Experimentally feasible inconsistency within foundations of quantum mechanics

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Abstract

It is our pleasure to say that some of theoretical analyses on quantum mechanics are scientifically beautifully and comprehensibly represented using the Kronecker delta notation. It happens that Nagata and Nakamura discuss a novel inconsistency within quantum mechanics when accepting we use the property of the Kronecker delta without extra assumptions about the reality of observables. Based on the argumentations, we propose an experimental accessible inconsistency in terms of imperfect source and detector. In more detail, we encounter an imperfect quantum state, the dark count, and the quantum efficiency, which cannot be avoidable from a real experimental situation. However, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers. One of the objectives of this paper is for us to remain wondering the extension of quantum mechanical axiom to concrete commuting observables themselves.

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I. INTRODUCTION

Quantum mechanics (cf. [1–7]) gives explanations for the microscopic behaviors of the nature. We see researches concerning the mathematical formulations of quantum mechanics. Mackey studies the mathematical foundations of quantum mechanics [8]. Gudder discusses on the quantum logic approach to quantum mechanics [9]. Conditional probability and the axiomatic structure of quantum mechanics are also reported by Guz [10].

We notice [4, 11] that von Neumann's mathematical model for quantum mechanics is logically successful. And the axiomatic system for the mathematical model is a consistent one. Thus, we cannot say that von Neumann's mathematical model has an inconsistency. What is the inconsistency to be discussed in this paper? We cannot expand the von Neumann's mathematical model more in handling real experimental data. Mathematically, von Neumann's model is logically consistent, which fact is true. However, von Neumann's theory is questionable in the sense that the mathematical model does not always expand to real experimental data. And there is the inconsistency if we apply the von Neumann's model to expanding even a simple physical situation. In short, von Neumann's mathematical model might not be useful in that case.

The inconsistency to be discussed in this paper is significant. von Neumann's mathematical model has the qualification to be true axiomatic system for quantum mechanics. Therefore, we cannot modify the axioms based on the nature of Matrix theory. Nevertheless, we encounter an inconsistency, probably due to the nature of Matrix theory, within von Neumann's theory.

Einstein, Podolsky, and Rosen discuss the incompleteness argument to quantum mechanics itself [12]. A hidden-variable interpretation of quantum mechanics is a topic of research [2, 3] and the no-hidden-variable theorem is discussed by Bell, Kochen, and Specker [13, 14]. The Kochen-Specker theorem based on the Kronecker delta is also discussed by Nagata, Patro, and Nakamura [15].

The Kronecker delta is necessary for quantum mechanics that is based on Matrix and Vector. The Kronecker delta is explained as follows: The two-variable function $\delta_{ll'}$ that takes the value 1 when $l = l'$ and the value 0 otherwise. If the elements of a square matrix are defined by the delta function, the matrix produced will be the identity matrix [16]. The name of Kronecker is also used in the Kronecker product. However, in this paper, we dare to use the concept of the Kronecker delta.

Recently, Nagata and Nakamura discuss a novel inconsistency within quantum mechanics when accepting we use the property of the Kronecker delta without extra assumptions about the reality of observables [17]. Based on the argumentations, here, we propose an experimental accessible inconsistency in terms of imperfect source and detector.

In more detail, we encounter an imperfect quantum state, the dark count, and the quantum efficiency, which cannot be avoidable from a real experimental situation. If we use the quantum predictions by $2N$ trials, then the inconsistency increases by an amount that grows linearly with $2N$. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers. One of the objectives of this paper is for us to remain wondering the extension of quantum mechanical axiom to concrete commuting observables themselves.

We define [17] an inconsistency as follows, when considering only two commuting Hermitian matrices:

1. Define two commuting Hermitian matrices A_1, A_2.
2. Define a two-variable function $f(X, Y)$, where f is an appropriate function and X, Y are two variables.
3. Derive a value of $f(A_1, A_2) = a$ by substituting A_1, A_2 into X, Y, respectively, without using the property of the Kronecker delta.
4. Introduce the property of the Kronecker delta.
5. Derive another value of $f(A_1, A_2) = b(\neq a)$ under the supposition that we use the property of the Kronecker delta.
6. We cannot assign simultaneously the same two values ("1" and "1") or ("0" and "0") for the two suppositions $f(A_1, A_2) = a$ and $f(A_1, A_2) = b$.
7. Confirm the inconsistency derived only by the two commuting Hermitian matrices A_1, A_2.

II. EXPERIMENTALLY FEASIBLE INCONSISTENCY WITHIN FOUNDATIONS OF QUANTUM MECHANICS

Let σ_z^1, σ_z^2 be two z-component Pauli observables, where they are also supposed to be commutative. They could be defined respectively as follows:

$$\sigma_z^1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \sigma_z^2 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be eigenstates of σ_z such that $\sigma_z|\uparrow\rangle = +1|\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -1|\downarrow\rangle$. The measured results of trials are either $+1$ or -1 in the ideal case.

When we consider a quantum optical experiment, we have the following relation with the photon polarization states:

$$\begin{aligned} |\uparrow\rangle &\leftrightarrow |H\rangle, \\ |\downarrow\rangle &\leftrightarrow |V\rangle, \end{aligned} \tag{2}$$

where $|H\rangle$ is a quantum state interpreted by a horizontally polarized photon and $|V\rangle$ is a quantum state interpreted by a vertically polarized photon.

Let us introduce the random noise admixture $\rho_{\text{noise}}(= \frac{1}{2}I)$ into the quantum states, where I is the two-dimensional identity operator. We consider the noisy quantum states emerged from an imperfect source as follows:

$$\begin{aligned} |\Psi^1\rangle\langle\Psi^1| &= (1-\epsilon)|\uparrow\rangle\langle\uparrow| + \epsilon \times \rho_{\text{noise}}, \\ |\Psi^2\rangle\langle\Psi^2| &= (1-\epsilon)|\downarrow\rangle\langle\downarrow| + \epsilon \times \rho_{\text{noise}}. \end{aligned} \tag{3}$$

The value of $\epsilon(\ll 1)$ is interpreted as the reduction factor of the contrast observed in the single-particle experiment. Then we have $\mathrm{tr}[|\Psi^1\rangle\langle\Psi^1|\sigma_z] = +1-\epsilon$ and $\mathrm{tr}[|\Psi^2\rangle\langle\Psi^2|\sigma_z] = -1+\epsilon$.

We might be an inconsistency when the first result is $+1-\epsilon$ by measuring Pauli observable σ_z^1 in the quantum state $|\Psi^1\rangle\langle\Psi^1|$, the second result is $-1+\epsilon$ by measuring the same Pauli observable σ_z^2 in the quantum state $|\Psi^2\rangle\langle\Psi^2|$, and then $[\sigma_z^1, \sigma_z^2] = 0$.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either $+1-\epsilon$ or $-1+\epsilon$. We suppose the number of trials of obtaining the result $-1+\epsilon$ is equal to the number of trials of obtaining the result $+1-\epsilon$. We can depict experimental data r_1, r_2 as follows: $r_1 = +1-\epsilon$ and $r_2 = -1+\epsilon$. Let us write V as follows:

$$V = \sum_{l=1}^{2} r_l. \tag{4}$$

We are very interested in the following value:

$$V \times V = \left(\sum_{l=1}^{2} r_l\right)^2 = \left(\sum_{l=1}^{2} r_l\right) \times \left(\sum_{l'=1}^{2} r_{l'}\right). \tag{5}$$

Surprisingly, we cannot define $V \times V$ as zero as shown below.

Without using the property of the Kronecker delta, we have

$$\begin{aligned} V \times V \times \delta_{ll'} &= \left(\sum_{l=1}^{2} r_l\right)^2 \delta_{ll'} \\ &= ((+1-\epsilon) + (-1+\epsilon))^2 \delta_{ll'} = 0 \times \delta_{ll'} = 0. \end{aligned} \tag{6}$$

We derive a necessary condition of the product $(V \times V \times \delta_{ll'})$ of the value V without using the property of the Kronecker delta. In this case, we have the calculation result as

$$(V \times V \times \delta_{ll'}) = 0. \tag{7}$$

This is the necessary condition without using the property of the Kronecker delta.

In the following, we evaluate another value of $(V \times V \times \delta_{ll'})$ and derive another necessary condition when we use the property of the Kronecker delta.

We introduce the property of the Kronecker delta then we have

$$\begin{aligned} & V \times V \times \delta_{ll'} \\ &= \left(\sum_{l=1}^{2} r_l\right)^2 \times \delta_{ll'} \\ &= \left(\sum_{l=1}^{2} r_l\right) \times \left(\sum_{l'=1}^{2} r_{l'}\right) \times \delta_{ll'} \\ &= (+1-\epsilon)^2 + (-1+\epsilon)^2 = 2(+1-\epsilon)^2. \end{aligned} \tag{8}$$

Clearly, we have the calculation result as

$$(V \times V \times \delta_{ll'}) = 2(+1-\epsilon)^2. \tag{9}$$

These argumentations are possible for the case that we utilize the property of the Kronecker delta. We cannot assign simultaneously the same two values ("1" and "1") or ("0" and "0") for the two suppositions (7) and (9). We derive the inconsistency when we utilize the property of the Kronecker delta.

In summary, we have been the inconsistency when the first result is $+1-\epsilon$ by measuring Pauli observable σ_z^1 in the quantum state $|\Psi^1\rangle\langle\Psi^1|$, the second result is $-1+\epsilon$ by measuring the same Pauli observable σ_z^2 in the quantum state $|\Psi^2\rangle\langle\Psi^2|$, and then $[\sigma_z^1, \sigma_z^2] = 0$.

III. DARK COUNT, QUANTUM EFFICIENCY, AND STRONG LAW OF LARGE NUMBERS

In a real experiment, a perfect detector is not feasible. There is an unforeseen effect that an imperfect detector does not count even though the particle indeed passes through the detector (the quantum efficiency). There is also an unforeseen effect that an imperfect detector counts even though the particle does not pass through the detector (the dark count). In this case, we increase measurement outcomes to $2N(\gg 1)$ and then we change such errors into trivial things. If we use the quantum predictions by $2N$ trials, then the inconsistency increases by an amount that grows linearly with $2N$. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

We might be an inconsistency when the odd number results are $+1-\epsilon$ by measuring Pauli observable σ_z^1 in the quantum state $|\Psi^1\rangle\langle\Psi^1|$, the even number results are $-1+\epsilon$ by measuring the same Pauli observable σ_z^2 in the quantum state $|\Psi^2\rangle\langle\Psi^2|$, and then $[\sigma_z^1, \sigma_z^2] = 0$.

We consider a value V which is the sum of $2N$ data in an experiment. The measured results of trials are either $+1-\epsilon$ or $-1+\epsilon$. We suppose the number of trials of obtaining the result $-1+\epsilon$ is N that is equal to the number (N) of trials of obtaining the result $+1-\epsilon$. We can depict experimental data $r_1, r_2, r_3, \ldots$ as follows: $r_1 = +1-\epsilon$, $r_2 = -1+\epsilon$, $r_3 = +1-\epsilon$ and so on. Let us write V as follows:

$$V = \sum_{l=1}^{2N} r_l. \tag{10}$$

Notice the following value:

$$V \times V = \left(\sum_{l=1}^{2N} r_l\right)^2 = \left(\sum_{l=1}^{2N} r_l\right) \times \left(\sum_{l'=1}^{2N} r_{l'}\right). \tag{11}$$

Again, we cannot define $V \times V$ as zero as shown below.

Without using the property of the Kronecker delta, we have

$$\begin{aligned} & V \times V \times \delta_{ll'} = \left(\sum_{l=1}^{2N} r_l\right)^2 \delta_{ll'} \\ & = ((+1-\epsilon) + (-1+\epsilon) + \ldots + (-1+\epsilon))^2 \delta_{ll'} \\ & = 0 \times \delta_{ll'} = 0. \end{aligned} \tag{12}$$

We derive a necessary condition of the product $(V \times V \times \delta_{ll'})$ of the value V without using the property of the Kronecker delta. In this case, we have the calculation result as

$$(V \times V \times \delta_{ll'}) = 0. \tag{13}$$

This is the necessary condition without using the property of the Kronecker delta.

In the following, we evaluate another value of $(V \times V \times \delta_{ll'})$ and derive another necessary condition when we use the property of the Kronecker delta.

We introduce the property of the Kronecker delta then we have

$$\begin{aligned} & V \times V \times \delta_{ll'} \\ & = \left(\sum_{l=1}^{2N} r_l\right)^2 \times \delta_{ll'} \\ & = \left(\sum_{l=1}^{2N} r_l\right) \times \left(\sum_{l'=1}^{2N} r_{l'}\right) \times \delta_{ll'} \\ & = (+1-\epsilon)^2 + (-1+\epsilon)^2 + \ldots + (-1+\epsilon)^2 \\ & = 2N(+1-\epsilon)^2. \end{aligned} \tag{14}$$

Clearly, we have the calculation result as

$$(V \times V \times \delta_{ll'}) = 2N(+1-\epsilon)^2. \tag{15}$$

These argumentations are possible for the case that we utilize the property of the Kronecker delta. We cannot assign simultaneously the same two values ("1" and "1") or ("0" and "0") for the two suppositions (13) and (15). We derive the inconsistency when we utilize the property of the Kronecker delta. If we use the quantum predictions by $2N$ trials, then the inconsistency increases by an amount that grows linearly with $2N$. In fact, such an error of the number of particles becomes less and less important as we increase trials more and more by using the strong law of large numbers.

In summary, we have been the inconsistency when the odd number results are $+1-\epsilon$ by measuring Pauli observable σ_z^1 in the quantum state $|\Psi^1\rangle\langle\Psi^1|$, the even number results are $-1+\epsilon$ by measuring the same Pauli observable σ_z^2 in the quantum state $|\Psi^2\rangle\langle\Psi^2|$, and then $[\sigma_z^1, \sigma_z^2] = 0$.

IV. CONCLUSIONS AND DISCUSSIONS

In conclusions, recently, Nagata and Nakamura have discussed a novel inconsistency within quantum mechanics when accepting we use the property of the Kronecker delta without extra assumptions about the reality of observables. Based on the argumentations, we have proposed an experimental accessible inconsistency in terms of imperfect source and detector.

In more detail, we have encountered an imperfect quantum state, the dark count, and the quantum efficiency, which cannot be avoidable from a real experimental situation. However, such an error of the number of particles has become less and less important as we increase trials more and more by using the strong law of large numbers. One of the objectives of this paper has been for us to remain wondering the extension of quantum mechanical axiom to concrete commuting observables themselves.

The inconsistency is derived only by commuting observables. Thus, non-commutativeness of Matrix theory is needless for the derivation of the inconsistency based on the property of the Kronecker delta. The important of deriving the inconsistency is only commutativeness of Matrix theory for our purpose. The property of the Kronecker delta mainly is related to commutativeness of Matrix and Vector theory of Linear algebra and we see such an inconsistency based on the Kronecker delta. It may be much likely that Matrix mechanics is not always efficient for describing properly quantum mechanics.

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The authors are in an applicable thought to ethical approval.

Competing interests

The authors state that there is no conflict of interest.

Author contributions

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