$X \ = \ \lambda_1 \ \tilde{x} \ = \ \lambda_1 \ \frac{m^2-n^2}{m^2+n^2}, \ \ Y \ = \ \lambda_1 \ \tilde{y} \ = \ \lambda_1 \ \frac{2mn}{m^2+n^2}; \ \ (\tilde{x})^2 + (\tilde{y})^2 = 1 \tag{9}$$

Thus, by inserting the expressions given by eq.(9) into the right hand side of eq.(7) one arrives at

$$x \ = \ \frac{2\lambda_1}{1+\lambda_1^2} \ \frac{m^2-n^2}{m^2+n^2}, \ \ y \ = \ \frac{2\lambda_1}{1+\lambda_1^2} \ \frac{2mn}{m^2+n^2}, \ \ z \ = \ \frac{\lambda_1^2 \ - \ 1}{\lambda_1^2 \ + \ 1} \tag{10a}$$

where λ_1 is a rational number (or integer), and m, n integers. One can verify from eq.(10a) that $x^2 + y^2 + z^2 \ = \ 1$. Therefore, rational points on the unit sphere can be obtained from the prior rational points $\tilde{x}, \tilde{y}$ on the unit circle (found in eqs.(3,4)) after performing the following scaling depicted in eq.(9) and inserting the values of X, Y into eq.(7). Consequently, one can obtain rational points on the unit sphere from the rational points on a unit circle via an *inverse* stereographic projection after a suitable scaling of the rational coordinates of the unit circle by a factor of $\lambda_1 \geq 1$.

Because we had established a *birational* map between the real line and the unit circle, by setting $r \equiv m/n$, one can rewrite in eq.(10a) that

$$\frac{m^2 - n^2}{m^2 + n^2} = \frac{r^2 - 1}{r^2 + 1}, \quad \frac{2mn}{m^2 + n^2} = \frac{2r}{r^2 + 1}, \quad r \equiv \frac{m}{n} \tag{10b}$$

leading to a pair of rational numbers on the unit circle which will be mapped to a rational point on the real axis, and vice versa. Similarly one has in eq.(10a) that both expressions

$$\frac{2\lambda_1}{1 + \lambda_1^2}, \quad \frac{\lambda_1^2 - 1}{\lambda_1^2 + 1} \tag{10c}$$

which have the *same* functional form as those in eq.(10b), lead also to a pair of rational numbers, and such that there is *no irrational* number λ that can yield a pair of rational numbers in eq.(10c). Since the product/ratio of two irrationals is *not* always irrational, one cannot exclude the possibility that an irrational value for r and λ might yield rational values in eqs.(10b,10c), but we have shown that this is not possible due to the fact that the *birational* maps between the real line and the unit circle establish a one-to-one correspondence among the countable-infinity number of rationals lying on the real line and the rational points in the unit circle.

Note that eq.(10a) is not the only combination available to us. By exchanging $\frac{2\lambda_1}{1+\lambda_1^2} \leftrightarrow \frac{\lambda_1^2 - 1}{\lambda_1^2 + 1}$ in eq.(10a) one arrives at

$$x' = \frac{\lambda_1^2 - 1}{1 + \lambda_1^2} \frac{m^2 - n^2}{m^2 + n^2}, \quad y' = \frac{\lambda_1^2 - 1}{1 + \lambda_1^2} \frac{2mn}{m^2 + n^2}, \quad z' = \frac{2\lambda_1}{\lambda_1^2 + 1} \tag{10d}$$

One can verify that after setting, for example, $m = 2, n = 1, \lambda_1 = 2$ in eqs.(10a,10d) yields the following combinations of the sums of three squares $(12)^2 + (16)^2 + (15)^2 = (25)^2$, and $(9)^2 + (12)^2 + (20)^2 = (25)^2$, respectively.

To sum up, by constraining λ_1 to be rational numbers, or integers, the expressions in eq.(10a) will generate *all* the possible rational points on the unit sphere. However there is *caveat* because this will not be the case for higher dimensional hyperspheres. The reason being that the product/ratio of two irrationals is *not* always irrational. We shall show at the end of this work that there are rational points on S^n that require *irrational* scaling factors λ. The most notorious example is the case of the hyper-sphere S^{24} associated with the Pythagorean 24-tuple of numbers inherent in the 24-dimensional Leech lattice.

This stereographic construction can be generalized to hyper-spheres S^n embedded in flat Euclidean R^{n+1} spaces of $n+1$-dim. The projection of the north pole of S^n onto the n-dimensional equatorial *hyperplane* obtained from the *intersection* of the straight line obtained by joining a point on S^n to the north pole leads to the following relationships among the coordinates X_n of the hyperplane and the $x_1, x_2, \cdots x_n, x_{n+1}$ coordinates on S^n

$$x_i = \frac{2X_i}{(\sum_{j=1}^n X_j^2) + 1}, \quad i = 1, 2, 3, \cdots, n. \quad x_{n+1} = \frac{(\sum_{j=1}^n X_j^2) - 1}{(\sum_{j=1}^n X_j^2) + 1} \tag{11}$$

one can verify that $x_1^2 + x_2^2 \ \cdots + x_n^2 + x_{n+1}^2 = 1$. Therefore, eq.(11) provides the rational coordinates of most of the rational points on the unit hyper-sphere S^n, if the X_i's are rational or integer numbers.

As an example let us find the rational points on S^3 following the same procedure as the one leading to eqs.(10a,10d). S^3 can be embedded into a 4-dimensional Euclidean space R^4. The 3-dim subspace R^3 can be seen as a hyperplane of R^4 that is defined by the algebraic equation $x_4 = 0$. Hence, the intersection of the hyperplane (R^3) with the unit S^3 is an equatorial great "circle" given by the unit sphere S^2. One just repeats the previous procedure and performs the scaling of the rational coordinates $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ on the unit sphere by $\lambda_2 \geq 1$ and sets

$$X_1 \ = \ \lambda_2 \, \tilde{x}_1, \quad X_2 \ = \ \lambda_2 \, \tilde{x}_2, \quad X_3 \ = \ \lambda_2 \, \tilde{x}_3 \tag{12}$$

$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are the *rational* coordinates of the points lying on the unit sphere S^2 and whose expressions are given by eqs.(10a). Therefore, after inserting the expressions in eq.(12) into eq.(11), when $i = 1, 2, 3$, yields most of the rational points (not *all* of them) on the unit hypersphere S^3 and which are given by the rational coordinates

$$\begin{aligned} x_1 \ &= \ \frac{2\lambda_2}{1+\lambda_2^2} \, \frac{2\lambda_1}{1+\lambda_1^2} \, \frac{m^2-n^2}{m^2+n^2} \\ x_2 \ &= \frac{2\lambda_2}{1+\lambda_2^2} \, \frac{2\lambda_1}{1+\lambda_1^2} \, \frac{2mn}{m^2+n^2} \\ x_3 \ &= \ \frac{2\lambda_2}{1+\lambda_2^2} \, \frac{\lambda_1^2-1}{\lambda_1^2+1} \\ x_4 \ &= \ \frac{\lambda_2^2-1}{\lambda_2^2+1} \end{aligned} \tag{13a}$$

with $\lambda_1, \lambda_2 \geq 1$ rational or integer numbers, and m, n integers. One can verify from eq.(13) that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. Therefore, eq.(13a) provides the rational coordinates of many points on the unit S^3.

Once again, by exchanging $\frac{2\lambda_2}{1+\lambda_2^2} \leftrightarrow \frac{\lambda_2^2-1}{\lambda_2^2+1}$ in eq.(13a) one arrives at another combination

$$\begin{aligned} x_1' \ &= \ \frac{\lambda_2^2-1}{1+\lambda_2^2} \, \frac{2\lambda_1}{1+\lambda_1^2} \, \frac{m^2-n^2}{m^2+n^2} \\ x_2' \ &= \frac{\lambda_2^2-1}{1+\lambda_2^2} \, \frac{2\lambda_1}{1+\lambda_1^2} \, \frac{2mn}{m^2+n^2} \\ x_3' \ &= \ \frac{\lambda_2^2-1}{1+\lambda_2^2} \, \frac{\lambda_1^2-1}{\lambda_1^2+1} \\ x_4' \ &= \ \frac{2\lambda_2}{\lambda_2^2+1} \end{aligned} \tag{13b}$$

Similarly, one can repeat the process leading to eqs.(13a,13b) from eq.(10a) to one originating from eq.(10d) instead, and arrive at the following

$$
\begin{aligned}
x''_1 &= \frac{2\lambda_2}{1+\lambda_2^2}\,\frac{\lambda_1^2-1}{1+\lambda_1^2}\,\frac{m^2-n^2}{m^2+n^2} \\
x''_2 &= \frac{2\lambda_2}{1+\lambda_2^2}\,\frac{\lambda_1^2-1}{1+\lambda_1^2}\,\frac{2mn}{m^2+n^2} \\
x''_3 &= \frac{2\lambda_2}{1+\lambda_2^2}\,\frac{2\lambda_1}{\lambda_1^2+1} \\
x''_4 &= \frac{\lambda_2^2-1}{\lambda_2^2+1} \qquad (13c)
\end{aligned}
$$

$$
\begin{aligned}
x'''_1 &= \frac{\lambda_2^2-1}{1+\lambda_2^2}\,\frac{\lambda_1^2-1}{1+\lambda_1^2}\,\frac{m^2-n^2}{m^2+n^2} \\
x'''_2 &= \frac{\lambda_2^2-1}{1+\lambda_2^2}\,\frac{\lambda_1^2-1}{1+\lambda_1^2}\,\frac{2mn}{m^2+n^2} \\
x'''_3 &= \frac{\lambda_2^2-1}{1+\lambda_2^2}\,\frac{2\lambda_1}{\lambda_1^2+1} \\
x'''_4 &= \frac{2\lambda_2}{\lambda_2^2+1} \qquad (13d)
\end{aligned}
$$

The $four$ combinations provided by eqs.(13a,13b,13c,13d) will generate $four$ families of rational points on the unit S^3. For example, by setting $m = 2, n = 1, \lambda_1 = \lambda_2 = 2$ into eq.(13a) one will have

$$
\begin{aligned}
(48)^2 + (64)^2 + (60)^2 + (75)^2 &= (125)^2 = 15625 \Rightarrow \\
(48/125)^2 + (64/125)^2 + (60/125)^2 + (75/125)^2 &= 1 \qquad (13e)
\end{aligned}
$$

Similarly, by inserting those values of $m = 2, n = 1, \lambda_1 = \lambda_2 = 2$ into the other eqs.(13b,13c,13d) one will generate the other combinations of rational points on the unit S^3.

One can continue this *recursion* process and derive the expressions for the rational coordinates of the rational points on the unit hyper-spheres S^n. We began with the intersection of lines containing two rational points with the unit circle and derived the eqs.(2,3,4) generating *all* the rational points on the unit circle. Then we inserted the *scaling* of these rational points on the unit circle into the expressions in eq.(7) in order to obtain *all* of the rational points of the unit sphere. Then we inserted the *scaling* of these rational points on the unit sphere S^2 into the expressions in eq.(11), when $i = 1, 2, 3$, in order to obtain most of the rational points on the unit S^3. This recursion process holds for all the other values of the dimensions of the hyper-spheres.

One may rewrite the Cartesian coordinates of the points of S^3 with unit radius (13) in terms of three angles $\theta_1, \theta_2, \theta_3$ as follows

$$x_1 = sin(\theta_3)\ sin(\theta_2)\ cos(\theta_1)$$

$$x_2 = sin(\theta_3)\ sin(\theta_2)\ sin(\theta_1)$$

$$x_3 = sin(\theta_3)\ cos(\theta_2)$$

$$x_4 = cos(\theta_3) \qquad (14a)$$

such that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

Therefore, from eq.(13a) one finds the respective values for three angles associated with the rational points in the unit S^3

$$cos(\theta_1) = \frac{m^2 - n^2}{m^2 + n^2}, \quad sin(\theta_1) = \frac{2mn}{m^2 + n^2}$$

$$sin(\theta_2) = \frac{2\lambda_1}{1 + \lambda_1^2}, \quad cos(\theta_2) = \frac{\lambda_1^2 - 1}{\lambda_1^2 + 1}$$

$$cos(\theta_3) = \frac{\lambda_2^2 - 1}{\lambda_2^2 + 1}, \quad sin(\theta_3) = \frac{2\lambda_2}{1 + \lambda_2^2}, \qquad (14b)$$

with $\lambda_1, \lambda_2, m, n$ integers. A similar procedure follows for the other combinations in eqs.(13b,13c,13d).

Setting $\theta_3 = \frac{\pi}{2}$ in (14a) leads to $x_4 = 0$, $sin(\theta_3) = 1$, and one recovers the standard expression for the x, y, z coordinates of the sphere S^2 where θ_1 is the azimuth angle and θ_2 is the zenith angle. S^2 is the great equatorial "circle" of S^3 corresponding to the angle $\theta_3 = \frac{\pi}{2}$. Likewise, S^1 is the great equatorial circle of S^2 corresponding to the angle $\theta_2 = \frac{\pi}{2}$.

The angular coordinates of S^4 are obtained after multiplying each term of eq.(14a) by $sin(\theta_4) : x_i \to x_i\ sin(\theta_4)$ *and* adding the extra coordinate $x_5 = cos(\theta_4)$ as follows

$$x_1 = sin(\theta_4)\ sin(\theta_3)\ sin(\theta_2)\ cos(\theta_1)$$

$$x_2 = sin(\theta_4)\ sin(\theta_3)\ sin(\theta_2)\ sin(\theta_1)$$

$$x_3 = sin(\theta_4)\ sin(\theta_3)\ cos(\theta_2)$$

$$x_4 = sin(\theta_4)\ cos(\theta_3)$$

$$x_5 = cos(\theta_4) \qquad (15)$$

such that $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1$. One can repeat this process for S^5, and so forth, for the other higher dimensional unit hyper-spheres by introducing more angles successively. The early study of the generalized angular momentum operators and Laplacians for hyper-spheres S^N can be found in [8]. A more recent treatment of the generalized Legendre polynomials for hyper-spheres can be found in [9].

As a result, the recursive process to derive the rational coordinates of the unit S^4 from those of a unit S^3 can be simply obtained by multiplying each term of eqs.(13a,13b,13c,13d) by $(2\lambda_3/\lambda_3^2+1)$, *and* adding $x_5 = (\lambda_3^2 - 1/\lambda_3^2+1)$, or vice versa, by exchanging $\frac{2\lambda_3}{\lambda_3^2+1} \leftrightarrow \frac{\lambda_3^2-1}{\lambda_3^2+1}$, generating in turn *eight* families of rational points on the unit S^4. This recursion process can be continued indefinitely with $S^5, S^6, \cdots S^n, S^{n+1}, \cdots$ by introducing the scaling factors, rational numbers, given $\lambda_4, \lambda_5, \cdots, \lambda_{n-1}, \lambda_n, \cdots$.

The Leech lattice can also be constructed in terms of the vector $(0, 1, 2, 3, \ldots, 22, 23, 24; 70)$ in the 26-dimensional even Lorentzian unimodular lattice $\Lambda_{25,1}$. The existence of such an integral vector of Lorentzian norm zero relies on the fact that $1^2 + 2^2 + \cdots + 24^2$ is a perfect square 70^2; the number 24 is the only integer bigger than 1 with this property (see cannonball problem [3]). This was conjectured by Édouard Lucas, but the proof came much later, based on elliptic functions [3]. Hence one has the Pythagorean 24-tuple of non-vanishing numbers

$$0^2 \ + \ 1^2 \ + \ 2^2 \ + \ 3^2 \ + \ \ldots \ + \ 24^2 \ = \ 70^2 \ \Rightarrow$$
$$x_1 \ = \ 0, \ \ x_2 \ = \ \frac{1}{70}, \ x_3 \ = \ \frac{2}{70}, \ x_4 \ = \ \frac{3}{70}, \cdots, x_{25} \ = \ \frac{24}{70} \tag{16}$$

such that $x_1^2 + x_2^2 + x_3^2 + \cdots + x_{25}^2 = 1$. To find the rational points in the unit S^{24} based on the recursive method requires introducing $\lambda_1, \lambda_2, \cdots, \lambda_{23}$ scaling factors. In particular, the last ratio $x_{25} = 24/70$ is given by

$$x_{25} \ = \ \frac{\lambda_{23}^2 - 1}{\lambda_{23}^2 + 1} \ = \ \frac{24}{70} \ \Rightarrow \ \lambda_{23} \ = \ \sqrt{\frac{47}{23}} \ = \ 1.429\cdots \tag{17}$$

One finds that the square λ_{23}^2 is rational but λ_{23} is *not* a rational number and such that the other key factor $(2\lambda_{23}/1 + \lambda_{23}^2)$ is irrational. The latter factor appears in the product

$$x_{24} \ = \ \frac{2\lambda_{23}}{1 + \lambda_{23}^2} \ \frac{\lambda_{22}^2 - 1}{\lambda_{22}^2 + 1} \ = \ \frac{23}{70} \tag{18}$$

From eqs.(17,18) one learns that

$$\frac{2\lambda_{23}}{1 + \lambda_{23}^2} \ = \ \frac{\sqrt{4324}}{70}, \ \ \frac{\lambda_{22}^2 - 1}{\lambda_{22}^2 + 1} \ = \ \frac{23}{\sqrt{4324}} \ \Rightarrow \lambda_{22} \ = \ \left(\frac{\sqrt{4324} + 23}{\sqrt{4324} - 23} \right)^{\frac{1}{2}} \tag{19}$$

From eqs.(17,19) one can verify that the irrational scaling factors $\lambda_{23}, \lambda_{22} > 1$ are greater than 1 as expected. One can continue this procedure and determine all of scaling factors as we descend down the ladder $x_{23}, x_{22}, \cdots, x_2, x_1 = 0$. The condition $x_1 = 0$ requires to choose $m = n \Rightarrow (2mn/m^2 + n^2) = 1$, and such that x_2 is given by the product involving all of the scaling factors

$$x_2 \ = \ \frac{1}{70} \ = \ \prod \frac{2\lambda_i}{\lambda_i^2 + 1}, \ \ \ i = 1, 2, 3, \cdots, 23 \tag{20}$$

Therefore to obtain all the above rational numbers in eq.(16) associated with the Leech lattice one requires *irrational* scaling factors for the λ's. As mentioned earlier, this is due to the fact that the product/ratio of two irrationals is *not* always irrational. To sum up, there are rational points on the unit S^n that are *not* captured by our formulae if one constrains the scaling factors λ's to be rational, or integer-valued. Irrational values would be required also to capture *all* the possible rational points on the unit S^n. The circle and the sphere are an exception.

2 The Bloch sphere and its extension for higher spin particles

In this section we shall find physical applications of the results found above. We shall show that the *rational* coordinates $x_1, x_2, x_3, \cdots, x_{N+1}$ of the points in the unit S^N embedded in a flat Euclidean R^{N+1} space encode the rational values of the *probabilities* of finding a spin-$\frac{N}{2}$ particle in a given quantum state (spin up, spin down in the special case of a spin-$\frac{1}{2}$ particle). A quantum superposition of two orthogonal spin-$\frac{1}{2}$ states (spin up, spin down) is given by

$$|\psi\rangle \ = \ \alpha \mid \uparrow\rangle \ + \ \beta \mid \downarrow\rangle, \ \ |\alpha|^2 \ + \ |\beta|^2 \ = \ 1 \tag{21}$$

where α, β are complex numbers. It is well known (to the experts) that the complex-valued ratios α/β and β/α admit an stereographic interpretation such that

$$\frac{\alpha}{\beta} \ = \ Z \ = \ X \ + \ iY \ = \ e^{i\phi} \ cotan(\frac{\theta}{2}) \ = \ [cos(\phi) \ + i \ sin(\phi)] \ cotan(\frac{\theta}{2}) \tag{22}$$

where X, Y are the planar coordinates of Q associated with the stereographic projection from the *north* pole of the unit sphere onto the equatorial plane obtained by joining the north pole to the point P in the unit sphere. The inverse of Z is

$$W \ = \ \frac{1}{Z} \ = \ X' \ + \ iY' \ = \ \frac{\beta}{\alpha} \ = \ e^{-i\phi} \ tan(\frac{\theta}{2}) \ = \ [cos(\phi) \ - \ isin(\phi)] \ tan(\frac{\theta}{2}) \tag{23}$$

where X', Y' are the planar coordinates of Q' associated with the stereographic projection from the *south* pole of the unit sphere onto the equatorial plane obtained by joining the south pole to the point P in the unit sphere. It is known that the Riemann sphere is obtained from the one-point-compactification of the complex plane C by adding the point at infinity and can be identified with the one-dimensional complex projective space CP^1. The Riemann sphere is a *one*-dimensional *complex line* that appears two-dimensional from the real numbers point of view and it is topologically equivalent to $C \bigcup \{\infty\}$.

Similarly, the two-dim complex projective space CP^2 can ben be obtained from C^2 by adding the complex line CP^1 at infinity to C^2. Thus CP^2 is topologically equivalent to $C^2 \bigcup \{CP^1\}_\infty$. CP^3 is topologically equivalent to $C^3 \bigcup \{CP^2\}_\infty$, and so forth. The complex projective spaces CP^N can also be interpreted as the cosets $SU(N+1)/U(N)$ with N-complex dimensions ($2N$ real dimensions).

The main physical application of this geometrical construction is that a quantum superposition of two spin-$\frac{1}{2}$ states (21) can be written in terms of the angles θ, ϕ as follows

$$|\psi\rangle \;=\; \alpha \mid \uparrow\rangle \;+\; \beta \mid \downarrow\rangle \;=\; cos(\frac{\theta}{2}) \mid \uparrow\rangle \;+\; e^{-i\phi}\; sin(\frac{\theta}{2}) \mid \downarrow\rangle, \;\; |\alpha|^2 \;+\; |\beta|^2 \;=\; 1 \qquad (24)$$

The "location" of the spin up state $\mid \uparrow\rangle$ is identified with the *north* pole $\theta = \phi = 0$, and the spin down state $\mid \downarrow\rangle$ with the *south* pole $\theta = \pi, \phi = 0$. The state $|\psi\rangle$ is associated (has a one-to-one correspondence) with the point P on the unit sphere given by the angles θ, ϕ. Since the quantum states of a projective Hilbert space are represented by *rays*, namely that the states $|\psi\rangle \sim \lambda|\psi\rangle$ are physically equivalent up to an arbitrary phase factor $\lambda = e^{i\xi}$, after multiplying both sides of eq.(24) by $\lambda = e^{i\xi}$ one arrives at the most general expression

$$e^{i\xi}\, |\psi\rangle \;=\; \alpha' \mid \uparrow\rangle + \beta' \mid \downarrow\rangle \;=\; e^{i\xi}\; \left(cos(\frac{\theta}{2}) \mid \uparrow\rangle \;+\; e^{-i\phi}\; sin(\frac{\theta}{2}) \mid \downarrow\rangle \right), \;\; |\alpha'|^2 + |\beta'|^2 \;=\; 1 \qquad (25)$$

involving 3 angles θ, ϕ, ξ associated with a unit S^3 which can be described by the condition $|\alpha'|^2 + |\beta'|^2 = 1$, with α', β' two complex numbers (4 real numbers) after setting $\alpha' = a_1 + ib_1; \beta' = a_2 + ib_2 \Rightarrow (a_1^2 + b_1^2) + (a_2^2 + b_2^2) = 1$. Note that one still recovers the complex ratios $Z = \alpha'/\beta' = \alpha/\beta; W = 1/Z$, respectively, depicted in eqs.(22,23), and which admit the stereographic projection interpretation.

This is where the notion of the Bloch sphere [5] comes into play. The Bloch sphere is the geometrical representation of the pure state space of a two-level quantum mechanical system; consequently the *pure* quantum spin states live on the unit sphere S^2, whereas the *mixed* states live inside the ball enclosed by S^2 [5]. The physical reason is that one can rotate the axes such that the new location of the north pole N' coincides with the point P that is represented by the state $|\Psi\rangle$ given by eq.(24). Hence $|\Psi\rangle$ will coincide now with the pure $\mid \uparrow\rangle'$ state. Alternatively, one can rotate the axes such that new location of the south pole S' coincides with the point P that is represented by the state $|\Psi\rangle$ given by eq.(24), and such that $|\Psi\rangle$ will coincide now with the pure $\mid \downarrow\rangle'$ state.

It is worth mentioning the Hopf fibrations of spheres [6]. Treat the phase factor $e^{i\xi}$ in (25) as a function of the angles θ, ϕ associated with a base point P on the two-dimensional sphere S^2. In this way a S^3 can be locally fibered over S^2 with S^1 being the fibers (since the phase $e^{i\xi}$ is the circle group of rotations). The natural metric on the Bloch sphere is the Fubini–Study metric. The mapping from the unit 3-sphere in the two-dimensional state space C^2 to the Bloch sphere is the Hopf fibration of S^3 over S^2 with each ray of spinors mapping to one point on the Bloch sphere [5]. One can repeat this process for the fibration of S^7 over S^4 with S^3 as fibers (which can be realized in terms of unit quaternions $\mathbf{H}$). And finally, the fibration of S^{15} over S^8 with S^7 fibers (which can be realized in terms of unit octonions $\mathbf{O}$)[2].

[2]A unit quaternion $\mathbf{q} = x_o + x_i e_i; i = 1,2,3$ obeys $q\bar{q} = x_o^2 + x_1^2 + x_2^2 + x_3^2 = 1$, which is the analytical expression for the unit S^3 embedded in R^4. And a unit octonion is given by $\mathbf{u} = y_o + y_j e_j; j = 1,2,3,\cdots,7$ and obeys $u\bar{u} = y_o^2 + y_1^2 + y_2^2 + \cdots + y_7^2 = 1$, which is the analytical expression for a unit S^7 embedded in R^8

Let us generalize the quantum superposition (21) to one involving $N+1$ spin-$\frac{N}{2}$ states ($2s+1 = N+1 \Rightarrow s = \frac{N}{2}$) by writing

$$|\Psi\rangle \;=\; \sum_{n=1}^{N+1} \alpha_n \; |\psi_n\rangle, \quad \sum_{n=1}^{N+1} |\alpha_n|^2 \;=\; 1 \tag{26}$$

The last condition is a result of the normalization condition $\langle\Psi||\Psi\rangle = 1$. The state $|\Psi\rangle$ belongs now to an $N+1$-level quantum mechanical system. Based on our construction of the rational points on the unit S^N given by the rational numbers $x_1, x_2, \cdots, x_{N+1}$ obeying $x_1^2 + x_2^2 + \cdots + x_{N+1}^2 = 1$, it allows to equate

$$|\alpha_1|^2 \;=\; a_1^2 + b_1^2 \;=\; x_1^2, \;\; |\alpha_2|^2 \;=\; a_2^2 + b_2^2 \;=\; x_2^2, \quad \ldots, |\alpha_{N+1}|^2 \;=\; a_{N+1}^2 + b_{N+1}^2 \;=\; x_{N+1}^2 \tag{27}$$

and read out automatically the *rational* solutions for the real and imaginary parts of all the complex coefficients $\alpha_n = a_n + ib_n$ given by

$$a_1 \;=\; x_1 \frac{m'^2 - n'^2}{m'^2 + n'^2}, \;\; b_1 \;=\; x_1 \frac{2m'n'}{m'^2 + n'^2}; \;\; a_2 \;=\; x_2 \frac{m'^2 - n'^2}{m'^2 + n'^2}, \;\; b_2 \;=\; x_2 \frac{2m'n'}{m'^2 + n'^2}; \;\; \ldots$$

$$a_{N+1} \;=\; x_{N+1} \frac{m'^2 - n'^2}{m'^2 + n'^2}, \;\; b_{N+1} \;=\; x_{N+1} \frac{2m'n'}{m'^2 + n'^2} \tag{28}$$

with m', n' integers which are not to be confused with m, n in eqs.(3,4).

It is possible to generalize the construction of the Bloch sphere to dimensions larger than two, but the geometry of such a "Bloch body" is more complicated than that of a ball [7].

Concluding, given the superposition of $N+1$ orthogonal states (qudits) in eq.(26), $|\alpha_n|^2$ is the *probability* of observing a spin-$\frac{N}{2}$ particle in a given $|\psi_n\rangle$ state, (spin up, spin down, qubits in the special case of a spin-$\frac{1}{2}$ particle), then it follows that the corresponding *rational* values of such probabilities are given by $x_1, x_2, \cdots, x_{N+1}$, respectively, where $x_n, n = 1, 2, \cdots, x_{N+1}$ are the rational coordinates of a unit S^N sphere embedded in a flat Euclidean R^{N+1} space. The probabilities are ≤ 1 by construction.

A heuristic reason why Fermat's last theorem is true could possibly be related to the nature of Quantum Mechanics. The complex coefficients α_n in eq.(26) are comprised of a real and imaginary part so one requires two real numbers (a_n, b_n) living in R^2 to specify α_n. In QM, Ψ is a complex-valued probability *amplitude*, thus it is the square $|\Psi|^2$ that furnishes the actual probability. Therefore $|\alpha_n|^2 = a_n^2 + b_n^2$ is the natural expression one encounters in QM. One does *not* encounter expressions like $a_n^p + b_n^p, p > 2$; nor $a_n^p + b_n^p + c_n^p$, etc .... it is only the expression corresponding to Fermat's last theorem that one encounters in QM. The role of integers (like the quantization of energy levels in the atom) clearly has also a QM connection. It is warranted to explore this QM interpretation further.

3 Concluding Remarks : Fermat Surfaces

As a generalization of hyper-spheres, one can define Fermat (hyper) surfaces $\mathcal{F}^N$ of any dimension N and order p as the locus of points obeying the following algebraic equation

$$x_1^p \ + \ x_2^p \ + \ x_3^p \ \cdots \ + \ x_{N+1}^p \ = \ r^p, \quad p \geq 3, \quad N \geq 2 \tag{29}$$

Many integer-valued solutions to (29) can be found. There are well known sums of cubes like

$$(3)^3 + (4)^3 + (5)^3 = (6)^3, \quad (11)^3 + (12)^3 + (13)^3 + (14)^3 = (20)^3 \tag{30}$$

Diophantine equations with sum of cubes and cube of sum can be found in [11].

The Jacobi-Madden equation [10] leads to many solutions, in particular to the following non-trivial sums of quartic powers

$$(5400)^4 + (1770)^4 + (2634)^4 + (955)^4 \ = \ (5491)^4 \tag{31a}$$

$$(31764)^4 + (27385)^4 + (48150)^4 + (7590)^4 \ = \ (51361)^4 \tag{31b}$$

The sums of quintic powers like

$$(27)^5 + (84)^5 + (110)^5 + (133)^5 \ = \ (144)^5 \tag{32}$$

were found by [12], and so forth. Extensive numerical computations for the search of rational points lying on Fermat (hyper) surfaces $\mathcal{F}^N$ of arbitrary dimensions N, and order p, obeying the following equations

$$\sum_{i=1}^{i=N+1} |x_i|^p \ = \ r^p, \quad p \geq 3, \quad N \geq 2 \tag{33}$$

with $|x_i|, r$ positive integers were provided by [12]. The non-zero entries $|x_i|, i = 1, 2, 3, \cdots, N, N+1$ were interpreted by the authors [12], [13] as the non-zero components of the so-called "Fermat vectors" whose tips correspond to the rational points lying on such Fermat (hyper) surfaces.

In this work we have described the procedure how to obtain rational points on circles, spheres and hyper-spheres based on stereographic projections. To figure out the geometric procedure to find the rational points of Fermat surfaces is another matter. Furthermore, we only studied real surfaces. It is warranted to explore the case of complex surfaces. Enriques and Kodaira classified compact complex surfaces into ten classes. The standard reference book for compact complex surfaces can be found in [14]. A complete classification of *complex* surfaces is still undergoing to our knowledge.

Acknowledgments

We thank M. Bowers for assistance and to Ramon Carbó-Dorca for many insightful and helpful discussions concerning the vast search for Fermat vectors in arbitrary dimensions and for bringing to our attention refs-[2,9].

Disclosure statement

There were no conflict of interests in this completion of this work.

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