DIVISIBLE CYCLIC NUMBERS

JULIAN BEAUCHAMP

ABSTRACT. There are known to exist a number of (multiplicative) cyclic numbers, but in this paper I introduce what appears to be a new kind of number, which we call *divisible cyclic numbers* (DCNs), examine some of their properties and give a proof of their cyclic property. It seems remarkable that I can find no reference to them anywhere. Given their simplicity, it would be extraordinary if they were hitherto unknown.

1. What is a Divisible Cyclic Number?

A DCN is a number, $\delta_{(n)}$, that is divisible by an integer divisor, n, without remainder in any of its cyclic permutations. For example, $485695_{(7)}$ is divisible by 7 in all its cyclic permutations:

$$485695 \rightarrow 856954 \rightarrow 569548 \rightarrow 695485 \rightarrow 954856 \rightarrow 548569.$$

$1265_{(11)}$ is also a DCN, divisible by 11 in all its permutations:

$$1265 \rightarrow 2651 \rightarrow 6512 \rightarrow 5126.$$

$786448_{(13)}$ is another, divisible by 13:

$$786448 \rightarrow 864487 \rightarrow 644878 \rightarrow 448786 \rightarrow 487864 \rightarrow 878644.$$

$518_{(37)}$ is yet another, divisible by 37:

$$518 \rightarrow 185 \rightarrow 851.$$

$2486628_{(2)}$ is another, divisible by 2:

$$24868 \rightarrow 48682 \rightarrow 86824 \rightarrow 68248 \rightarrow 82486.$$

$63417_{(3)}$ is another, divisible by 3:

$$63417 \rightarrow 34176 \rightarrow 41763 \rightarrow 17634 \rightarrow 76341.$$

2. What is the digit-length of a Divisible Cyclic Number?

When n is prime, the minimum digit-length of $\delta_{(p)}$ is closely related to the integer sequence in OEIS A002371 which gives the period of decimal expansion of $1/(n$th prime) (0 by convention for the primes 2 and 5) and begins:

a(n) = 0, 1, 0, 6, 2, 6, 16, 18, 22, 28, 15, 3, 5, 21, 46, 13, 58, 60, 33, 35, 8, 13, 41, 44, 96, 4, 34, 53, 108, 112, 42, 130, 8, 46,...

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For example, when $p = 13$ (the 6th prime number), then a(6) = 6, because $1/13 = 0.\overline{076923}$ has a period of 6, which is also the minimum digit-length of $\delta_{(13)}$. Or when $p = 17$ (the 7th prime number), then a(7) = 16, because $1/17 = 0.\overline{0588235294117647}$ has a period of 16, which is also the minimum digit-length of $\delta_{(17)}$. Unfortunately, there is no known general method for generating sequence A002371.

However, perhaps more usefully, this sequence also gives the smallest solution for m for the modular equation:

$$10^m - 1 \equiv 0 \pmod{p}.$$

For example, when $p = 19$, then $m = 18$ is the smallest value of m that satisfies the equation, which is also the minimum digit-length of $\delta_{(19)}$. Or when $p = 37$, $m = 3$, which is also the minimum digit-length of $\delta_{(37)}$.

This equation is useful because it also accommodates composite numbers, n, when $\gcd(n, 2^a 5^b) = 1$, $a, b \geq 0$. So replacing p with n we get:

$$10^m - 1 \equiv 0 \pmod{n}.$$

If the divisor, n, is composite, then the minimum digit-length is equal to the largest digit-length of its prime factors. For example, 21 divides $10^6 - 1$ without remainder. It shares a minimum digit-length with the largest digit-length of its two prime factors (i.e. 7, for which $m = 6$). Therefore any 6-digit number (or $6x$-digit number) that divisible by 21 is a DCN.

3. Trivial cases

We may consider as trivial divisors 2, 3 and 5.

First, 3 may be considered trivial since every multiple of 3 is cyclic regardless of digit-length and in any digit permutation.

We may also consider 2 as trivial. Regardless of digit-length *any* number whose digits are all even is divisibly-cyclic by 2. An odd digit will render it non-cyclic.

We may also consider 5 as trivial. Regardless of digit-length any number containing only digits 5, or 5s and zeros, is divisibly cyclic by 5. Any other digit will render it non-cyclic. From these last two cases it follows that (in base 10) if n has the form $2^a 5^b$ $(a, b > 0)$, then no DCNs exist.

However, when the divisor is $2x$, $3x$ or $5x$, then the digit-length will be determined by the largest digit-length of the prime factors of x.

Notice also that if we allow leading zeros every integer (which does not have the form $2^a 5^b$, $a, b > 0$) is divisibly-cyclic. For example, take $26_{(13)}$:

$$000026 \to 000260 \to 002600 \to 026000 \to 260000 \to 600002.$$

4. Proof of the Cyclic Property

Theorem 4.1. *Prove that for any DCN divisible by* n*,* $\gcd(n, 2^a 5^b) = 1$ *(*$a, b > 0$*), all other cyclic permutations are also divisible by* n*. We will work rotating permutations backwards (i.e. shifting from right to left).*

Proof. Let $\delta_{(n)}^{[k]}$ be any DCN, divisible without remainder by a divisor, n (which also divides $10^m - 1$, where m equals the number of digits of $\delta_{(n)}^{[k]}$), and where $[k]$ represents the kth permutation and let $\delta_{(i)}^{[k+1]}$ be the next permutation, where i represents the unknown divisor of the next permutation; and let z be the final digit of $\delta_{(n)}^{[k]}$. We wish to prove that $i = n$. To find the next permutation, we subtract the last digit, z from $\delta_{(n)}^{[k]}$ and add $z * 10^m$, such that:

$$\delta_{(n)}^{[k]} - z + (z * 10^m) = 10\delta_{(i)}^{[k+1]} \tag{4.1}$$

$$\Rightarrow \delta_{(n)}^{[k]} + z(10^m - 1) = 10\delta_{(i)}^{[k+1]} \tag{4.2}$$

Since both summands, $\delta_{(n)}^{[k]}$ and $z(10^m - 1)$, are divisible by n, it follows that $10\delta_{(i)}^{[k+1]}$ must also be divisible by n. But since 10 is not divisible by n, then $\delta_{(i)}^{[k+1]}$ must be. Therefore $i = n$. □

So if an integer, n, divides divides $10^m - 1$, we can be certain that it will also be the divisor of a DCN with m digits (or a multiple of m).

5. Creating new DCNs

To create new DCNs, we can carry out the following operations:

a) add n (or a multiple of n) to a known DCN, as long as the new DCN has the same digit-length (or multiple digit-length);

b) multiply a known DCN by any integer, as long as the new DCN has the same digit-length (or multiple digit-length);

c) concatenate 2 or more existing DCNs to create a new one. For example, 851, 629 and 851629 are all DCNs divisible by 37;

d) incatenate 2 or more existing DCNs to create a new one. For example, 851, 629 and 8[629]51 are DCNs divisible by 37.

e) swap single digits (or sub-strings of the same digit-length) if their difference is divisible by the divisor. For example, $745892_{(7)}$ and $745829_{(7)}$ (since $9 - 2 = 7$). Or $719589_{(13)}$ and $758199_{(13)}$ (since $58 - 19 = 39$).

Proof. Using a similar line of argument to the proof above, we show how any 2 individual digits can be swapped. This time, let z and y be digits to swap (from any position), where r and s correspond to the (base-10)position of z and y respectively, and where $s > r$, such that:

$$\delta_{(n)}^{[k]} - 10^r z - 10^s y + 10^s z + 10^r y = 10\delta_{(i)}^{[k+1]} \tag{5.1}$$

$$\Rightarrow \delta_{(n)}^{[k]} + 10^r[10^{(s-r)}z - z - 10^{(s-r)}y + y] = 10\delta_{(i)}^{[k+1]} \tag{5.2}$$

$$\Rightarrow \delta_{(n)}^{[k]} + 10^r[z(10^{(s-r)} - 1) - y(10^{(s-r)} + 1)] = 10\delta_{(i)}^{[k+1]} \tag{5.3}$$

$$\Rightarrow \delta_{(n)}^{[k]} + 10^r(10^{(s-r)} - 1)[z - y] = 10\delta_{(i)}^{[k+1]}. \tag{5.4}$$

Since $\delta_{(n)}^{[k]}$ is divisible by n, then $i = n$ iff $10^r(10^{(s-r)} - 1)[z - y]$ is divisible by n. But since $10^r(10^{(s-r)} - 1)$ is not divisible by n (since $s - r < m$), it follows that $i = n$ only when $[x - y]$ is divisible by n. □

6. Mirror-Images?

Sometimes, the mirror image of a DCN produces another. For example, the following pairs are mirror images of each other:
$886325_{(11)}$ and $523688_{(11)}$; $4058429852554185_{(17)}$ and $5814552589248504_{(17)}$;
$897164591235_{(33)}$ and $532195461798_{(33)}$; $794848028436_{(77)}$ and $634820848497_{(77)}$.

But the following pairs are not (the second in each pair is not divisible by the divisor of the first):
$785134_{(7)}$ and 431587; $4058429852554168_{(17)}$ and 8614552589248504;
$47896322_{(73)}$ and 22369874.

Is this accidental or is there a reason for this?

7. Can a DCN remain divisible under any permutation of digits?

A friend of mine has wondered whether a $\delta_{(n)}$ exists (for all n coprime with 2,3,5) that remains divisible by n under any permutation of its digits, and also the opposite, whether a $\delta_{(n)}$ exists that remains *in*divisible by n when digits are permuted.

Chester, UK
Email address: julianbeauchamp47@gmail.com