Mathematics for Incompletely Predictable Problems required to prove Riemann hypothesis, Modified Polignac's and Twin prime conjectures

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Abstract

As two different but related infinite-length equations through analytic continuation, Hasse principle is satisfied by Riemann zeta function as a certain type of equation that generates all infinitely-many trivial zeros but this principle is not satisfied by its proxy Dirichlet eta function as a dissimilar type of equation that generates all infinitely-many nontrivial zeros. Based on two seemingly different location that are in fact identical, all nontrivial zeros are mathematically located on critical line or geometrically located on Origin point. Thus we prove location for complete Set nontrivial zeros to be critical line confirming Riemann hypothesis to be true. Sieve of Eratosthenes as a certain type of infinite-length algorithm is exactly constituted by an Arbitrarily Large Number of (self-)similar infinite-length sub-algorithms that are specified by every even Prime gaps. Modified Hasse principle is satisfied by this algorithm and its sub-algorithms that perpetually generate the Arbitrarily Large Number of all Odd Primes. Thus we prove Set even Prime gaps with corresponding Subsets Odd Primes all have cardinality Arbitrarily Large in Number confirming Modified Polignac's and Twin prime conjectures to be true.

Keywords: Dirichlet eta function, Godel's incompleteness theorems, Gram's Law, Hasse principle, Polignac's conjecture, Prime-Composite identifier grouping, Prime-Composite quotient, Riemann hypothesis, Riemann zeta function, Rosser's Rule, Twin prime conjecture

MSC Classification: 11M26 , 11A41

Contents

1 Introduction

Conceived in 1859, Riemann hypothesis proposed all infinitely many nontrivial zeros from Riemann zeta function (via *proxy* Dirichlet eta function) are located on its critical line. Respectively conceived in 1849 and 1846, Polignac's conjecture proposed there are infinitely many Odd Primes derived from each and every even Prime gaps 2, 4, 6, 8, 10.... and Twin prime conjecture proposed there are infinitely many (Odd) Twin Primes derived from even Prime gap 2. Thus Twin prime conjecture is simply a subset of Polignac's conjecture.

The correct and complete mathematical arguments required to prove famous open problems in Number theory of Riemann hypothesis, Polignac's and Twin prime conjectures are best depicted when these problems are regarded as Incompletely Predictable Problems. It is informative for the required mathematical arguments that are "centered" around Axiom 1 to be broadly based on **Mathematics for Incompletely Predictable Problems** (MIPP) with Summary provided in subsection 1.2. Solving these problems is assisted by formulating the Completely and Incompletely Predictable entities as "Universal Principles" (Lemma 1) that are applicable to relevant (sub)sets and (sub)tuples. The novel classification of Countably Infinite Set into three subtypes (Lemma 2), p-adic absolute values, modular arithmetic, Hasse and Modified Hasse principles are applied at various places in this paper when required.

As two different but related infinite-length equations through analytic continuation, Hasse principle is satisfied by Riemann zeta function as a certain type of equation that generates all infinitely-many trivial zeros [located *outside* the $0 < \sigma < 1$-critical strip] but this principle is not satisfied by its *proxy* Dirichlet eta function as a dissimilar type of equation that generates all infinitely-many nontrivial zeros [located *inside* the $0 < \sigma < 1$-critical strip on the $\sigma = \frac{1}{2}$-critical line]. As two seemingly different location that are in fact identical, all nontrivial zeros are mathematically located on [one-dimensional] $\sigma = \frac{1}{2}$-critical line or geometrically located on [zero-dimensional] $\sigma = \frac{1}{2}$-Origin point. Thus we prove location for complete Set nontrivial zeros to be critical line confirming Riemann hypothesis to be true.

We ignore the solitary even Prime number 2. As a certain type of infinite-length algorithm, Sieve-of-Eratosthenes is exactly constituted by an Arbitrarily Large Number of (self-)similar infinite-length sub-algorithms that are specified by every even Prime gaps 2, 4, 6, 8, 10.... Again as a certain type of infinite-length algorithm, Complement-Sieve-of-Eratosthenes is exactly constituted by two inversely related infinite-length sub-algorithms that are specified by odd Gap 1-Composites and even Gap 2-Composites. Modified Hasse principle is satisfied by these two algorithms and their associated sub-algorithms that perpetually generate all Odd Primes and Gap 2-Even Composites [that are both Arbitrarily Large in Numbers and overall equal to each other], and Gap 1-Even Composites and Gap 1-Odd Composites [that are both Infinitely Many in Numbers and equal to each other]. Thus we prove Set even Prime gaps with uniquely associated Subsets Odd Primes all have cardinality Arbitrarily Large in Number confirming Modified Polignac's and Twin prime conjectures to be true [and with our proofs being fully consistent with Prime number theorem].

The p_1 commencing values are defined as being constituted from the entire set of prime numbers 2, 3, 5, 7, 11, 13.... Incorporating all the integers between 0 and p-1 inclusive, the p_1 residue classes of 0 mod p_1, 1 mod p_1, 2 mod p_1,..., p_1-2 mod p_1, p_1-1 mod p_1 from modular arithmetic are used to delineate Admissible Prime k-tuplets from Inadmissible Prime $(k+1)$-tuples [in Proposition 1, subsection 6.1] whereby all these created tuplets and tuples are regarded as the **"overlapping and incomplete" (Sub)Tuples Classification of consecutive**

primes that cannot be used to either prove or disprove Modified Polignac's and Twin prime conjectures. On the contrary, the **"non-overlapping and complete" (Sub)Sets Classification of grouped primes** is used by us to prove Modified Polignac's and Twin prime conjectures.

The p-adic number system for any prime number p extends the ordinary arithmetic of rational numbers in a different way from the extension of rational number system to real and complex number systems. The p-adic expansion of rational numbers also incorporate all the integers between 0 and p-1 inclusive. We apply p-adic absolute values to Prevalences of Nontrivial zeros, Primes and Composites in subsection 3.1. *We insightfully deduce the computed (infinite-length) patterns of p-adic absolute values are specific for* Odd Primes [as generated by (sub-)algorithms of Sieve-of-Eratosthenes from all even Prime gaps 2, 4, 6, 8, 10... in total, and from each and every even Prime gap in sub-totals]; Composites [as generated by (sub-)algorithms of Complement-Sieve-of-Eratosthenes as Composites in total, and even Gap 2-Composites and odd Gap 1-Composites in sub-totals]; and Nontrivial zeros [as generated by equation Dirichlet eta function only at (unique) $\sigma = \frac{1}{2}$ critical line whereby this $\sigma = \frac{1}{2}$ value is mutually exclusive and independent of all other $\sigma \neq \frac{1}{2}$ values associated with infinitely-many (non-unique) $\sigma \neq \frac{1}{2}$ non-critical lines in the $0 < \sigma < 1$ critical strip].

1.1 General notations, (Sub)Sets versus (Sub)Tuples Classification

Common abbreviations used in this paper: CP = Completely Predictable, IP = Incompletely Predictable, FL = finite-length, IL = infinite-length, CFS = countably finite set, CIS = countably infinite set, IM = infinitely many, ALN = Arbitrarily Large Number.

List of abbreviations incorporating relevant definitions [that also include $s = \sigma \pm it$]:
·**CP entities**: Completely Predictable entities which will manifest CP *independent* properties.
·**IP entities**: Incompletely Predictable entities which will manifest IP *dependent* properties.
·ζ**(s)**: $f(n)$ Riemann zeta function containing variable n, and parameters t and σ will generate [via its *proxy* Dirichlet eta function] Zeroes when $\sigma = \frac{1}{2}$ and virtual Zeroes when $\sigma \neq \frac{1}{2}$.
·η**(s)**: $f(n)$ Dirichlet eta function, as the analytic continuation of ζ(s), containing variable n, and parameters t and σ will generate Zeroes when $\sigma = \frac{1}{2}$ and virtual Zeroes when $\sigma \neq \frac{1}{2}$.
·**sim-**η**(s)**: $f(n)$ simplified Dirichlet eta function, derived by applying Euler formula to η(s), containing variable n, and parameters t and σ will generate Zeroes when $\sigma = \frac{1}{2}$ and virtual Zeroes when $\sigma \neq \frac{1}{2}$.
·**DSPL**: $F(n)$ Dirichlet Sigma-Power Law = $\int$ sim-$\eta(s)dn$ containing variable n, and parameters t and σ will generate Pseudo-zeroes when $\sigma = \frac{1}{2}$ and virtual Pseudo-zeroes when $\sigma \neq \frac{1}{2}$ whereby the (virtual) Zeros = (virtual) Pseudo-zeros – $\frac{\pi}{2}$ relationship allows (virtual) Pseudo-zeros to (virtual) Zeros conversion and *vice versa*.
·**NTZ**: Nontrivial zeros located on one-dimensional (mathematical) $\sigma = \frac{1}{2}$-critical line are precisely equivalent to **G[x=0,y=0]P**: Gram[x=0,y=0] points as Origin intercept points which are located at zero-dimensional (geometrical) $\sigma = \frac{1}{2}$-Origin point [as per Figure 6]. These entities, mathematically defined by $\sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + Im\{\eta(s)\} = 0$, are generated by equation G[x=0,y=0]P-η(s) containing exponent $\frac{1}{2}$ when $\sigma = \frac{1}{2}$.
·**GP or G[y=0]P**: 'usual' or 'traditional' Gram points = Gram[y=0] points = x-axis intercept points that are [multiple-positioned] located on one-dimensional x-axis line are generated by equation G[y=0]P-η(s) when $\sigma = \frac{1}{2}$. These entities are mathematically defined by $\sum$ReIm$\{\eta(s)\}$ = Re$\{\eta(s)\}$ + 0, or simply Im $\{\eta(s)\} = 0$. Riemann hypothesis is usefully stated

as none of the [additional] virtual G[x=0]P generated by equation G[x=0]P-η(s) when $\sigma \neq \frac{1}{2}$ – as demonstrated by Figure 11 for $\sigma = \frac{1}{3}$ – can be constituted by t transcendental number values that [incorrectly] coincide with t transcendental number values for NTZ when $\sigma = \frac{1}{2}$.
·**G[x=0]P**: Gram[x=0] points = y-axis intercept points that are [multiple-positioned] located on one-dimentional y-axis line are generated by equation G[x=0]P-η(s) when $\sigma = \frac{1}{2}$. These entities are mathematically defined by $\sum \text{ReIm}\{\eta(s)\} = 0 + \text{Im}\{\eta(s)\}$, or simply $\text{Re}\{\eta(s)\} = 0$.
·**virtual NTZ**: virtual nontrivial zeros or **virtual G[x=0,y=0]P**: virtual Gram[x=0,y=0] points. These are virtual Origin intercept points located at the multiple-positioned virtual Origin points which are generated by equation G[x=0,y=0]P-η(s) containing exponent values $\neq \frac{1}{2}$ when $\sigma \neq \frac{1}{2}$. We note that each virtual NTZ when $\sigma < \frac{1}{2}$ in Figure 7 equates to an [additional] negative virtual G[y=0]P located at IP varying positions on horizontal axis, and each virtual NTZ when $\sigma > \frac{1}{2}$ in Figure 8 equates to an [additional] positive virtual G[y=0]P located at IP varying positions on horizontal axis. We observe overall less virtual G[x=0]P when $\sigma > \frac{1}{2}$, and overall more virtual G[x=0]P when $\sigma < \frac{1}{2}$.

·**Sieve-of-Eratosthenes**: As symbolically denoted by $p_{n+1} = 2 + \sum_{i=1}^{n} g_i$ with $g_n = p_{n+1} - p_n$, its derived (sub-)algorithms will faithfully generate all prime numbers.

·**Complement-Sieve-of-Eratosthenes**: As symbolically denoted by $c_{n+1} = 4 + \sum_{i=1}^{n} c_i$ with $g_n = c_{n+1} - c_n$, its derived (sub-)algorithms will faithfully generate all composite numbers.

Remark 1. *Important deductions on* ***(Sub)Sets Classification of grouped Primes*** *(*$\mathbb{P}$*),* ***Composites*** *(*$\mathbb{C}$*)* ***and Integers*** *(*$\mathbb{Z}$*) versus* ***(Sub)Tuples Classification of consecutive Primes****:*

$\mathbb{Z}$ {0, 1, 2, 3, 4, 5...} = Non-$\mathbb{P}$ Non-$\mathbb{C}$ {0, 1} + $\mathbb{P}$ {2, 3, 5, 7, 11, 13, 17, 19...} + $\mathbb{C}$ {4, 6, 8, 9, 10, 12, 14, 15...}. Available gaps between any two consecutive numbers to **non-overlappingly classify all** $\mathbb{Z}$**,** $\mathbb{P}$ **and** $\mathbb{C}$ **as mutually exclusive sets or subsets**: CIS-IM-linear odd Gap 1-$\mathbb{Z}$ (or simply Gap 1-$\mathbb{Z}$); CFS odd Gap 1-$\mathbb{P}$ (or simply Gap 1-$\mathbb{P}$); CIS-ALN-decelerating even Gap 2i-$\mathbb{P}$ (or simply Gap 2i-$\mathbb{P}$) with i = 1, 2, 3, 4, 5...; CIS-IM-accelerating odd Gap 1-$\mathbb{C}$ (or simply Gap 1-$\mathbb{C}$); CIS-ALN-decelerating even Gap 2-$\mathbb{C}$ (or simply Gap 2-$\mathbb{C}$).
· We refer to the CIS-ALN-decelerating p_1 commencing values obtained from all $\mathbb{P}$ 2, 3, 5, 7, 11, 13.... For each p_1 commencing value [with some caveats in Proposition 1, subsection 6.1]; we can **overlappingly classify consecutive primes as the [non-mutually exclusive] Admissible Prime** k**-tuplets and Inadmissible Prime** $(k+1)$**-tuples** of increasing lengths.
· (Finite-Length) Admissible Prime k-tuplets & tuples, and (Finite-Length) Inadmissible Prime k-tuples represent finite ordered list of k consecutive primes. Only when $k = 2$, the two consecutive primes in all Prime 2-tuplets [viz, representing all Gap 2-Odd $\mathbb{P}$] and in all Prime 2-tuples [viz, representing all Gap 4-Odd $\mathbb{P}$, Gap 6-Odd $\mathbb{P}$, Gap 8-Odd $\mathbb{P}$, Gap 10-Odd $\mathbb{P}$,...] are always uniquely **non-overlapping** and admissible. Otherwise when $k > 2$, some of the k consecutive primes as subtuples from Prime k-tuplets or Prime k-tuples could **overlappingly** recur in different Prime k-tuplets [that are always admissible] or in different Prime k-tuples [that can be either admissible or inadmissible].

(I) All $\mathbb{P}$ = 2, 3, 5, 7, 11,... as CIS-ALN-decelerating; Even $\mathbb{P}$ or Gap 1-Even $\mathbb{P}$ = 2 as CFS
· (i) Odd $\mathbb{P}$ or Gap 2i-Odd $\mathbb{P}$ = 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97,... as CIS-ALN-decelerating

· (ii) please refer to List of twin primes, cousin primes, sexy primes, etc in section 2 as derived from individual even Prime gaps 2, 4, 6, 8, 10... in an *ad infinitum* manner.
· For i = 1, 2, 3, 4, 5...; All Odd $\mathbb{P}$ or Gap 2i-Odd $\mathbb{P}$ [as Set] = Gap 2-Odd $\mathbb{P}$ + Gap 4-Odd $\mathbb{P}$ + Gap 6-Odd $\mathbb{P}$ + Gap 8-Odd $\mathbb{P}$ + Gap 10-Odd $\mathbb{P}$... [as Subsets]. This important *Set* = $\sum$ *(ALN of Subsets)* **non-overlapping** mutually exclusive relationship with the Set and its derived Subsets of prime numbers all having cardinality CIS-ALN-decelerating **confirms Modified Polignac's and Twin prime conjectures to be true**.

(II) All $\mathbb{C}$ = 4, 6, 8, 9, 10, 12, 14, 15, 16... as CIS-IM-accelerating
· (i) Even $\mathbb{C}$ = 4, 6, 8, 10, 12, 14, 16... as CIS-IM-accelerating
· (ii) Odd $\mathbb{C}$ = 9, 15, 21, 25, 27, 33, 35... as CIS-IM-accelerating
· (iii) Gap 1-Even $\mathbb{C}$ = 8, 14, 20, 24, 26, 32, 34, 38, 44... as CIS-IM-accelerating
· (iv) Gap 1-Odd $\mathbb{C}$ = 9, 15, 21, 25, 27, 33, 35, 39, 45... as CIS-IM-accelerating
· (v) Gap 2-Even $\mathbb{C}$ = 4, 6, 10, 12, 16, 18, 22, 28, 30, 36... as CIS-IM-decelerating
· Important set and subsets **non-overlapping** mutually exclusive relationships for composite numbers: (1) All $\mathbb{C}$ = Even $\mathbb{C}$ + Odd $\mathbb{C}$ = Gap 1-Even $\mathbb{C}$ + Gap 1-Odd $\mathbb{C}$ + Gap 2-Even $\mathbb{C}$. (2) Even $\mathbb{C}$ = Gap 1-Even $\mathbb{C}$ + Gap 2-Even $\mathbb{C}$. (3) Odd $\mathbb{C}$ = Gap 1-Odd $\mathbb{C}$. (4) Gap 1-Even $\mathbb{C}$ = Gap 1-Odd $\mathbb{C}$. (5) Even $\mathbb{C}$ > Odd $\mathbb{C}$, Gap 1-Even $\mathbb{C}$ + Gap 1-Odd $\mathbb{C}$ or Gap 1-Even $\mathbb{C}$ or Gap 1-Odd $\mathbb{C}$ > Gap 2-Even $\mathbb{C}$. (6) $\mathbb{P}$-$\mathbb{C}$ **dependent connection**: Gap 2-Even $\mathbb{C}$ = All Odd $\mathbb{P}$.

1.2 Summary of Mathematics for Incompletely Predictable Problems

On the overall objective to rigorously derive the Algorithm-type proofs for Modified Polignac's and Twin prime conjectures [see Definition 1 in subsection 7.1] and the Equation-type proof for Riemann hypothesis, we apply infinitesimal numbers at two places using the following colloquially-stated propositions with their formal proofs[27] provided in section 7:

Proposition 3: *In the limit of never reaching a [nonexisting] zero* conceptually seen as Prevalences of both even Prime gaps and the associated [positive and negative] Odd Primes never becoming zero whereby arbitrarily large number of different even Prime gaps that uniquely accompany all Odd Primes in totality will never stop recurring. Foundation Figure 1 allows Geometrical-Mathematical interpretation for positive Odd Primes.

Proposition 4: *In the limit of reaching an [existing] zero* conceptually seen as [entire $-\infty < t < +\infty$] trajectory of Dirichlet eta function, proxy for Riemann zeta function, touching (symbolic) zero-dimensional $\sigma = \frac{1}{2}$-Origin point only when parameter $\sigma = \frac{1}{2}$ whereby all nontrivial zeros [mathematically] located on (symbolic) one-dimensional $\sigma = \frac{1}{2}$-critical line will [geometrically] declare themselves in totality as corresponding Origin intercept points. Foundation Figure 12 allows Geometrical-Mathematical interpretation for $0 < t < +\infty$ range.

Remark 2. ***Important relationships & deductions from Gram Points, Primes, Composites, Prime-Composite identifier grouping and Prime-Composite quotient:***

· At $\sigma = \frac{1}{2}$-critical line; Gram Points = Solitary CP 1^{st} G[y=0]P as Rational number {0} + All IP CIS-IM-linear G[y=0]P, G[x=0]P and G[x=0,y=0]P as Irrational numbers. Overall, IP CIS-IM-linear G[y=0]P = IP CIS-IM-linear G[x=0]P = IP CIS-IM-linear G[x=0,y=0]P. In $0 < \sigma < 1$-critical strip (with logical reasoning given in Remark 5), (i) unique solitary (Co-linear) Equation G[x=0,y=0]P-η(s) at $\sigma = \frac{1}{2}$-critical line [with optimal "formula symmetry"] is independent of (ii) non-unique infinitely many (Co-linear) Equations G[x=0,y=0]P-η(s) at $\sigma \neq \frac{1}{2}$-noncritical lines [without optimal "formula symmetry"]. Both (i) and (ii) generate

mutually exclusive and parallel co-linear lines (co-lines) that never cross over one another. ***The complete absence of G[x=0,y=0]P at all CIS-IM-linear*** $\sigma \neq \frac{1}{2}$***-noncritical lines equates to Riemann hypothesis being true.*** As other phenomena happening on $\sigma = \frac{1}{2}$-critical line, the perpetually applicable Gram's Law and Rosser's Rule on Nontrivial zeros-Gram points relationships, and their [expected] intermittently occurring violations on an eternal basis, do not contradict our derived proof for Riemann hypothesis to be true.

· We arbitrarily classify CIS-ALN-decelerating even Prime gaps 2, 4, 6, 8, 10... as small Prime gaps 2 & 4, and large Prime gaps ≥ 6. (i) Small or smaller Prime gaps tend to appear amongst the smaller range of integers. (ii) Large or larger Prime gaps tend to appear amongst the larger range of integers. Both the former (i) and the later (ii) should overall appear in a perpetual manner amongst the entire range of integers. First appearance of an even Prime gap do not always occur in an orderly manner e.g. 31^{st} $\mathbb{P}$ 127 [with even Prime gap 14] first appear before 35^{th} $\mathbb{P}$ 149 [with even Prime gap 10] and 43^{rd} $\mathbb{P}$ 191 [with even Prime gap 12]. However once a particular even Prime gap does first appear, it must always perpetually reappear albeit with decreasing frequency amongst the ever increasing range of integers.

· (i) IP CIS-IM-accelerating Gap 1-Even $\mathbb{C}$ = IP CIS-IM-accelerating Gap 1-Odd $\mathbb{C}$. (ii) IP CIS-ALN-decelerating Gap 2-Even $\mathbb{C}$ = IP CIS-ALN-decelerating Gap 2i-Odd $\mathbb{P}$. From the inversely related (i) [with its two unique subsets of Gap 1-Even $\mathbb{C}$ and Gap 1-Odd $\mathbb{C}$ that **fully comply with Composite number theorem**] and (ii) [with its one unique subset of Gap 2-Even $\mathbb{C}$ and ALN unique subsets from Gap 2i-Odd $\mathbb{P}$ that **fully comply with Prime number theorem**], the **Prime-Composite quotient** is stated below in two equivalent formats:

(1) $$\lim_{x\to\infty} \frac{\text{CIS-ALN-decelerating Gap } 2i\text{-Odd } \mathbb{P} + \text{CIS-ALN-decelerating Gap 2-Even } \mathbb{C}}{\text{CIS-IM-accelerating Gap 1-Even } \mathbb{C} + \text{CIS-IM-accelerating Gap 1-Odd } \mathbb{C}} = 0$$

(2) $$\text{CIS-ALN-decelerating Gap } 2i\text{-Odd } \mathbb{P} + \text{CIS-ALN-decelerating Gap 2-Even } \mathbb{C} \sim \frac{1}{\text{CIS-IM-accelerating Gap 1-Even } \mathbb{C} + \text{CIS-IM-accelerating Gap 1-Odd } \mathbb{C}}$$

· In between any two given Odd $\mathbb{P}$, even Prime gap [= 2 + Σ(Number of all Gap 1-Even $\mathbb{C}$ + Number of all Gap 1-Odd $\mathbb{C}$)]. Hence Gap 1-Even $\mathbb{C}$ and Gap 1-Odd $\mathbb{C}$ do not exist for Gap 2-Odd $\mathbb{P}$ (twin primes). For Gap 2i-Odd $\mathbb{P}$ with CIS-ALN-decelerating i = 1, 2, 3, 4, 5...; the initial five computed **Prime-Composite identifier groupings** [see section 5] are listed:–
When i = 1, CIS-ALN-decelerating Gap 2-Odd $\mathbb{P}$ is given by Gap 2-Even $\mathbb{C}$, Gap 2-Odd $\mathbb{P}$.
When i = 2, CIS-ALN-decelerating Gap 4-Odd $\mathbb{P}$ is given by Gap 2-Even $\mathbb{C}$, Gap 4-Odd $\mathbb{P}$, Gap 1-Even $\mathbb{C}$, Gap 1-Odd $\mathbb{C}$.
When i = 3, CIS-ALN-decelerating Gap 6-Odd $\mathbb{P}$ is given by Gap 2-Even $\mathbb{C}$, Gap 6-Odd $\mathbb{P}$, Gap 1-Even $\mathbb{C}$, Gap 1-Odd $\mathbb{C}$, Gap 1-Even $\mathbb{C}$, Gap 1-Odd $\mathbb{C}$.
When i = 4, CIS-ALN-decelerating Gap 8-Odd $\mathbb{P}$ is given by Gap 2-Even $\mathbb{C}$, Gap 8-Odd $\mathbb{P}$, Gap 1-Even $\mathbb{C}$, Gap 1-Odd $\mathbb{C}$, Gap 1-Even $\mathbb{C}$, Gap 1-Odd $\mathbb{C}$, Gap 1-Even $\mathbb{C}$, Gap 1-Odd $\mathbb{C}$.
When i = 5, CIS-ALN-decelerating Gap 10-Odd $\mathbb{P}$ is given by Gap 2-Even $\mathbb{C}$, Gap 10-Odd $\mathbb{P}$, Gap 1-Even $\mathbb{C}$, Gap 1-Odd $\mathbb{C}$, Gap 1-Even $\mathbb{C}$, Gap 1-Odd $\mathbb{C}$, Gap 1-Even $\mathbb{C}$, Gap 1-Odd $\mathbb{C}$, Gap 1-Even $\mathbb{C}$, Gap 1-Odd $\mathbb{C}$. $\cdots$ for all other remaining i.
· If we also include Gap 2-Even $\mathbb{C}$ between two consecutive Odd $\mathbb{P}$, then even Prime gap [= 1 + Σ(Number of all Gap 1-Even $\mathbb{C}$ + Number of all Gap 1-Odd $\mathbb{C}$ + solitary Gap 2-Even $\mathbb{C}$)]. The generalized sequence $\{n!+2, n1+3, n1+4, ..., n!+n\}$ using factorial function give rise to precisely $n-1$ consecutive $\mathbb{C}$ since 1^{st} term $n!+2$ is divisible by 2, 2^{nd} term $n!+3$ is divisible by 3,..., $(n-1)^{th}$ term $n!+n$ is divisible by n. For any n = 2, 4, 6, 8, 10, 12..., there is (even Prime

gap – 1) = $n-1$ = 1, 3, 5, 7, 11, 13... "with length of at least $n-1$" thus reflecting the permitted total number of $n-1$ consecutive $\mathbb{C}$ in between two consecutive Odd $\mathbb{P}$. This implies arbitrarily large even Prime gaps are possible whereby any particular even Prime gap chosen from 2, 4, 6, 8, 10... can be derived from the same grouping with or without including Gap 2-Even $\mathbb{C}$ as respectively denoted by (even Prime gap – 1) or (even Prime gap – 2). **Two important $\mathbb{P}$-$\mathbb{C}$ constraints:** (i) Prime gaps of $n-1$ numbers can occur at numbers much smaller than $n!$ e.g. first prime gap of size larger than 14 occurs between Odd $\mathbb{P}$ 523 and Odd $\mathbb{P}$ 541, while 15! is the vastly larger number 1307674368000. (ii) Although computed n consecutive $\mathbb{C}$ = 2, 4, 6, 8, 10... [≡ (forbidden) odd Prime gaps 1, 3, 5, 7, 9...] based on our generalized sequence using factorial function are mathematically possible; all these specific n consecutive $\mathbb{C}$ will simply *not* correctly represent the (permitted) total number of $n-1$ consecutive $\mathbb{C}$ in between any two consecutive Odd $\mathbb{P}$. **All even Prime gaps 2, 4, 6, 8, 10... with associated unique Odd $\mathbb{P}$ that perpetually reappear [albeit with decreasing frequency] along the number line equates to Modified Polignac's and Twin prime conjectures being true.**

· ***Probability (Gap 2*i*-Odd $\mathbb{P}$ from any i = 1, 2, 3, 4, 5... value that abruptly terminates) = 0 equates to Modified Polignac's and Twin prime conjectures being true.*** In particular, this statistical statement is fully validated by application of Prime number theorem for Arithmetic progressions in Axiom 1, section 6 that confirms the Set and derived Subsets of Gap 2i-Odd $\mathbb{P}$ are CIS-ALN-decelerating [and support the generalized and ordinary Riemann hypothesis].

2 Infinite-length or Finite-length equations, sub-equations, algorithms and sub-algorithms

We adopt the abbreviations from subsection 1.1. Conceptually, IL (sub-)algorithms or IL (sub-)equations and FL (sub-)algorithms or FL (sub-)equations will respectively generate infinitely-many and finitely-many entities. All the FL (sub-)algorithms or FL (sub-)equations are CP but the IL (sub-)algorithms or IL (sub-)equations can be either CP or IP. Here, we can validly regard equation Dirichlet eta function (*proxy* for Riemann zeta function), and algorithms Sieve-of-Eratosthenes [for prime numbers] and Complement-Sieve-of-Eratosthenes [for composite numbers] as "IP IL number generators". Not least to maintain Dimensional analysis homogeneity and to conserve Total number of elements (cardinality) [as supported by the outlined *Proof by Contradiction* in Remark 4], it is a ***sine qua non*** *Pre-requisite Mathematical Condition that a parent IP IL algorithm which is precisely constituted by its IP IL sub-algorithms or a parent IP IL equation which is precisely constituted by its IP IL sub-equations must generally all be wholly IP IL [and not be mixed IP IL and CP FL].*

Prime counting function Prime-$\pi(x)$ = number of prime numbers $\leq$ x. As literally an infinite-scale stepped-mathematical function contributing to tuples and subtuples from Admissible Prime k-tuplets/tuples and Inadmissible Prime k-tuples, there are three possible trajectories from Prime-$\pi(x)$ whereby we also use even Prime gaps 6n as common randomly chosen examples – viz. for n = 1, 2, 3..., even Prime gaps = 6, 12, 18... [multiples of 6].

(a) **Accelerating primes:** Prime gap_{i+2} – Prime gap_{i+1} > Prime gap_{i+1} – Prime gap_i occurring an arbitrarily large number of times e.g. Admissible Prime 3-tuplet (p, p+2, p+6) with smallest possible diameter = 6, Admissible Prime 3-tuple (p+6, p+10, p+16) ≡ (p, p+4, p+10) with [not the smallest possible] diameter = 10 that is derived from Admissible Prime 18-tuplet (p, p+4, p+6, p+10, p+16, p+18, p+24, p+28, p+30, p+34, p+40, p+46, p+48,

p+54, p+58, p+60, p+66, p+70) with smallest possible diameter = 70, and Admissible Prime 3-tuple (p, p+6, p+18) from ([p-24], [p-22], [p-10], p, p+6, p+18, [p+42], [p+50]) with [not the smallest possible] diameter = 18 occurring at consecutive primes (22391, 22397, 22409) with position of first p = 2506.

(b) **Decelerating primes:** Prime gap_{i+2} – Prime gap_{i+1} < Prime gap_{i+1} – Prime gap_i occurring an arbitrarily large number of times e.g. Admissible Prime 3-tuplet (p, p+4, p+6) with smallest possible diameter = 6, Admissible Prime 3-tuple (p+20, p+26, p+30) ≡ (p, p+6, p+10) with [not the smallest possible] diameter = 10 that is derived from Admissible Prime 9-tuplet (p, p+2, p+6, p+8, p+12, p+18, p+20, p+26, p+30) with smallest possible diameter = 30, and Admissible Prime 3-tuple (p, p+18, p+30) from ([p-26], [p-22], [p-12], p, p+18, p+30, [p+50], [p+54]) with [not the smallest possible] diameter = 30 occurring at consecutive primes (10193, 10211, 10223) with position of first p = 1252.

(c) **Steady primes:** Prime gap_{i+2} – Prime gap_{i+1} = Prime gap_{i+1} – Prime gap_i that should occur an arbitrarily large number of times [albeit on extremely rare occasions] and can only involve prime gaps 6n. For instance, the Admissible Prime 3-tuple (p, p+6, p+12) from ([p-2], p, p+6, p+12, [p+18], [p+28], [p+36]) with [not the smallest possible] diameter = 12 occurring at consecutive primes (63691, 63697, 63703) with position of first p = 6386; and Admissible Prime 3-tuple (p, p+18, p+36) from ([p-2], p, p+18, p+36, [p+54], [p+60]) with [not the smallest possible] diameter = 36 occurring at consecutive primes (76543, 76561, 76579) with position of first p = 7531. An exception is the solitary Inadmissible Prime 3-tuple (p, p+2, p+4) with smallest diameter = 4 occurring at consecutive primes (3, 5, 7) ≡ cummulative prime gaps (0, 2, 4). We can explain using either (3, 5, 7) tuple or (0, 2, 4) tuple why this particular Prime 3-tuple is inadmissible, and we choose the former tuple. k = 3, prime q ≤ k $\implies$ prime q = 2 and 3 which are required for modular q. For modular 2: 3 ≡ 1 (mod 2), 5 ≡ 1 (mod 2), 7 ≡ 1 (mod 2) $\implies$ these three primes did not take on all two residue values 0 and 1 [considered as success]. However, for modular 3: 3 ≡ 0 (mod 3), 5 ≡ 2 (mod 3), 7 ≡ 1 (mod 3) $\implies$ these three primes did take on all three residue values 0, 1 and 2 [considered as failure]. By definition, this failure occurrence $\implies$ the three primes are inadmissible since they would always include a multiple of 3 and therefore could not all be prime unless one of the numbers is 3 itself with finite one prime placement.

For i = 1, 2, 3, 4, 5,..., n; relevant algorithm and sub-algorithms from Sieve of Eratosthenes computed for the following mutually exclusive but dependent prime numbers all as rational numbers endowed with the solitary odd Prime gap 1 for even prime number 2, and the initial even Prime gaps 2, 4 and 6 for odd Twin primes, odd Cousin primes and odd Sexy primes:

(a) For IP IL algorithm [Gap 2, 4, 6, 8, 10...]-Sieve of Eratosthenes $p_{n+1} = 3 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd $\mathbb{P}$ {3, 5, 7, 11, 13, 17, 19...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd $\mathbb{P}$ is denoted by g_n = $(n + 1)^{st}$ Odd $\mathbb{P}$ – $(n)^{th}$ Odd $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 2, 2, 4, 2, 4, 2....

(b) For CP FL sub-algorithm [Gap 1]-Sieve of Eratosthenes $p_{n+1} = 2 + \sum_{i=1}^{n} g_i$ [where n = 1 and not ALN] that faithfully generates the first and only Even $\mathbb{P}$ {2} with cardinality CFS of 1, the solitary n^{th} odd prime gap between two successive primes is denoted by $g_n = (n + 1)^{st}$ Odd $\mathbb{P}$ – $(n)^{th}$ Even $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n = 3 - 2 = 1$.

(c) For IP IL sub-algorithm [Gap 2]-Sieve of Eratosthenes $p_{n+1} = 3 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd twin $\mathbb{P}$ {3, 5, 11, 17, 29, 41, 59...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd twin $\mathbb{P}$ is denoted by g_n = $(n + 1)^{st}$ Odd twin $\mathbb{P}$ – $(n)^{th}$ Odd twin $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 2, 6, 6, 12, 12, 18....

(d) For IP IL sub-algorithm [Gap 4]-Sieve of Eratosthenes $p_{n+1} = 7 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd cousin $\mathbb{P}$ {7, 13, 19, 37, 43, 67...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd cousin $\mathbb{P}$ is denoted by g_n = $(n + 1)^{st}$ Odd cousin $\mathbb{P}$ – $(n)^{th}$ Odd cousin $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 6, 6, 8, 6, 24....

(e) For IP IL sub-algorithm [Gap 6]-Sieve of Eratosthenes $p_{n+1} = 23 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd sexy $\mathbb{P}$ {23, 31, 47, 53, 61, 73, 83...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd sexy $\mathbb{P}$ is denoted by g_n = $(n + 1)^{st}$ Odd sexy $\mathbb{P}$ – $(n)^{th}$ Odd sexy $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 8, 16, 6, 8, 12, 10....

With n = ALN or, traditionally, ∞; rigorous algorithm-type proof for Modified Polignac's and Twin prime conjectures can be stated here as two statements. Statement 1: All known prime numbers = IP IL algorithm (a) + CP FL sub-algorithm (b). Statement 2: IP IL algorithm (a) = IP IL sub-algorithm (c) + IP IL sub-algorithm (d) + IP IL sub-algorithm (e) +... [that involves all even Prime gaps 2, 4, 6, 8, 10...] whereby all the (sub-)algorithms in Statement 2 can be mathematically used to create *self-similar fractal objects based on their corresponding Prevalences* [that are geometrically never identical to each other as outlined in section 3].

There are three types of Gram points when $\sigma = \frac{1}{2}$ and two types of virtual Gram points when $\sigma \neq \frac{1}{2}$. With nontrivial zeros being a type of Gram points, there is zero probability that any of the countably infinitely many nontrivial zeros (Gram[x=0,y=0] Points or Origin intercept Points) can be located away from [geometrical] Origin point, which correspond to [mathematical] critical line. This statement is precisely equivalent to Riemann hypothesis.

As *proxy* function for Riemann zeta function in $0 < \sigma < 1$ critical strip, Dirichlet eta function at the (unique) $\sigma = \frac{1}{2}$-critical line generates all x-axis intercept points as *usual* Gram points or Gram[y=0] points, all y-axis intercept points as Gram[x=0] points, and all Origin intercept points as Gram[x=0,y=0] points or nontrivial zeros. When treated as equation and sub-equation, Dirichlet eta function can conceptually be used to calculate all these mutually exclusive but dependent entities endowed with t-valued irrational (transcendental) numbers except for initial Gram[y=0] point endowed with a t-valued rational number:

(a) Considered for $t = 0$ to $+\infty$, Dirichlet eta function as IP IL equation will faithfully generate all above-mentioned three types of Gram points that are endowed with t-valued irrational (transcendental) numbers except for first Gram[y=0] point.

(b) Considered only for $t = 0$, Dirichlet eta function as CP FL sub-equation will faithfully generate the first and only Gram[y=0] point that is endowed with t-valued rational number 0.

2.1 The extended and generalized Riemann hypothesis, Hasse principle for equations and Modified Hasse principle for algorithms

Being an integral part of L-functions and modular forms database (LMFDB), an L-function is a Dirichlet series with an Euler product and a functional equation e.g. Riemann zeta function,

Dirichlet L-functions, L-functions of elliptic curves. Convolution involves manipulations of the Satake parametes of the L-function at good places e.g. symmetric powers, exterior powers, Rankin-Selberg convolution. The top half of the diagram in '*The LMFDB universe*'[13] is based on Langlands program, which predicts that any motivic object corresponds to an automorphic object via their L-functions. Usually convergent on a half-plane, an L-series is a Dirichlet series that may give rise to an L-function via analytic continuation. Riemann zeta function $\zeta(s)$ is a meromorphic function on complex plane associated with one of several categories of mathematical objects. Via analytic continuation, it gives rise to Dirichlet eta function $\eta(s)$ [a special case of polylogarithm function]. We compare and contrast [*as our personal views using current knowledge*] the extended and generalized Riemann hypothesis [which are generally considered to be true] with the original Riemann hypothesis.

Suppose K is a number field (a finite-dimensional field extension of the rationals Q) with ring of integers O_κ (this ring is the integral closure of the integers Z in K). If a is an ideal of O_κ, other than the zero ideal, we denote its norm by Na. The Dedekind zeta-function of κ is then defined by $\zeta_K(s) = \sum_a \frac{1}{(Na)^s}$ for every complex number s with real part > 1. The sum extends over all non-zero ideals a of O_κ. Dedekind zeta-function satisfies a functional equation and can be extended by analytic continuation to the whole complex plane. The resulting function encodes important information about the number field κ. The extended Riemann hypothesis asserts that for every number field K and every complex number s with $\zeta_K(s) = 0$: if the real part of s is between 0 and 1, then it is in fact $\frac{1}{2}$. The ordinary Riemann hypothesis follows from the extended one if we take the number field to be Q, with ring of integers Z.

Hurwitz zeta function is one of the many zeta functions formally defined for complex variables s with $Re(s) > 1$ and $a \neq 0, -1, -2, -3, ...$ by $\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$. This series is absolutely convergent for given values of s and a, and can be extended to a meromorphic function defined for all $s \neq 1$. With $a = 1$, Riemann zeta function is then $\zeta(s, 1)$. With using rational arguments, Hurwitz zeta function may be expressed as a linear combination of Dirichlet L-functions and *vice versa*. For Dirichlet L-functions which is defined as $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$; the generalized Riemann hypothesis asserts that, for every Dirichlet character χ and every complex number s with $L(\chi, s) = 0$, if s is not a negative real number, then the real part of s is $\frac{1}{2}$. The case $\chi(n) = 1$ for all n yields the ordinary Riemann hypothesis.

Where p is a prime number, analytic or arithmetic p-adic zeta function [or the more general p-adic L-function] is a function analogous to Riemann zeta function [or the more general L-functions], but whose domain (e.g. the p-adic integers Z_p, a profinite p-group, or a p-adic family of Galois representations) and target (e.g. the p-adic numbers Q_p or its algebraic closure) are p-adic. Via p-adic interpolation of special values of L-functions, a [analytic] p-adic L-function, also known as p-adic Riemann zeta function $\zeta_p(s)$, is constructed by Kubota-Leopoldt using Kummer's congruences for Bernoulli numbers whereby values at negative odd integers are those of Riemann zeta function at negative odd integers (up to an explicit correction factor). The main conjecture of Iwasawa theory (now a theorem due to Barry Mazur and Andrew Wiles) is the statement that the [analytic] Kubota-Leopoldt p-adic L-function $L_p(s, \chi)$ and an [arithmetic] p-adic L-function [as analogue constructed by Iwasawa theory which can be potentially sourced from the arithmetic of cyclotomic fields,

or more generally, certain Galois modules over towers of cyclotomic fields or even more general towers] are essentially the same. This type of conjecture represent formal statements concerning the philosophy that special values of L-functions contain arithmetic information.

The [analytic] Kubota-Leopoldt p-adic L-function $L_p(s,\chi)$ interpolates the Dirichlet L-function with Euler factor at p removed. More precisely, $L_p(s,\chi)$ is the unique continuous function of p-adic number s such that $L_p(1-n,\chi) = (1-\chi(p)p^{n-1})L(1-n,\chi)$ for positive integers n divisible by $p-1$. The right hand side is just the usual Dirichlet L-function, except that Euler factor at p is removed, otherwise it would not be p-adically continuous. The continuity of right hand side is closely related to Kummer congruences. When n is not divisible by $p-1$, this does not usually hold; instead $L_p(1-n,\chi) = (1-\chi\omega^{-n}(p)p^{n-1})L(1-n,\chi\omega^{-n})$ for positive integers n. Here χ is twisted by a power of the Teichmuller character ω.

The p-adic L-functions can also be perceived as the p-adic measures (or the p-adic distributions) on p-profinite Galois groups. The translation between this point of view and the original point of view of Kubota-Leopoldt (as Q_p-valued functions on Z_p) is via the Mazur-Mellin transform (and class field theory). One can also construct analytic p-adic L-functions for totally real fields. Via analytic continuation, Dirichlet L-function is given by $L(s,\chi) = \sum_n \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-\chi(p)p^{-s}}$. At negative integers, it is given by $L(1-n,\chi) = -\frac{B_{n,\chi}}{n}$ where $B_{n,\chi}$ is a generalized Bernoulli number defined by $\sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!} = \sum_{a=1}^{f} \frac{\chi(a)te^{at}}{e^{ft}-1}$ for χ a Dirichlet character with conductor f.

A polynomial can be expressed more concisely by using summation notation $\sum_{k=0}^{n} a_k x^k$; viz, it can either be zero or can be written as the sum of a finite number of non-zero terms. Each term consists of the product of a number, called the coefficient of the term; and a finite number of indeterminates raised to non-negative integer powers. Some subtypes of polynomials are Laurent polynomials [involving negative integer powers], trigonometric polynomials, matrix polynomials and exponential polynomials. As opposed to an infinite series, a polynomial can also be regarded as a finite series with operation of adding finitely many quantities.

An infinite series is the operation of adding infinitely many quantities whereby it can be constituted by the two broad groups of power series and harmonic series. A power series is viewed as generalization of polynomials since it is essentially an infinite polynomial that allows infinitely many non-zero terms to occur with finite number of indeterminates raised to non-negative integer powers [as well as fractional or negative integer powers]. A harmonic series is the infinite series formed by summing all positive unit fractions whereby Riemann zeta function [manifesting non-converging or diverging behavior in $0 < \sigma < 1$ critical strip] is a (non-alternating) harmonic series. Then Dirichlet eta function [manifesting converging behavior in $0 < \sigma < 1$ critical strip] is a (alternating) harmonic series; viz, the infinite series formed by summing all positive and negative unit fractions.

Some special cases of power series: A geometric series is the sum of an infinite number of terms that have a constant ratio between successive terms. Puiseux series are a generalization of power series that allow for negative and fractional exponents of the indeterminate. Laurent series of a complex function $f(z)$ is a representation of that function as a power series which

includes terms of negative degree. A function can be represented as a power series if it is complex differentiable in an open set. This is applicable to functions such as $sin\ x$, $cos\ x$, e^x, $sinh\ x$ and $cosh\ x$ for domain of applicability or convergence interval $(-\infty, \infty)$; and $ln(1+x)$, $arcsin\ x$ and $arctan\ x$ for domain of applicability or convergence interval $(-1, 1)$.

Hasse principle is the original idea that one can find an integer solution to a polynomial equation with rational coefficients by using Chinese remainder theorem to piece together solutions modulo powers of each different prime number. This is handled by examining the equation in completions of rational numbers: real numbers and p-adic numbers.

Real numbers $\mathbb{R}$ = Rational numbers $\mathbb{Q}$ + Irrational numbers $\mathbb{R}/\mathbb{Q}$. When Hasse principle or Modified Hasse principle is satisfied, local solutions derived from equations or algorithms are given as $\mathbb{R}$ **and** p-adics numbers $\mathbb{Q}_{\mathbb{P}}$ with their global solutions given as $\mathbb{Q}$ [since $\mathbb{Q}$ embed in $\mathbb{R}$ and $\mathbb{Q}_{\mathbb{P}}$; a global solution yields local solutions at each prime, and *vice versa*]. For equations or algorithms having global solutions given as $\mathbb{Q}/\mathbb{P}$, their local solutions can only be given as $\mathbb{R}$ **but not** p-adics numbers $\mathbb{Q}_{\mathbb{P}}$, and thus Hasse principle or Modified Hasse principle is not satisfied. When these principles are satisfied, we can conceptually consider *eligible polynomials, power series or harmonic series be generally regarded as representing* ***certain types of p-adic (sub-)equations*** and *eligible algorithms and sub-algorithms be generally regarded as representing* ***certain types of p-adic (sub-)algorithms***.

Our formal generic version of *Hasse or local-global principle* states that *certain types of equations* have [global] rational solutions *if and only if* they have [local] solutions in the real numbers and in the p-adic numbers for each prime p. We reveal the following examples: For $-\infty < x < +\infty$, $y = 2x$ equation [that generates all positive and negative even numbers] and $y = 2x - 1$ equation [that generates all positive and negative odd numbers] have finite (solitary) rational solution at, respectively, $x = 0$ and $x = \frac{1}{2}$.

When considering $s = \sigma + it$, the CP IL equation Riemann zeta function [through its functional equation given by Eq. (3)] as a (non-alternating) harmonic series, has a simple zero at each even negative integer $s = -2n = -2, -4, -6, -8, -10...$ that exactly correspond to all trivial zeros [with cardinality of $\aleph_0$] given as the (infinitely many) rational solutions. Based on Hasse principle being satisfied, the p-adic Riemann zeta function [w.r.t. mathematically obtaining infinitely many Completely Predictable trivial zeros *outside the* $0 < \sigma < 1$ *critical strip*] can be conceptually created.

When considering $s = \sigma \pm it$, the IP IL equation Dirichlet eta function [*proxy* function for Riemann zeta function via analytic continuation] as a (alternating) harmonic series, has nontrivial zeros [with cardinality of $\aleph_0$] given as the infinitely many t-valued irrational (transcendental) solutions [**which are not rational solutions**] **only** at $\sigma = \frac{1}{2}$ critical line – this is Riemann hypothesis. Then as *not satisfying* Hasse principle, the p-adic Dirichlet eta function [w.r.t. mathematically obtaining infinitely many Incompletely Predictable nontrivial zeros *inside the* $0 < \sigma < 1$ *critical strip*] cannot be conceptually created.

Our formal generic version of *Modified Hasse or local-global principle* states that *certain types of algorithm and its sub-algorithms* have [global] rational solutions *if and only if* they have [local] solutions in the real numbers and in the p-adic numbers for each prime p.

We ignore even prime number 2. The algorithms Sieve-of-Eratosthenes generates all Primes as Odd Primes, and Complement-Sieve-of-Eratosthenes generates all Composites as Even and Odd Composites. For $i = 1, 2, 3, 4, 5...$, there are an arbitrarily large number of sub-algorithms derived from Gap $2i$-Sieve-of-Eratosthenes [c.f. List in section 2] that generates

all Gap 2-Odd Twin Primes (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43)... depicted as paired Odd Twin Prime and its next Odd Prime + all Gap 4-Odd Cousin Primes (7, 11), (13, 17), (19, 23), (37, 41), (43, 47), (67, 71)... depicted as paired Odd Cousin Prime and its next Odd Prime + all Gap 6-Odd Sexy Primes (23, 29), (31, 37), (47, 53), (53, 59), (61, 67), (73, 79), (83, 89)... depicted as paired Odd Sexy Prime and its next Odd Prime +...], etc. For Modified Polignac's and Twin prime conjectures to be true [w.r.t. primes (and composites) being rational numbers]; then all these (sub-)algorithms must generate relevant (sub-)sets of Incompletely Predictable primes (and composites) [with cardinality of $\aleph_0$] as (infinitely many) rational solutions. We say all these (sub-)algorithms which can be *conceptually perceived as* ***certain types of p-adic (sub-)algorithms*** must *satisfy* Modified Hasse principle in that one can always find all the (infinitely many) rational solutions to these IP IL (sub-)algorithms.

3 The Prevalences of Nontrivial zeros, Primes and Composites as Incompletely Predictable entities

We adopt the abbreviations from subsection 1.1. The sets of numbers generated using power (exponent) such as 2 or $\frac{1}{2}$, even numbers, odd numbers, etc are morphologically constituted by *Completely Predictable numbers* in the sense that these sets of numbers are actually ***not*** random and do not behave like one. The sets of nontrivial zeros, primes, composites, etc are morphologically constituted by *Incompletely Predictable numbers* [or *pseudo-random numbers*] in the sense that these sets of numbers are actually ***not*** random but behave like one. The word *number* [singular noun] or *numbers* [plural noun] in reference to CP even and odd numbers, IP prime and composite numbers, IP Gram points and virtual Gram points can be interchanged with the word *entity* [singular noun] or *entities* [plural noun].

Lemma 1. *We can formally define the elements from (sub)sets and (sub)tuples as Completely Predictable or Incompletely Predictable entities.*

Proof. A set is a collection of zero (viz, the empty set) or more elements (viz, a finite set with a finite number of elements or an infinite set with an infinite number of elements). A singleton refers to a finite set with a single element. A set can be any kind of mathematical objects: numbers, symbols, points in space, lines, other geometrical shapes, variables, or even other sets whereby these [mutable] non-repeating elements are not arranged in an unique order. A subset can be a [smaller] finite set derived from its [larger] parent finite set or its [larger] parent infinite set. A subset can also be a [smaller] infinite set derived from its [larger] parent infinite set. A tuple, which can potentially be subdivided into subtuples, is a finite ordered list (sequence) of elements whereby these [immutable] non-repeating elements are arranged in an unique order. Thus a tuple or a subtuple is regarded as a special type of finite set with the extra imposed restriction. As shown below using worked examples:
CP simple equation or algorithm generates CP numbers e.g. even numbers 0, 2, 4, 6, 8, 10... or odd numbers 1, 3, 5, 7, 9, 11.... A generated CP number is **locationally defined** as a number whose i^{th} position is *independently* determined by simple calculations without needing to know related positions of all preceding numbers – this is a **Universal Property**.
IP complex equation or algorithm generates IP numbers e.g. prime numbers 2, 3, 5, 7, 11, 13... or composite numbers 4, 6, 8, 9, 10, 12.... A generated IP number is **locationally defined** as a number whose i^{th} position is *dependently* determined by complex calculations with needing

to know related positions of all preceding numbers – this is a **Universal Property**.
We notice that the elements in (sub)sets and (sub)tuples when generated by equations or algorithms will precisely constitute the relevant entities or numbers of interest.

The proof is now complete for Lemma 1□.

Lemma 2. *We can validly classify the countably infinite sets as accelerating, linear or decelerating subtypes.*

Proof. We provide the following required mathematical arguments.

Cardinality: With increasing size, arbitrary Set [or Subset] **X** can be countably finite set (**CFS**), countably infinite set (**CIS**) or uncountably infinite set (**UIS**). Denoted as ||**X**|| in this paper, the cardinality of Set **X** measures *number of elements* in Set **X**. E.g., Set **negative Gram[y=0] point** as constituted by a [solitary] rational ($\mathbb{Q}$) t-value of 0 instead of a usual transcendental ($\mathbb{R} - \mathbb{A}$) t-value has CFS of negative Gram[y=0] point with this particular ||**negative Gram[y=0] point**|| = 1, Set even Prime number ($\mathbb{P}$) has CFS of solitary even $\mathbb{P}$ 2 with ||**even** $\mathbb{P}$|| = 1, Set Natural numbers ($\mathbb{N}$) has CIS of $\mathbb{N}$ with $||\mathbb{N}|| = \aleph_0$, and Set Real numbers ($\mathbb{R}$) has UIS of $\mathbb{R}$ with $||\mathbb{R}|| = \mathfrak{c}$ (cardinality of the continuum). Then with ||**CIS**|| = $\aleph_0$ = [countably] infinitely many elements; we provide a novel classification for CIS based on its number of elements (cardinality) manifesting linear, accelerating or decelerating property thus constituting the three subtypes of CIS.

CIS-IM-accelerating: CIS with its cardinality given by ||**CIS-IM-accelerating**|| = $\aleph_0$-accelerating = [countably] infinitely many elements that will (overall) acceleratingly reach an *infinity value*. Examples: CP integers 0, 1, 4, 9, 16... generated by simple equation $y = x^2$ for x = 0, 1, 2, 3, 4... and CP values obtained from natural exponential function $y = e(x)$; and IP composite numbers 4, 6, 8, 9, 10... faithfully generated by complex Complement-Sieve-of-Eratosthenes algorithm [which is equivalent to simply discarding 0, 1, and all generated prime numbers via Sieve-of-Eratosthenes algorithm from the set of integers 0, 1, 2, 3, 4, 5...].

CIS-IM-linear: CIS with its cardinality given by ||**CIS-IM-linear**|| = $\aleph_0$-linear = [countably] infinitely many elements that will (overall) linearly reach an *infinity value*. Examples: CP entities 0, 1, 2, 3, 4, 5... [representing all positive integer numbers] generated by simple equation $y = x$ for x = 0, 1, 2, 3, 4...; CP entities 0, 2, 4, 6, 8, 10... [representing all positive even numbers] generated by simple equation $y = 2x$ for x = 0, 1, 2, 3, 4...; CP entities 1, 3, 5, 7, 9, 11... [representing all positive odd numbers] generated by simple equation $y = 2x - 1$ for x = 1, 2, 3, 4, 5...; and IP nontrivial zeros, Gram[y=0] points and Gram[x=0] points (all given as $\mathbb{R} - \mathbb{A}$ t-values) generated from complex equation Riemann zeta function via its *proxy* Dirichlet eta function. These IP entities will inevitably manifest IP perpetual repeating violations (failures) in Gram's Law and Rosser's Rule occuring infinitely many times. E.g., the former will give rise to Set **negative Gram[y=0] points** whereby CIS negative Gram[y=0] points is constituted by $\mathbb{R} - \mathbb{A}$ t-values and is classified as having ||**negative Gram[y=0] points**|| = ||**CIS-IM-linear**|| = $\aleph_0$-linear.

CIS-IM-decelerating or **CIS-ALN-decelerating**: CIS with its cardinality given by ||**CIS-ALN-decelerating**|| = $\aleph_0$-decelerating = [countably] arbitrarily large number of elements that will (overall) deceleratingly reach an *Arbitrarily Large Number* value. Examples: CP entities 0, 1, $\sqrt{2}$, $\sqrt{3}$, 2, $\sqrt{5}$... generated by simple equation $y = \sqrt{x}$ for x = 0, 1, 2, 3, 4, 5... and CP values obtained from natural logarithm function $y = ln(x)$; and IP prime numbers 2, 3, 5, 7, 11... faithfully generated by complex Sieve-of-Eratosthenes algorithm.

The proof is now complete for Lemma 2□.

Endowed with *t*-valued transcendental (irrational) numbers, we define Gram points and virtual Gram points which are generated by Riemann zeta function (via *proxy* Dirichlet eta function) as follows: Three types of Gram points refer to Origin intercept points as Gram[x=0,y=0] points or nontrivial zeros, x-axis intercept points as Gram[y=0] points and y-axis intercept points as Gram[x=0] points when $\sigma = \frac{1}{2}$ (i.e. the solitary critical line). Two types of virtual Gram points refer to virtual x-axis intercept points as virtual Gram[y=0] points and virtual y-axis intercept points as virtual Gram[x=0] points when $\sigma \neq \frac{1}{2}$ (i.e. the infinitely many non-critical lines) whereby virtual Origin intercept points do not exist.

Let **n** denotes *number*. Congruence **n** ≡ 0 (mod 2) holds for even **n** {0, 2, 4, 6, 8, 10...} and congruence **n** ≡ 1 (mod 2) holds for odd **n** {1, 3, 5, 7, 9, 11...}. For i = all integers ≥ 0 [to allow inclusion of *zeroeth* or 0^{th} even number = 0] or for i = all integers ≥ 1; both the i^{th} position of nominated CP numbers or entities and the i^{th} position of nominated IP numbers or entities are simply given by i. Apart from the very first Gram[y=0] point and the very first virtual Gram[y=0] point both occurring at t = 0, we note all infinitely many Gram points and infinitely many virtual Gram points will consist of *t*-valued transcendental numbers whose i^{th} positions are IP with infinitely many digits after the decimal point in each transcendental number again being IP. Important caveat: The choice of index *n* for Gram[y=0] points [or *usual / traditional* Gram point] is crude and confusing as it is historically chosen in such a way that this index is 0 at the first value which is larger than the smallest positive [equating to 1^{st}] nontrivial zero (occurring at imaginary part *t* = 14.134725...) located on $\sigma = \frac{1}{2}$ critical line of Riemann zeta function. Thus when using the notation i^{th} Gram[y=0] point; our utilized position index i = 1, 2, 3, 4, 5, 6, 7... will now correspond to traditional position index *n* = –3, –2, –1, 0, 1, 2, 3.... The initial few Gram[y=0] points are 0, 3.436218..., 9.666908..., 17.845599..., 23.170282..., 27.670182..., etc. The initial few nontrivial zeros are 14.134725..., 21.022040..., 25.010858..., 30.424876..., 32.935062..., 37.586178..., etc.

CP simple equation or algorithm generates CP numbers. Reiterating from Lemma 1: A generated CP number is **locationally defined** as a number whose i^{th} position is *independently* determined by simple calculations without needing to know related positions of all preceding numbers. We supply the example using even and odd numbers.

E-O Pairing: For i = 1, 2, 3,..., ∞; let i^{th} Even numbers = E_i and i^{th} Odd numbers = O_i, and i^{th} even number gaps = $eGap_i$ and i^{th} odd number gaps = $oGap_i$. We ignore $E_0 = 0$. The positions of E_i and O_i are CP, and are independent from each other.

E_i	2		4		6		8		10		12	.....
$eGap_i$		2		2		2		2		2		2

We can precisely, easily and independently calculate $E_5 = (2X5) = 10$ and $O_5 = (2X5)–1 = 9$.

O_i	1		3		5		7		9		11	.....
$oGap_i$		2		2		2		2		2		2

IP complex equation or algorithm generates IP numbers. Reiterating from Lemma 1: A generated IP number is **locationally defined** as a number whose i^{th} position is *dependently* determined by complex calculations with needing to know related positions of all preceding numbers. We supply the example using prime and composite numbers (and note analogous examples can be readily created using nontrivial zeros, Gram[y=0] points and Gram[x=0] points when $\sigma = \frac{1}{2}$).

P-C Pairing: For i = 1, 2, 3,..., ∞; let i^{th} Prime numbers = P_i and i^{th} Composite numbers = C_i, and i^{th} prime number gaps = $pGap_i$ and i^{th} composite number gaps = $cGap_i$. The positions

of P_i and C_i are IP, and are dependent on each other.

P_i	2		3		5		7		11		13	.....
$pGap_i$		1		2		2		4		2		4

We can precisely, tediously and dependently compute $C_6 = 12$ and $P_6 = 13$: 2 is 1^{st} prime, 3 is 2^{nd} prime, 4 is 1^{st} composite, 5 is 3^{rd} prime, 6 is 2^{nd} composite, 7 is 4^{th} prime, 8 is 3^{rd} composite, 9 is 4^{th} composite, 10 is 5^{th} composite, 11 is 5^{th} prime, 12 is 6^{th} composite, 13 is 6^{th} prime, etc. Our desired integer 12 is the 6^{th} composite and integer 13 is the 6^{th} prime.

C_i	4		6		8		9		10		12	.....
$cGap_i$		2		2		1		1		2		2

We concisely define (*rolling* and *cumulative*) Prevalence of nontrivial zeros [including logical deduction that support Riemann hypothesis to be true], and (*rolling* and *cumulative*) Prevalences of prime and composite numbers [including logical deductions that support Modified Polignac's and Twin prime conjectures to be true]. We provide in subsection 3.1 the p-adic absolute values for these eternal entities when calculated as *rolling* Prevalences.

We analyze the data of all CIS-IM-linear computed nontrivial zeros (NTZ) when extrapolated out over a wide range of $t \geq 0$ real number values. We can symbolically define nontrivial zeros counting function NTZ-π(t) = number of NTZ $\leq$ t with t assigned to having real number values which are conveniently designated by 10^n whereby n = 1, 2, 3, 4, 5.... The *cumulative* **Prevalence of nontrivial zeros** = NTZ-π(t) / t = NTZ-π(t) / (10^n) when t = 0 to 10^n, whereby denominator t is [artificially] regarded as having integer number values. We conceptually define all consecutive NTZ gaps as i^{th} t-valued NTZ – $(i\text{-}1)^{th}$ t-valued NTZ. Thus there are CIS-IM-linear computed NTZ gaps. The numbers of NTZ between $10^0 - 10^1$ [interval = 9], $10^1 - 10^2$ [interval = 90], $10^2 - 10^3$ [interval = 900], $10^3 - 10^4$ [interval = 9000], $10^4 - 10^5$ [interval = 90000], $10^5 - 10^6$ [interval = 900000], $10^6 - 10^7$ [interval = 9000000], $10^7 - 10^8$ [interval = 90000000]... are 0, 29, 620, 9493, 127927, 1609077, 19388979, 226871900... with corresponding *rolling* **Prevalence of nontrivial zeros** = 0, 0.322, 0.689, 1.055, 1.421, 1.788, 2.154, 2.521... $\Longrightarrow$ *rolling* **Prevalence of nontrivial zeros** seems to overall fluctuatingly increase by around 0.366 in a "linear" manner. This limited observation alone suggests Cardinality of nontrivial zeros = $\|$**CIS-IM-linear**$\|$ = $\aleph_0$-linear.

In comparison, we further notice here the numbers of NTZ between $10^0 - 10^1$ [interval = 9], $10^0 - 10^2$ [interval = 99], $10^0 - 10^3$ [interval = 999], $10^0 - 10^4$ [interval = 9999], $10^0 - 10^5$ [interval = 99999], $10^0 - 10^6$ [interval = 999999], $10^0 - 10^7$ [interval = 9999999], $10^0 - 10^8$ [interval = 99999999]... are 0, 29, 649, 10142, 138069, 1747146, 21136125, 248008025... with corresponding *cumulative* **Prevalence of nontrivial zeros** = 0, 0.293, 0.650, 1.014, 1.381, 1.747, 2.114, 2.480...

Using different σ-valued Riemann zeta function, we define co-linear lines or co-lines as any two generated independent and mutually exclusive parallel (curved) lines that will never cross over one another. For $-\infty < t < +\infty$ that give rise to $0 < t < +\infty$ positive part or image having positive [t-valued] nontrivial zeros and $-\infty < t < 0$ negative counterpart or mirror image having negative [t-valued] nontrivial zeros, the synthesized **Proposition** is: Each and every t-valued nontrivial zeros that can be depicted geometrically as t-valued Gram[x=0,y=0] points or t-valued Origin intercept points will only materialize when parameter σ in Riemann zeta function [precisely] reaches the [exact] value of $\frac{1}{2}$.

The synthesized **Corollary** is: Nontrivial zeros will never materialize when parameter σ in Riemann zeta function [imprecisely] reach an [inexact] infinitesimal small number $\frac{1}{\infty}$ value

less than or more than (but not equal to) $\frac{1}{2}$. Therefore Riemann hypothesis is true since this $\sigma = \frac{1}{2}$-associated Proposition and its $\sigma \neq \frac{1}{2}$-associated Corollary constitute a set of mutually exclusive correct and complete mathematical arguments on nontrivial zeros location being the unique $\sigma = \frac{1}{2}$ critical line (but not the non-unique $\sigma \neq \frac{1}{2}$ non-critical lines), and this is moreover valid for the entire range of $-\infty < t < +\infty$.

Riemann zeta function's co-linear lines, three types of Gram points, two types of virtual Gram points, and perpetually recurring violations (failures) of Gram's Law and Rosser's Rule must [smoothly] manifest Mirror symmetry and Law of continuity (subsection 4.1) when considered for entire range of $-\infty < t < +\infty$ as positive and negative real number line having its end boundaries delineated by interval $(-\infty, +\infty)$.

We analyze the data of all CIS-IM-accelerating computed composite numbers when extrapolated out over a wide range of x ≥ 4 integer values. We define composite counting function Composite-π(x) = number of composites ≤ x with x conveniently assigned to having odd number values of the form $10^n - 1$ whereby n = 1, 2, 3, 4, 5.... The *cumulative* **Prevalence of all composite numbers** = Composite-π(x) / x = Composite-π(x) / $(10^n - 4)$ when x = 4 to $10^n - 1$. CIS-IM-accelerating composite numbers in totality all have either odd composite gap 1 or even composite gap 2. All the odd integers which are not prime numbers are odd composite numbers, consecutively given as 9, 15, 21, 25, 27, 31.... Except for 0 and 2, all the remaining even integers are even composite numbers, consecutively given as 4, 6, 8, 10, 12, 14, 16.... We can also create *cumulative* **Prevalences** separately for the two subsets of even and odd composite numbers with the former subset always being larger than the later subset because, with exception of even prime number 2, all prime numbers are odd numbers.

We analyze the data of all CIS-ALN-decelerating computed prime numbers when extrapolated out over a wide range of x ≥ 2 integer values. We define prime counting function Prime-π(x) = number of primes ≤ x with x conveniently assigned to having odd number values of the form $10^n - 1$ whereby n = 1, 2, 3, 4, 5.... The *cumulative* **Prevalence of all prime numbers** = Prime-π(x) / x = Prime-π(x) / $(10^n - 2)$ when x = 2 to $10^n - 1$. Prime gaps for all odd prime numbers are constituted by CIS-ALN-decelerating even Prime gaps 2, 4, 6, 8, 10.... We can also create *cumulative* **Prevalence of twin primes with prime gap 2**, **Prevalence of cousin primes with prime gap 4**, **Prevalence of sexy primes with prime gap 6**, etc.

n	1	2	3	4	5	6	7
Prev_{allP}	0.4	0.25	0.168	0.1229	0.09592	0.078498	0.0664579
Prev_{Pgap2}	0.2	0.08	0.035	0.0205	0.01224	0.008169	0.0058980
Prev_{Pgap4}	0.1	0.08	0.040	0.0202	...	...	...
Prev_{Pgap6}	0.0	0.07	0.044	0.0299	...	...	...

n	8	9	10	11	...
Prev_{allP}	0.05761455	0.050847534	0.0455052511	0.04118054813	...
Prev_{Pgap2}	0.00440312	...	...	...	...

The terms *Prevalence* and *Proportion* are interchangeable. For n = 1, 2, 3, 4, 5... in the expression x = 2 to $10^n - 1$, we obtain above tabulated calculations with captured manifestation of decelerating properties on the relationship: Prevalence of all primes (Prev_{allP}) = Prevalence of twin primes with prime gap 2 (Prev_{Pgap2}) + Prevalence of cousin primes with prime gap 4 (Prev_{Pgap4}) + Prevalence of sexy primes with prime gap 6 (Prev_{Pgap6}) +.... We can compare and contrast with computed *rolling* **Prevalence of Prime numbers**: between 0 to 9 = 4 1-digit Primes per 10 1-digit integers = 0.4, between 10 to 99 = 21 2-digit Primes per 90 2-digit

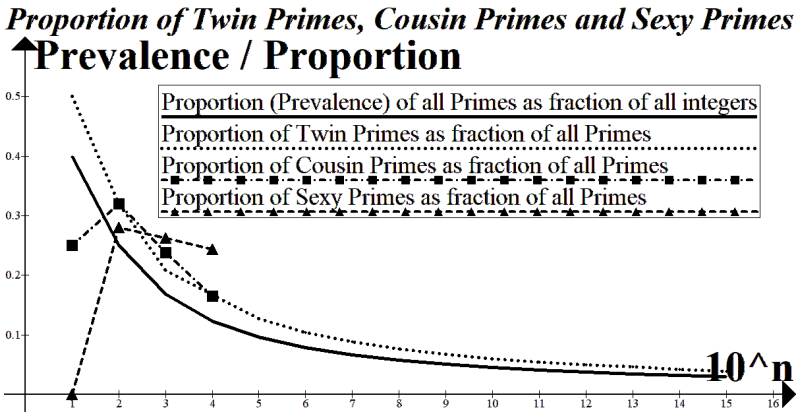

Fig. 1 Proportion (Prevalence) of Twin primes, Cousin primes [as partial calculations] and Sexy Primes [as partial calculations] with Proportion (Prevalence) of all Primes included. The n = 1, 2, 3, 4, 5, 6, 7, 8... in 10^n that is denoted with horizontal x-axis implies the scale of this axis is non-linearly depicted using increasing powers of 10.

integers = 0.233, between 100 to 999 = 143 3-digit Primes per 900 3-digit integers = 0.159, between 1000 to 9999 = 1061 4-digit Primes per 9000 4-digit integers = 0.118, between 10000 to 99999 = 8363 5-digit Primes per 90000 5-digit integers = 0.093, between 100000 to 999999 = 68906 6-digit Primes per 900000 6-digit integers = 0.076, etc.

n	1	2	3	4	5	6	7	8
$\frac{Prev_{Pgap2}}{Prev_{allP}}$	0.5	0.32	0.2083	0.1668	0.1276	0.1041	0.08875	0.07642
$\frac{Prev_{Pgap4}}{Prev_{allP}}$	0.25	0.32	0.2381	0.1644	...	...	...	...
$\frac{Prev_{Pgap6}}{Prev_{allP}}$	0.0	0.28	0.2619	0.2433	...	...	...	...

We next calculate in above table [which are then graphically depicted in Figure 1] the Proportion of Twin primes with prime gap 2, Cousin primes with prime gap 4, and Sexy primes with prime gap 6. These are respectively derived using relevant ratios $\frac{Prev_{Pgap2}}{Prev_{allP}}$, $\frac{Prev_{Pgap4}}{Prev_{allP}}$ and $\frac{Prev_{Pgap6}}{Prev_{allP}}$ with the Proportion of all Primes also depicted for comparison. The Proportion of Twin primes, Cousin primes [as partial calculations] and Sexy Primes [as partial calculations] with the Proportion (Prevalence) of all Primes included in Figure 1 clearly depict these Proportions to deceleratingly reach an infinitesimal small number value $\frac{1}{\infty}$ just above 0 [but never reaches 0] as $n \to \infty$. When utilizing the same 10^n number system as specified by variable n in Figure 1; the Prev_{allC} = 0.5, 0.75, 0.832, 0.8771, 0.9041, 0.9215, 0.9335, 0.9424... will fully reflect the statement Integers = {0, 1} + all Primes {2, 3, 5, 7, 11, 13...} + all Composites {4, 6, 8, 9, 10, 12...}. Then the Proportion (Prevalence) of all Composites will reciprocally depict acceleratingly reaching an infinitesimal small number value $\frac{1}{\infty}$ just below 1 [but never reaches 1] as $n \to \infty$.

We can validly ignore the solitary even Prime number 2 [at x = 2] and simply regard all Primes here as odd primes. Then there is an arbitrarily large number of all Primes for x = 3 to $10^n - 1$. We deduce the average prime gaps in relation to arbitrarily large number of all even

Prime gaps 2, 4, 6, 8, 10, 12, 14... must overall and individually manifest the asymptotically zero behavior of natural logarithm. Proportion of all Primes is known to deceleratingly reach an infinitesimal small number value $\frac{1}{\infty}$ [but never 0] as $n \to \infty$. Then for cases of even Pime gaps 2, 4, 6, 8, 10... we infer the following deduction based on the all-important condition *Proportions of Twin primes, Cousin primes, Sexy primes, etc are all coupled to Proportion of all Primes [conceptually] as self-similar fractal objects* displayed in Figure 1: Whereas the Proportions of Twin primes, Cousin primes, Sexy primes, etc and Proportion of all Primes are coupled together [conceptually] as self-similar *fractal objects* [that are never identical], so must both the well-defined Proportions of Twin primes, Cousin primes, Sexy primes, etc and Proportion of all Primes always manifest deceleratingly reaching an infinitesimal small number value $\frac{1}{\infty}$ [but never 0] as $n \to \infty$.

The above synopsis involves positive odd prime numbers as an overall set and individual subsets of odd prime numbers derived from even Prime gaps 2, 4, 6, 8, 10... [treated overall and individually as part or image] in regards to mathematical arguments on even Prime gaps that prove Modified Polignac's and Twin prime conjectures to be true. The mathematical arguments on even Prime gaps can equivalently be applied to negative odd prime numbers [with corresponding overall set and individual subsets treated as counterpart or mirror image], thus also proving Modified Polignac's and Twin prime conjectures to be true.

All above-mentioned overall set and individual subsets of positive and negative odd primes must [jaggedly] manifest Mirror symmetry and Law of continuity (subsection 4.1) when considered for entire range of positive and negative integer number line with its end boundaries delineated by interval $(-\infty, +\infty)$.

3.1 The *p*-adic absolute values applied to Prevalences of Nontrivial zeros, Primes and Composites

A p-adic number is an extension of the field of rationals such that congruences modulo powers of a fixed prime p are related to proximity in ***p*-adic metric**. Any nonzero rational number x can be represented by $x = (p^a r)/s$ where p is a prime number, r and s are integers not divisible by p, and a is a unique integer. We firstly define the p-adic norm of x by $|x|_p = p^{-a}$, and secondly define the p-adic norm $|0|_p = 0$.

Since zero terms can always be added at the beginning, every rational x has an [essentially] unique p-adic expansion $x = \sum_{j=m}^{\infty} a_j p^j$, with m an integer, a_j the integers between 0 and p-1 inclusive, and where the sum is convergent with respect to p-adic valuation. If $x! \neq 0$ and $a_m! \neq 0$, then the expansion is unique. Then for p a prime and n a positive integer, $|n!|_p = p^{(-(n-A_p(n))/(p-1))}$, where the p-adic expansion of n is $n = a_0 + a_1 p + a_2 p^2 + ... + a_L p^L$, and $A_p(n) = a_0 + a_1 + ... + a_L$. For sufficiently large n, $|n!|_p \leq p^{-n/(2p-2)}$. The p-adic valuation on Q gives rise to the p-adic metric $d(x, y) = |x - y|_p$, which in turn gives rise to the p-adic topology. It can be shown that the rationals, together with the p-adic metric, do not form a complete metric space. The completion of this space can therefore be constructed, and the set of p-adic numbers Q_p is defined to be this completed space.

Consider the *rolling* **Prevalence of Nontrivial zeros** between 1 to 10, 10 to 100, 100 to 1000,... obtained from section 3: $\frac{0}{9} = 0{\cdot}3^{-2}$, $\frac{29}{90} = 29^1{\cdot}2^{-1}{\cdot}3^{-2}{\cdot}5^{-1}$, $\frac{620}{900} = 31^1{\cdot}3^{-2}{\cdot}5^{-1}$, $\frac{9493}{9000} =$

$11^1 \cdot 863^1 \cdot 2^{-3} \cdot 3^{-2} \cdot 5^{-3}$, $\frac{127927}{90000} = 19^1 \cdot 6733^1 \cdot 2^{-4} \cdot 3^{-2} \cdot 5^{-4}$, $\frac{1609077}{900000} = 257^1 \cdot 2087^1 \cdot 2^{-5} \cdot 3^{-1} \cdot 5^{-5}$, $\frac{19388979}{9000000} = 2154331^1 \cdot 2^{-6} \cdot 5^{-6}$, $\frac{226871900}{90000000} = 2268719^1 \cdot 2^{-5} \cdot 3^{-2} \cdot 5^{-5}$,.... They have corresponding p-adic absolute values as follows:

$|\frac{0}{9}|_2 = 0$,

$|\frac{29}{90}|_{29} = \frac{1}{29} / |\frac{29}{90}|_2 = 2 / |\frac{29}{90}|_3 = 9 / |\frac{29}{90}|_5 = 5$,

$|\frac{620}{900}|_{31} = \frac{1}{31} / |\frac{620}{900}|_3 = 9 / |\frac{620}{900}|_5 = 5$,

$|\frac{9493}{9000}|_{11} = \frac{1}{11} / |\frac{9493}{9000}|_{863} = \frac{1}{863} / |\frac{9493}{9000}|_2 = 8 / |\frac{9493}{9000}|_3 = 9 / |\frac{9493}{9000}|_5 = 125$,

$|\frac{127927}{90000}|_{19} = \frac{1}{19} / |\frac{127927}{90000}|_{6733} = \frac{1}{6733} / |\frac{127927}{90000}|_2 = 16 / |\frac{127927}{90000}|_3 = 9 / |\frac{127927}{90000}|_5 = 625$,

$|\frac{1609077}{900000}|_{257} = \frac{1}{257} / |\frac{1609077}{900000}|_{2087} = \frac{1}{2087} / |\frac{1609077}{900000}|_2 = 32 / |\frac{1609077}{900000}|_3 = 3 / |\frac{1609077}{900000}|_5 = 3125$,

$|\frac{19388979}{9000000}|_{2154331} = \frac{1}{2154331} / |\frac{19388979}{9000000}|_{2087} = \frac{1}{2087} / |\frac{19388979}{9000000}|_2 = 64 / |\frac{19388979}{9000000}|_5 = 15625$,

$|\frac{226871900}{90000000}|_{2268719} = \frac{1}{2268719} / |\frac{226871900}{90000000}|_2 = 64 / |\frac{226871900}{90000000}|_3 = 9 / |\frac{226871900}{90000000}|_5 = 15625$,...

Consider the *rolling* **Prevalence of Prime numbers** between 1 to 9 as 1-digit Primes, 10 to 99 as 2-digit Primes, 100 to 999 as 3-digit Primes, 1000 to 9999 as 4-digit Primes, 10000 to 99999 as 5-digit Primes, 100000 to 999999 as 6-digit Primes,... obtained from Appendix C: $\frac{4}{9} = 2^2 \cdot 3^{-2}$, $\frac{21}{90} = 3^1 \cdot 7^1 \cdot 2^{-1} \cdot 3^{-2} \cdot 5^{-1}$, $\frac{143}{900} = 11^1 \cdot 13^1 \cdot 2^{-2} \cdot 3^{-2} \cdot 5^{-2}$, $\frac{1061}{9000} = 1061^1 \cdot 2^{-3} \cdot 3^{-2} \cdot 5^{-3}$, $\frac{8363}{90000} = 8363^1 \cdot 2^{-4} \cdot 3^{-2} \cdot 5^{-4}$, $\frac{68906}{900000} = 2^1 \cdot 131^1 \cdot 263^1 \cdot 2^{-5} \cdot 3^{-2} \cdot 5^{-5}$,... They have corresponding p-adic absolute values as follows:

$|\frac{4}{9}|_2 = \frac{1}{4} / |\frac{4}{9}|_3 = 9$,

$|\frac{21}{90}|_7 = \frac{1}{7} / |\frac{21}{90}|_2 = 2 / |\frac{21}{90}|_3 = 3 / |\frac{21}{90}|_5 = 5$,

$|\frac{143}{900}|_{11} = \frac{1}{11} / |\frac{143}{900}|_{13} = \frac{1}{13} / |\frac{143}{900}|_2 = 4 / |\frac{143}{900}|_3 = 9 / |\frac{143}{900}|_5 = 25$,

$|\frac{1061}{9000}|_{1061} = \frac{1}{1061} / |\frac{1061}{9000}|_2 = 8 / |\frac{1061}{9000}|_3 = 9 / |\frac{1061}{9000}|_5 = 125$,

$|\frac{8363}{90000}|_{8363} = \frac{1}{8363} / |\frac{8363}{90000}|_2 = 16 / |\frac{8363}{90000}|_3 = 9 / |\frac{8363}{90000}|_5 = 625$,

$|\frac{68906}{900000}|_{131} = \frac{1}{131} / |\frac{68906}{900000}|_{263} = \frac{1}{263} / |\frac{68906}{900000}|_2 = 16 / |\frac{68906}{900000}|_3 = 9 / |\frac{68906}{900000}|_5 = 3125$,...

Consider the *rolling* **Prevalence of Composite numbers** between 1 to 9 as 1-digit Composites, 10 to 99 as 2-digit Composites, 100 to 999 as 3-digit Composites, 1000 to 9999 as 4-digit Composites, 10000 to 99999 as 5-digit Composites, 100000 to 999999 as 6-digit Composites,... obtained from Appendix C: $\frac{4}{9} = 2^2 \cdot 3^{-2}$, $\frac{69}{90} = 3^1 \cdot 23^1 \cdot 2^{-1} \cdot 3^{-2} \cdot 5^{-1}$, $\frac{757}{900}$

$= 757^1 \cdot 2^{-2} \cdot 3^{-2} \cdot 5^{-2}$, $\frac{7939}{9000} = 17^1 \cdot 467^1 \cdot 2^{-3} \cdot 3^{-2} \cdot 5^{-3}$, $\frac{81637}{90000} = 81637^1 \cdot 2^{-4} \cdot 3^{-2} \cdot 5^{-4}$, $\frac{831094}{900000} = 2^1 \cdot 11^1 \cdot 37^1 \cdot 1021^1 \cdot 2^{-5} \cdot 3^{-2} \cdot 5^{-5}$,... They have corresponding p-adic absolute values as follows:
$|\frac{4}{9}|_2 = \frac{1}{4}$ / $|\frac{4}{9}|_3 = 9$,
$|\frac{69}{90}|_{23} = \frac{1}{23}$ / $|\frac{69}{90}|_2 = 2$ / $|\frac{69}{90}|_3 = 3$ / $|\frac{69}{90}|_5 = 5$,
$|\frac{757}{900}|_{757} = \frac{1}{757}$ / $|\frac{757}{900}|_2 = 4$ / $|\frac{757}{900}|_3 = 9$ / $|\frac{757}{900}|_5 = 25$,
$|\frac{7939}{9000}|_{17} = \frac{1}{17}$ / $|\frac{7939}{9000}|_{467} = \frac{1}{467}$ $|\frac{7939}{9000}|_2 = 8$ / $|\frac{7939}{9000}|_3 = 9$ / $|\frac{7939}{9000}|_5 = 125$,
$|\frac{81637}{90000}|_{81637} = \frac{1}{81637}$ / $|\frac{81637}{90000}|_2 = 16$ / $|\frac{81637}{90000}|_3 = 9$ / $|\frac{81637}{90000}|_5 = 625$,
$|\frac{831094}{900000}|_{11} = \frac{1}{11}$ / $|\frac{831094}{900000}|_{37} = \frac{1}{37}$ / $|\frac{831094}{900000}|_{1021} = \frac{1}{1021}$ / $|\frac{831094}{900000}|_2 = 16$ / $|\frac{831094}{900000}|_3 = 9$ / $|\frac{831094}{900000}|_5 = 3125$,...

Progressively computed *rolling* Prevalences on infinitely-many Primes, Composites and nontrivial zeros must all be eternal entities. Primes and Composites are complementary numbers as manifested by the *reciprocal* patterns of [unique] p-adic absolute values (of Infinite-Length) obtained for their respective *rolling* Prevalences. As eternal entities, then the patterns of [unique] p-adic absolute values (of Infinite-Length) obtained for Odd Primes generated by each of their corresponding even Prime gaps 2, 4, 6, 8, 10... as *rolling* Prevalences are also possible. We deduce as **Proposition** there can only be one [unique] pattern of p-adic absolute values (of Infinite-Length) obtained for *rolling* Prevalence of Nontrivial zeros when the only one [unique] $\sigma = \frac{1}{2}$ condition is met. Then as **Corollary**, the pattern of p-adic absolute value for *rolling* Prevalence of Nontrivial zeros must perpetually be zero when the [non-unique] $\sigma \neq \frac{1}{2}$ conditions are met.

4 Intersection of Riemann zeta function, Dirichlet eta function and Sieve of Eratosthenes

Riemann zeta function $\zeta(s)$ is a function of complex variable s ($= \sigma \pm \imath t$) that continues sum of infinite series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$ for $\mathrm{Re}(s) > 1$, and its analytic continuation elsewhere. Containing no nontrivial zeros, $\zeta(s)$ is defined only in $1 < \sigma < \infty$ region where it is absolutely convergent. The common convention is to write s as $\sigma + \imath t$ with $\imath = \sqrt{-1}$, and with σ and t real. Valid for $\sigma > 0$, we write $\zeta(s)$ as $Re\{\zeta(s)\} + \imath Im\{\zeta(s)\}$ and note that $\zeta(\sigma + \imath t)$ when $0 < t < +\infty$ is the complex conjugate of $\zeta(\sigma - \imath t)$ when $-\infty < t < 0$.

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \end{aligned} \tag{1}$$

$$= \Pi_{p\ prime} \frac{1}{(1-p^{-s})}$$
$$= \frac{1}{(1-2^{-s})} \cdot \frac{1}{(1-3^{-s})} \cdot \frac{1}{(1-5^{-s})} \cdot \frac{1}{(1-7^{-s})} \cdot \frac{1}{(1-11^{-s})} \cdots \frac{1}{(1-p^{-s})} \cdots$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \tag{2}$$

Eq. (2) alternating harmonic series Dirichlet eta function $\eta(s)$ that faithfully generates all three types of Gram points as three dependent CIS-IM-linear Incompletely Predictable entities when $\sigma = \frac{1}{2}$ must act as *proxy* function for Eq. (1) non-alternating harmonic series Riemann zeta function $\zeta(s)$ in critical strip ($0 < \sigma < 1$) containing critical line ($\sigma = \frac{1}{2}$) because $\zeta(s)$ only converges when $\sigma > 1$. This implies $\zeta(s)$ is undefined to left of $\sigma > 1$ region in the critical strip which then requires $\eta(s)$ representation. They are related to each other as $\zeta(s) = \gamma \cdot \eta(s)$ or equivalently as $\eta(s) = \frac{1}{\gamma} \cdot \zeta(s)$ with proportionality factor $\gamma = \frac{1}{(1-2^{1-s})}$.

$$\zeta(s) = 2^s \pi^{s-1} \ \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s) \tag{3}$$

$\zeta(s)$ satisfies Eq. (3) as functional equation, whereby $\Gamma(s)$ is the gamma function. As an equality of meromorphic functions valid on whole complex plane, Eq. (3) relates values of $\zeta(s)$ at points s and $1-s$; in particular, it relates even positive integers with odd negative integers. Owing to the zeros of sine function, the functional equation implies that $\zeta(s)$ has a simple zero at each even negative integer $s = -2n = -2, -4, -6, -8, -10...$, known as trivial zeros of $\zeta(s)$. When s is an even positive integer, the product $\sin(\frac{\pi s}{2})\Gamma(1-s)$ on the right is non-zero because $\Gamma(1-s)$ has a simple pole, which cancels the simple zero of the sine factor.

4.1 Mirror symmetry and Law of continuity

Mirror symmetry is a state's geometrical property that a point of symmetry, an axis (line) of symmetry or a plane of symmetry will split the state (a line, figure or object) in half; whereby these halves as image and mirror image are identical to, or indistinguishable from, each other. We define Law of continuity as a heuristic principle that *"whatever succeeds for the finite [local state], also succeeds for the infinite [general state]"*, whereby locally and generally there must not be a break in the state (a line or figure or object), and nothing passes from one state to another without passing through all the intermediate states.

Remark 3. *Figure 2 [regarding positive & negative primes and composites] and Figure 3 [regarding Co-linear Riemann zeta function for positive & negative range] manifest perfect Mirror symmetry and fully comply with Law of Continuity. The following are valid comments:*

Whereas the continuous-like equation Riemann zeta function $\zeta(s)$ Eq. (1) [via *proxy* Dirichlet eta function $\eta(s)$ Eq. (2)] for $s = \sigma \pm t$ range that generate mutually exclusive CIS-IM-linear σ-valued co-lines are mathematically regarded as smoothly *continuous everywhere*

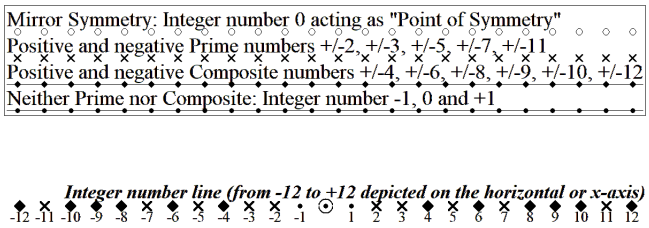

Fig. 2 Narrow range of positive & negative prime and composite numbers plotted together on integer number line generated using Sieve-of-Eratosthenes and complement-Sieve-of-Eratossthenes. The combined [positive] image and [negative] mirror image will conceptually represent a one-dimensional line (state) having perfect Mirror symmetry with integer number 0 acting as the Point of symmetry.

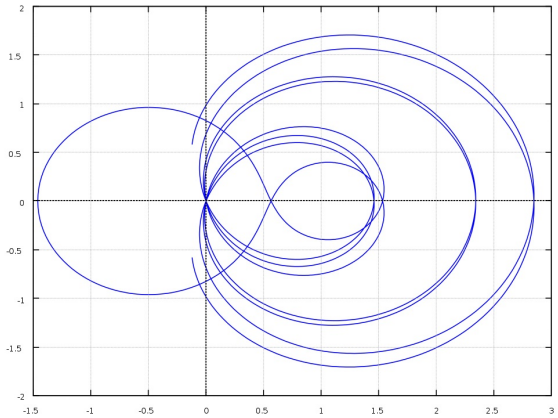

Fig. 3 OUTPUT for $\sigma = \frac{1}{2}$ as Gram points. Polar graph of $\zeta(\frac{1}{2}+\imath t)$ depicted as a two-dimensional figure plotted along critical line for real values of t running between –30 and +30 [viz, for $s = \sigma \pm t$ range], horizontal axis: $Re\{\zeta(\frac{1}{2}+\imath t)\}$, and vertical axis: $Im\{\zeta(\frac{1}{2}+\imath t)\}$. Total presence of all Origin intercept points at the Origin. There is manifestation of perfect Mirror symmetry about the horizontal x-axis acting as the line of symmetry in this figure (state).

thus obeying Law of continuity; so must the discrete-like algorithms Sieve-of-Eratosthenes and Complement-Sieve-of-Eratosthenes that generate mutually exclusive Primes and Composites be conceptually regarded as jaggedly *continuous everywhere* thus also obeying Law of continuity. CIS-ALN-decelerating Primes and CIS-IM-accelerating Composites are dependent and complementary entities. In $\zeta(s)$ Eq. (1), equivalent Euler product formula with product over prime numbers [instead of summation over natural numbers] also represents $\zeta(s) \implies$ all primes and, by default, composites are intrinsically encoded in $\zeta(s)$. Since via

analytic continuation, $\eta(s) = \frac{1}{\gamma} \cdot \zeta(s)$ [*proxy* function for $\zeta(s)$ in $0 < \sigma < 1$- critical strip]; then all primes and, by default, composites are also intrinsically encoded in $\eta(s)$ Eq. (2).

The involved states that manifest Mirror symmetry and obey Law of continuity are: (i) For $i = 1, 2, 3, 4, 5...$; positive (+ve) and negative (–ve) primes derived from $P_{i+1} = P_i \pm PGap_i$ with $P_1 = \pm 2$ [symbolizing Sieve-of-Eratosthenes] and +ve and –ve composites derived from $C_{i+1} = C_i \pm CGap_i$ with $C_1 = \pm 4$ [symbolizing Complement-Sieve-of-Eratosthenes], and (ii) never-ending supply of co-linear lines (with range $-\infty < t < +\infty$ giving rise to $s = \sigma \pm t$) from mutually exclusive and independent equations of Riemann zeta function [via *proxy* Dirichlet eta function] having solitary $\sigma = \frac{1}{2}$ value and infinitely-many $\sigma \neq \frac{1}{2}$ values.

CIS-IM-linear Integers {0, ±1, ±2, ±3, ±4, ±5...} = **CFS** {0, ±1} + **CIS-ALN-decelerating** Prime numbers {±2, ±3, ±5, ±7, ±11...} + **CIS-IM-accelerating** Composite numbers {±4, ±6, ±8, ±9, ±10...}. We validly ignore even Prime number ±2. The two discrete-like algorithms that generate unique Set (all +ve and –ve odd primes) and Set (all +ve and –ve composites) can be further subdivided into various sub-algorithms that generate unique Subsets (+ve and –ve odd primes) and Subsets (+ve and –ve composites). These (sub)sets listed under Remark 1 in subsection 1.1 [depicted as +ve entities] must belong to either CIS-ALN-decelerating or CIS-IM-accelerating. They contain primes and composites [as mutually exclusive and dependent discrete zero-dimensional (0-D) entities] characterized in a perpetual manner under +ve integer part or image and –ve integer counterpart or mirror image.

Via *proxy* Dirichlet eta function as its analytic continuation, Riemann zeta function is a continuous-like equation forming an infinite number of continuous 1-D co-linear lines as specified by their corresponding endowed σ values. For $-\infty < t < +\infty$ (constituting $0 < t < +\infty$ part or image and $-\infty < t < 0$ counterpart or mirror image), the generated solitary $\sigma = \frac{1}{2}$-specified co-linear line contains three mutually exclusive and dependent discrete 0-D entities of nontrivial zeros, Gram[y=0] points and Gram[x=0] points which are all given by variable t having ± transcendental number values. Similarly for $-\infty < t < +\infty$ (constituting $0 < t < +\infty$ part or image and $-\infty < t < 0$ counterpart or mirror image), the generated infinitely many $\sigma \neq \frac{1}{2}$-specified co-linear lines will each contains two mutually exclusive and dependent discrete 0-D entities of virtual Gram[y=0] points and virtual Gram[x=0] points which are all given by variable t having ± transcendental number values.

Major differentiating property: Our two algorithms that generate discrete zero-dimensional (0-D) primes and composites when plotted as connected [jagged] lines are only defined at two end-points a,b but not for interval [a,b] whereas our functions with different σ values that generate infinitely-many continuous [smooth] one-dimensional (1-D) co-linear lines are defined at two end-points a,b as well as for interval [a,b]. The algorithms and functions on interval $(-\infty, +\infty)$ as integer number line and real number number line are individually classified as *well-defined jagged continuous algorithm and well-defined smooth continuous function* whose definition assigns it a unique interpretation or value. Respectively, they are conceptually and mathematically *continuous everywhere*.

To colloquially interpret primes or composites never having any discontinuity, we say there must [theoretically] never be any existing *"hidden"* primes or composites yet-to-be-discovered in between any two (consecutive) primes or composites when [faithfully] generated by their *jagged* continuous algorithms. One just have to consider either +ve or

–ve primes, composites and t-valued nontrivial zeros when proving their associated open problems with Mathematical-Geometrical interpretations on Law of continuity given below.

A continuous-like equation such as Riemann zeta function (via its *proxy* Dirichlet eta function) with designated σ value obtained from anywhere in the $0 < \sigma < 1$ critical strip [that is bisected by the $\sigma = \frac{1}{2}$ critical line] is **smoothly** defined by the following properties:
1. Function $f(x)$ is defined at point $x = a$ and is smoothly continuous at that point.
2. The limit of function $f(x)$ should be smoothly defined at point $x = a$.
3. The value of function $f(x)$ at that point, i.e. $f(a)$ must smoothly equal the value of the limit of $f(x)$ at $x = a$.
The co-linear line in Figure 3 with all Gram points [and intermittent occurrences of less frequent violations (failures) of Gram's Law and much less frequent violations (failures) of Rosser's Rule] is (smoothly) continuous everywhere [viz, continuous locally and generally].

Analogically, a discrete-like algorithm such as Sieve-of-Eratosthenes or Complement-Sieve-of-Eratosthenes is **jaggedly** defined by the following properties:
1. Algorithm $A(x)$ is defined at point $x = a$ and is jaggedly continuous at that point.
2. The limit of algorithm $A(x)$ should be jaggedly defined at point $x = a$.
3. The value of algorithm $A(x)$ at that point, i.e. $A(a)$ must jaggedly equal the value of the limit of $A(x)$ at $x = a$.
The two algorithms that generate all +ve and –ve primes or composites are (jaggedly) continuous everywhere [viz, continuous locally and generally] as observed in Figure 2.

5 Prime-Composite identifier grouping and Co-linear Riemann zeta function

Let E = even numbers, O = odd numbers, $\mathbb{P}$ = prime numbers, even Prime gap_i = O-$\mathbb{P}_{i+1}$ – O-$\mathbb{P}_i$ = 2, 4, 6, 8, 10, 12..., Composite gap_i = $\mathbb{C}_{i+1} - \mathbb{C}_i$ = 1, 2. For even Prime gaps 4, 6, 8, 10, 12..., we can generate the orderly consecutive numbers as sequence {Gap 2-E-$\mathbb{C}_1$, O-$\mathbb{P}_i$, Gap 1-E-$\mathbb{C}_2$, Gap 1-O-$\mathbb{C}_3$, Gap 1-E-$\mathbb{C}_4$, Gap 1-O-$\mathbb{C}_5$,..., Gap 1-E-$\mathbb{C}_{n-2}$, Gap 1-O-$\mathbb{C}_{n-1}$, Gap 2-E-$\mathbb{C}_n$, O-$\mathbb{P}_{i+1}$}. The cardinality of sub-sequence {Gap 1-E-$\mathbb{C}_2$, Gap 1-O-$\mathbb{C}_3$, Gap 1-E-$\mathbb{C}_4$, Gap 1-O-$\mathbb{C}_5$,..., Gap 1-E-$\mathbb{C}_{n-2}$, Gap 1-O-$\mathbb{C}_{n-1}$} = even Prime gap_i – 2 = n – 2. However for twin primes; this sub-sequence [as an empty set or null set] do not exist with its cardinality = 0 since even Prime gap 2 – 2 = 0. With cardinality of this sub-sequence given by the involved even Prime gap minus 2; we conveniently define $\mathbb{P}$-$\mathbb{C}$ **identifier grouping** as Gap 2-E-$\mathbb{C}_1$, O-$\mathbb{P}_i$, Gap 1-E-$\mathbb{C}_2$, Gap 1-O-$\mathbb{C}_3$, Gap 1-E-$\mathbb{C}_4$, Gap 1-O-$\mathbb{C}_5$,..., Gap 1-E-$\mathbb{C}_{n-2}$, Gap 1-O-$\mathbb{C}_{n-1}$ for Arbitrarily Large Number of even Prime gaps 4, 6, 8, 10, 12... with caveat $\mathbb{P}$-$\mathbb{C}$ **identifier grouping** for even Prime gap 2 is an exception given by Gap 2-E-$\mathbb{C}_1$, O-$\mathbb{P}_i$. ***For ("different") n = 1, 2, 3, 4, 5...; [decelerating] size of equally distributed Gap 2n-O-*** $\mathbb{P}$ ***and Gap 2-E-*** $\mathbb{C}$ ***is inversely proportional to [accelerating] size of equally distributed Gap 1-E-*** $\mathbb{C}$ ***and Gap 1-O-*** $\mathbb{C}$. Gap 2-E-$\mathbb{C}_n$ is now acting as the new Gap 2-E-$\mathbb{C}_1$ for O-$\mathbb{P}_{i+1}$ in the following perpetually repeating cycles of O-$\mathbb{P}_i$ to O-$\mathbb{P}_{i+1}$ with a [usually] different even Prime gap_i [except for rare recurring cases of two or more consecutive O-$\mathbb{P}$ having two or more identical consecutive even Prime gaps involving 6 and multiples of 6].

We simply have no choice but to accept *There is zero probability that appearances of* $\mathbb{P}$-$\mathbb{C}$ ***identifier grouping*** *when computed as Cardinality 0 for Gap 2-Twin primes, Cardinality 2 for Gap 4-Cousin primes, Cardinality 4 for Gap 6-Sexy primes, etc should ever stop or terminate*

in a discriminatory manner over the large range of integer numbers, thus confirming Modified Polignac's and Twin prime conjectures. For Riemann zeta function via *proxy* Dirichlet eta function, we simply have no choice but to accept the **solitary $\sigma = \frac{1}{2}$-critical line connection** with all nontrivial zeros thus confirming Riemann hypothesis.

Useful deductions regarding prime numbers: For n = 1, 2, 3, 4, 5...; CIS-ALN-decelerating Gap 2n-O-$\mathbb{P}$ + CIS-ALN-decelerating Gap 2-E-$\mathbb{C}$ is inversely proportional to CIS-IM-accelerating Gap 1-E-$\mathbb{C}$ + CIS-IM-accelerating Gap 1-O-$\mathbb{C}$. The Arbitrarily Large Number of CIS-ALN-decelerating Gap 2-O-$\mathbb{P}$, CIS-ALN-decelerating Gap 4-O-$\mathbb{P}$, CIS-ALN-decelerating Gap 6-O-$\mathbb{P}$... must all constitute valid subsets of odd prime numbers. Law of continuity aesthetically implies each and every Arbitrarily Large Number of even Prime gaps 2, 4, 6, 8, 10, 12... must repeatedly exist *without discontinuities* [albeit not always appearing as first occurrences of the relevant associated odd prime numbers and thus not always complying with the prescribed naturally occurring ascending order for even numbers].

Remark 4. *To confirm algorithm and sub-algorithms from Sieve-of-Eratosthenes must all be Incompletely Predictable and of Infinite Length, we refer to the equally distributed Gap 1-E-$\mathbb{C}$ and Gap 1-O-$\mathbb{C}$ existing as (i) recurring sets with varying different cardinality of 2, 4, 6, 8, 10... that correspond to even Prime gaps 4, 6, 8, 10, 12... and (ii) recurring null sets with non-varying same cardinality of 0 that correspond to even Prime gap 2. There is zero probability that any of these sets will [discriminatorily] become countably finite set for some particular even Prime gap(s). We rephrase the relevant parts from section 2 below.*

Incompletely Predictable complex Sieve-of-Eratosthenes (sub)algorithms [as Σ(Gap 2n-Sieve-of-Eratosthenes algorithm) = Gap 2-Sieve-of-Eratosthenes sub-algorithm + Gap 4-Sieve-of-Eratosthenes sub-algorithm + Gap 6-Sieve-of-Eratosthenes sub-algorithm +... for n = 1, 2, 3, 4, 5...] will faithfully generate all ALN of Odd Primes. Applying **Proof by Contradiction** to theoretical situation of, for instance, Modified Twin prime conjecture being false; one would [falsely] contends the Gap 2-Sieve-of-Eratosthenes sub-algorithm [= Σ(Gap 2n-Sieve-of-Eratosthenes algorithm) – Gap 4-Sieve-of-Eratosthenes sub-algorithm – Gap 6-Sieve-of-Eratosthenes sub-algorithm –... for n = 1, 2, 3, 4, 5...] can only generate a CFS of twin primes. By logical deduction, this Gap 2-Sieve-of-Eratosthenes sub-algorithm is then strictly regarded [incorrectly] as a simple Completely Predictable sub-algorithm having Finite-Length instead of [correctly] as a complex Incompletely Predictable sub-algorithm having Infinite-Length. By logical contradiction, Modified Twin prime conjecture is then consequently true in that there must be an Arbitrarily Large Number of twin primes. *Ditto* for all other remaining Gap 2n-Sieve-of-Eratosthenes sub-algorithms derived from n = 2, 3, 4, 5, 6... by employing the above similar line of arguments to reach the same conclusion.

Useful deductions regarding nontrivial zeros: For a given function (equation) $y = f(x)$, there may be no geometrical symmetry in the given equation whereby this equation *may or may not* intercept the Origin point; or there may be one or more geometrical symmetry in the given equation about the X-axis, Y-axis, Diagonal, or Origin point whereby this equation *may or may not* intercept the Origin point. For a given equation, these types of symmetry can be correspondingly tested by replacing y with $-y$, x with $-x$, both y with x and x with y, or both x with $-x$ and y with $-y$.

With Dirichlet eta function acting as *proxy* function for Riemann zeta function, we deduce using first principle that the infinitely many ***t*-valued** Origin intercept points [which faithfully represents all ***t*-valued** nontrivial zeros] of Riemann zeta function will only be generated when its parameter $\sigma = \frac{1}{2}$ [which represents the solitary critical line] but not when its parameter $\sigma \neq \frac{1}{2}$ [which represents the infinitely many non-critical lines]. This is notwithstanding another simple deduction that Riemann zeta functions when endowed with any σ values between 0 and 1 [viz, in $0 < \sigma < 1$ critical strip which is bisected by $\sigma = \frac{1}{2}$ critical line into two regions $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$] will always behave mathematically as **different independent co-linear equations** [all *without geometrical symmetry* if we only consider the range of either $0 < t < +\infty$ or $-\infty < t < 0$] whereby they generate mutually exclusive co-linear lines that geometrically never cross over one another.

$$\sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}2^{\frac{1}{2}}\cos(t\ln(2n)+\frac{1}{4}\pi)-\sum_{n=1}^{\infty}(2n-1)^{-\frac{1}{2}}2^{\frac{1}{2}}\cos(t\ln(2n-1)+\frac{1}{4}\pi)=0 \tag{4}$$

$$\sum_{n=1}^{\infty}(2n)^{-\frac{2}{5}}2^{\frac{1}{2}}\cos(t\ln(2n)+\frac{1}{4}\pi)-\sum_{n=1}^{\infty}(2n-1)^{-\frac{2}{5}}2^{\frac{1}{2}}\cos(t\ln(2n-1)+\frac{1}{4}\pi)=0 \tag{5}$$

Euler formula can be stated as $e^{\imath n} = \cos n + \imath \cdot \sin n$. Applying this formula to f(n) η(s) will result f(n) simplified η(s). Eq. (4) is f(n) simplified η(s) at $\sigma = \frac{1}{2}$ that will incorporate all nontrivial zeros [as Zeroes]. There is total absence of (non-existent) virtual nontrivial zeros [as virtual Zeroes]. Eq. (5) is f(n) simplified η(s) at $\sigma = \frac{2}{5}$ that will incorporate all (non-existent) virtual nontrivial zeros [as virtual Zeroes]. There is total absence of nontrivial zeros [as Zeroes]. Upon inspecting Eq. (4) that manifest *exact* Dimensional analysis homogeneity when $\sigma = \frac{1}{2}$ whereby Σ(all fractional exponents) = $2(-\sigma)$ = exact negative whole number of –1 [as opposed to Eq. (5) that manifest *inexact* Dimensional analysis homogeneity when $\sigma = \frac{2}{5}$ whereby Σ(all fractional exponents) = $2(-\sigma)$ = inexact negative fractional number of $-\frac{4}{5}$]; we deduce only Dirichlet eta function containing parameter $\sigma = \frac{1}{2}$ will mathematically depict the [optimal] "formula symmetry" on Σ(all fractional exponents) as an exact negative whole number. This formula symmetry is not equivalent to geometrical symmetry about the X-axis, Y-axis, Diagonal, or Origin point that do not exist for any Dirichlet eta function when only considered for either $-\infty < t < 0$ or $0 < t < +\infty$. With full range of t variable being $-\infty < t < +\infty$ whereby we conventionally adopt positive range $0 < t < +\infty$, this simple observation confirms only $\sigma = \frac{1}{2}$-Dirichlet eta function will perpetually and geometrically intercept Origin point as Origin intercept points (i.e. will perpetually and mathematically lie on critical line as nontrivial zeros) an infinite number of times.

Remark 5. *Logical reasoning to analogically confirm statements on "formula symmetry":*

The **co-linear mathematical equations** having geometrical symmetry $y = a\cos(x)^b - c\cos(x)^d$ = zero [≡ solitary Origin intercept point at the Origin point] occur only when the unique solitary $a = c$ condition is met. This represents the **Proposition**: Optimal "formula symmetry" at solitary unique $\sigma = \frac{1}{2}$-critical line will always produce nontrivial zeros as Origin intercept points. Then, $y = a\cos(x)^b - c\cos(x)^d \neq$ zero [≡ nil Origin intercept point

at the Origin point] occur when the non-unique multiple $a \neq c$ conditions are met. These represent the **Corollary**: Nil optimal "formula symmetry" at infinitely many non-unique $\sigma \neq \frac{1}{2}$-noncritical lines will never produce nontrivial zeros as Origin intercept points. We notice that it is immaterial whether the $b = d$ or $b \neq d$ conditions are met for both situations.

We also deduce that occurrences of infinitely many violations (failures) of Gram's Law and Rosser's Rule resulting in altered appearances of Gram points [w.r.t. nontrivial zeros] in $\sigma = \frac{1}{2}$-Dirichlet eta function do not contradict the above findings in Remark 5 since any possible solutions for $\sigma \neq \frac{1}{2}$-Dirichlet eta functions as true x-axis intercept points [w.r.t. true Origin intercept points] or as true Gram points [w.r.t. true nontrivial zeros] is a geometrical or mathematical impossibility. Here, the term *Gram points* denote Gram[y=0] points (or x-axis intercept points). From Appendix A, we note Gram's Law is the tendency for nontrivial zeros of Riemann-Siegel function $Z(t)$ to alternate with Gram[y=0] points when $\sigma = \frac{1}{2}$. The first violation (failure) of Gram's Law occurs at n = 126. Rosser's Rule states that every Gram block contains the expected number of roots as Gram[y=0] points when $\sigma = \frac{1}{2}$. The first violation (failure) of Rosser's Rule occurs at the much larger n = 13999525.

Remark 6. *Incorporating classification of countably infinite sets into three subtypes, we outline the simple and complex properties manifested by Completely Predictable and Incompletely Predictable entities:*

As an example of simple property, x-axis intercept points for simple function *sin* n are Completely Predictable to "linearly" occur infinitely many times when n = all positive and negative multiples of π. Here are examples of complex properties: As stated by Gram's Law, x-axis intercept points for complex function Riemann-Siegel function $Z(t)$ or Riemann zeta function [via its proxy Dirichlet eta function] "linearly" occur infinitely many times as Incompletely Predictable t-values that represent usual positive Gram[y=0] points which tend to alternate with nontrivial zeros. As unique Incompletely Predictable events "linearly" occurring infinitely many times, there are intermittent observable *various geometric variants* of two consecutive (positive first and then negative) Gram[y=0] points that is alternatingly followed by two consecutive nontrivial zeros. These events denote violations (failures) of Gram's Law. Violations (failures) of Rosser's Rule refer to the much less frequent intermittently occurring Incompletely Predictable observable *various geometric variants* of reduction in expected number of t-values for certain x-axis intercept points. "Linearly" occurring infinitely many times, each of these events gives rise to two missing Gram[y=0] points or, equivalently, to two extra nontrivial zeros. *Plus Gap 2 Composite Number Continuous Law* and *Plus-Minus Gap 2 Composite Number Alternating Law*[26] outlined in section 8 are two overall Incompletely Predictable properties seen when we dependently combine [deceleratingly-occurring] primes and [acceleratingly-occurring] composites with associated prime gaps and composite gaps for critical analysis.

Occurring over 2000 years ago (c. 300 BC), ancient Euclid's theorem on infinitude of prime numbers using *reductio ad absurdum* (proof by contradiction) is earliest known but not the only proof for this simple problem. Since then dozens of proofs have been devised such as three chronologically listed: Goldbach's Proof using Fermat numbers (written in a letter

to Swiss mathematician Leonhard Euler, July 1730), Furstenberg's Topological Proof[6], and Filip Saidak's Proof[20]. The strangest candidate is Furstenberg's Topological Proof.

In 2013, Yitang Zhang proved a landmark result showing some unknown even number $N < 70$ million such that this condition holds: There are CIS-ALN-decelerating Odd Primes that differ by N between each other[32]. By optimizing Zhang's bound, subsequent Polymath Project collaborative efforts using a new refinement of GPY sieve in 2014 lowered N to 246; and assuming Elliott-Halberstam conjecture and its generalized form further lower N to 12 and 6, respectively. Intuitively, N has more than one valid values such that the same condition holds for each N value. With different methods, we can at most lower N to 2 and 4 in regards to Odd Primes having small prime gaps 2 & 4 with each uniquely generating CIS-ALN-decelerating Odd Primes. We anticipate there are all remaining prime gaps in regards to Odd Primes with large prime gaps ≥ 6 that are denoted by corresponding $N \geq 6$ values whereby each large prime gap will generate its own unique CIS-ALN-decelerating Odd Primes.

5.1 Inverse functions of ln(x) with e(x) and li(x) with Ei(x)

We start with the conditional statement "If P, then Q" which is notated as P $\rightarrow$ Q. The converse of the conditional statement is "If Q, then P" which is notated as Q $\rightarrow$ P. The contrapositive of the conditional statement is "If not Q, then not P" which is notated as ~Q $\rightarrow$ ~P. The inverse of the conditional statement is "If not P then not Q" which is notated as ~P $\rightarrow$ ~Q. An inverse function (or anti-function) is a function that "reverses" another function: if the function f applied to an input x gives a result of y, then applying its inverse function g to y gives the result x, i.e., $g(y) = x$ if and only if $f(x) = y$. Not all functions have an inverse. The inverse function of f is also denoted as f^{-1} and it exists if and only if f is bijective. Since a function is a special type of binary relation, many of the properties of an inverse function correspond to following three properties of converse relations:

(i) Uniqueness. If an inverse function exists for a given function f, then it is unique. This follows since the inverse function must be the converse relation, which is completely determined by f.

(ii) Symmetry. There is a symmetry between a function and its inverse. Specifically, if f is an invertible function with domain X and codomain Y, then its inverse f^{-1} has domain Y and image X, and the inverse of f^{-1} is the original function f. In symbols, for functions f: X $\rightarrow$ Y and f^{-1}: Y $\rightarrow$ X, $f^{-1} \circ f = \mathrm{id}_X$ and $f \circ f^{-1} = \mathrm{id}_Y$. This statement is a consequence of the implication that for f to be invertible it must be bijective.

(iii) Self-inverses. If X is a set, then the identity function on X is its own inverse: ${\mathrm{id}_X}^{-1} = \mathrm{id}_X$. More generally, a function $f : X \rightarrow X$ is equal to its own inverse, if and only if the composition $f \circ f$ is equal to id_X. Such a function is called an involution. The involutory nature of inverse can be concisely expressed by $\left(f^{-1}\right)^{-1} = f$. The inverse of $g \circ f$ is $\left(f^{-1}\right) \circ \left(g^{-1}\right)$. The inverse of a composition of functions is given by $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Notice that the order of g and f have been reversed; to undo f followed by g, we must first undo g, and then undo f. For a function $f : X \rightarrow Y$, its inverse $f^{-1} : Y \rightarrow X$ admits an explicit description: it sends each element $y \in Y$ to the unique element $x \in X$ such that $f(x) = y$.

As the base of natural logarithm, irrational (transcendental) number e is a mathematical constant approximately equal to 2.71828. It is the limit of $(1 + \frac{1}{n})^n$ as n approaches ∞ and can

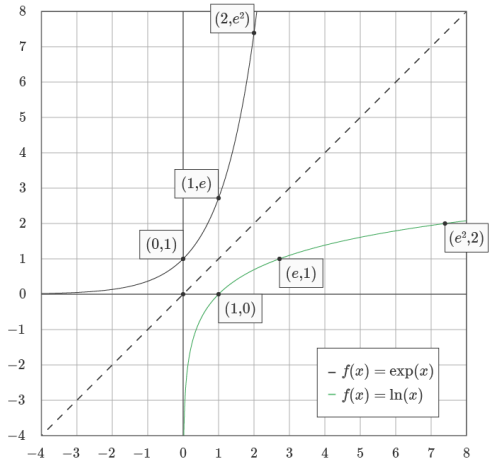

Fig. 4 The natural logarithm function $log_e x$ or $ln(x)$ and natural exponential function exp(x) or e^x. The graphs of $log_e x$ and its inverse e^x are symmetric with respect to line y = x thus geometrically denoting diagonal symmetry of these two functions.

also be calculated as sum of infinite series $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$. We outline the important properties of natural logarithm function and natural exponential function with their connections to logarithmic integral function and exponential integral function as inverse or pseudo-inverse functions. The natural logarithm function, if considered as a real-valued function of a positive real variable, is the inverse of exponential function, leading to the following identities: $e^{\ln x} = x$ if x is strictly positive, $\ln e^x = x$ if x is any real number.

As shown in Figure 4, the natural logarithm ln(x) has a vertical asymptote of x = 0 [y-axis] as x approaches 0 [with ln(x) becoming $-\infty$]. Its inverse function e(x) has a horizontal asymptote of y = 0 [x-axis] as x approaches $-\infty$ [with e(x) becoming 0]. With the slope of horizontal line being 0, and the slope of vertical line being an undefined value; we recognize the slope of ln(x) becomes an infinitesimal small number ($+\frac{1}{\infty}$) that approaches 0 as x grows towards ∞ but the slope of its inverse function e(x) becomes an infintely large number ($+\infty$) that approaches an undefined value as x grows towards ∞. Extrapolations: **(1) Slope for Prevalence of all Primes as fraction of all integers.** We recognize the slope of ln(x) can symbolically denote, for instance, the (decelerating) slope for Prevalence of all Primes will approach 0 as an infinitesimal small number value [but never becomes 0] as x grows towards ∞. **(2) Slope for Prevalence of all Composites as fraction of all integers.** Similarly, the slope of e(x) can symbolically denote, for instance, the (accelerating) slope for Prevalence of all Composites will approach an undefined value as an infinite large number value [but never becomes an undefined value] as x grows towards ∞.

Like all logarithms, the natural logarithm maps multiplication of positive numbers into addition: $\ln(x \cdot y) = \ln x + \ln y$. Logarithms can be defined for any positive base other than 1, not only e. However, logarithms in other bases differ only by a constant multiplier from the natural logarithm, and can be defined in terms of the latter, $\log_b x = \ln x / \ln b = \ln x \cdot \log_b e$.

Their properties are $\ln 1 = 0$; $\ln e = 1$; $\ln(xy) = \ln x + \ln y$ for $x > 0$ and $y > 0$; $\ln(x/y) = \ln x - \ln y$; $\ln(x^y) = y \ln x$ for $x > 0$; $\ln x < \ln y$ for $0 < x < y$; $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$; $\lim_{\alpha \to 0} \frac{x^\alpha - 1}{\alpha} = \ln x$ for $x > 0$; $\frac{x-1}{x} \leq \ln x \leq x - 1$ for $x > 0$; $\ln(1 + x^\alpha) \leq \alpha x$ for $x \geq 0$ and $\alpha \geq 1$.

The real natural exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ can be characterized in a variety of equivalent ways. It is commonly defined by Taylor series $\exp x := \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!} + \cdots$. By way of binomial theorem and power series definition, the exponential function can also be defined as the limit: $\exp x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$. It can be shown that every continuous, nonzero solution of the functional equation $f(x + y) = f(x)f(y)$ is an exponential function, $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto e^{kx}$, with $k \in \mathbb{R}$. The exponential function satisfies exponentiation identity $e^{x+y} = e^x e^y$ for all $x, y \in \mathbb{R}$, which, along with definition $e = \exp(1)$, shows that factors $e^n = \underbrace{e \times \cdots \times e}_{n \text{ factors}}$ for positive integers n, and relates exponential function to the elementary notion of exponentiation. The base of ***natural*** exponential function, its value at 1, $e = \exp(1)$ is a ubiquitous mathematical constant called Euler's number approximately equal to 2.71828 – this number also acts as base of ***natural*** logarithm function.

The [analogical] logarithmic integral function $\mathrm{li}(x)$ is defined as $\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t}$. The function $1/(\ln t)$ has a singularity at $t = 1$, and the integral for $x > 1$ is interpreted as a Cauchy principal value, $\mathrm{li}(x) = \lim_{\varepsilon \to 0+} \left(\int_0^{1-\varepsilon} \frac{dt}{\ln t} + \int_{1+\varepsilon}^{x} \frac{dt}{\ln t} \right)$. The $\mathrm{li}(x)$ function is related to its inverse exponential integral function $\mathrm{Ei}(x)$ via equation $\mathrm{li}(x) = \mathrm{Ei}(\ln x)$, and is valid for $x > 0$. This identity provides a series representation of $\mathrm{li}(x)$ as $\mathrm{li}(e^u) = \mathrm{Ei}(u) = \gamma + \ln|u| + \sum_{n=1}^{\infty} \frac{u^n}{n \cdot n!}$ for $u \neq 0$, where $\gamma \approx 0.57721\ 56649\ 01532...$ is the Euler-Mascheroni constant. A more rapidly convergent series by Ramanujan is $\mathrm{li}(x) = \gamma + \ln\ln x + \sqrt{x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (\ln x)^n}{n!\, 2^{n-1}} \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2k+1}$. The asymptotic behavior for $x \to \infty$ is $\mathrm{li}(x) = O\left(\frac{x}{\ln x}\right)$ where O is big O notation. The full asymptotic expansion is $\mathrm{li}(x) \sim \frac{x}{\ln x} \sum_{k=0}^{\infty} \frac{k!}{(\ln x)^k}$ or $\frac{\mathrm{li}(x)}{x/\ln x} \sim 1 + \frac{1}{\ln x} + \frac{2}{(\ln x)^2} + \frac{6}{(\ln x)^3} + \cdots$. This gives more accurate asymptotic behaviour: $\mathrm{li}(x) - \frac{x}{\ln x} = O\left(\frac{x}{(\ln x)^2}\right)$.

In perspective, the [100% accurate] perfect Prime-$\pi(x)$ stepped-mathematical function being *wrapped around* by [less-than-100% accurate] approximate li(x) smooth-mathematical function infinitely many times via this *sign of difference* changes implies li(x) is the most efficient approximate mathematical function. Contrast this with the *crude* [less-than-100% accurate] approximate $\frac{x}{\ln x}$ smooth-mathematical function whereby studied values diverge away from Prime-$\pi(x)$ at increasingly greater rate for larger range of prime numbers.

6 Prime number theorem and Composite number theorem with Prime-Composite quotient

A number base, consisting of any whole number greater than 0, is the number of digits or combination of digits that a number system uses to represent numbers e.g. decimal number system or base 10, binary number system or base 2, octal number system or base 8, hexadecimal number system or base 16. As $x \to \infty$, various derived properties of Prime counting function, Prime-$\pi(x)$ [= number of primes up to x] occur in, for instance, Prime number theorem for Arithmetic Progressions, Prime-$\pi(x; b, a)$ [= number of primes up to x with last digit of primes given by a in base b]. For any choice of digit a in base b with gcd(a,b) = 1: Prime-$\pi(x; b, a) \sim \frac{\text{Prime-}\pi(x)}{\phi(b)}$. Here, Euler's totient function $\phi(n)$ is defined as the number of positive integers $\leq n$ that are relatively prime to (i.e., do not contain any factor in common with) n, where 1 is counted as being relatively prime to all numbers. Then each of the last digit of primes given by digit a in base b as $x \to \infty$ is equally distributed between the permitted choices for digit a with this result being valid for, and is independent of, any chosen base b.

Numbers with their last digit ending in (i) 1, 3, 7 or 9 [which can be either primes or composites] constitute ~40% of all integers; and (ii) 0, 2, 4, 5, 6 or 8 [which must be composites] constitute ~60% of all integers. We validly ignore the only single-digit even prime number 2 and odd prime number 5. We note ≥ 2-digit Odd Primes can only have their last digit ending in 1, 3, 7 or 9 but not in 0, 2, 4, 5, 6 or 8. These are given as the **complete List**:
The last digit of Odd Primes having their Prime gaps with last digit ending in 2 [viz, Gap 2, Gap 12, Gap 22, Gap 32...] can only be 1, 3 or 9 [but not (5) or 7] as three choices.
The last digit of Odd Primes having their Prime gaps with last digit ending in 4 [viz, Gap 4, Gap 14, Gap 24, Gap 34...] can only be 1, 3 or 7 [but not (5) or 9] as three choices.
The last digit of Odd Primes having their Prime gaps with last digit ending in 6 [viz, Gap 6, Gap 16, Gap 26, Gap 36...] can only be 3, 7 or 9 [but not (5) or 1] as three choices.
The last digit of Odd Primes having their Prime gaps with last digit ending in 8 [viz, Gap 8, Gap 18, Gap 28, Gap 38...] can only be 1, 7 or 9 [but not (5) or 3] as three choices.
The last digit of Odd Primes having their Prime gaps with last digit ending in 0 [viz, Gap 10, Gap 20, Gap 30, Gap 40...] can only be 1, 3, 7 or 9 [but not (5)] as four choices.

Axiom 1. ***Applications of Prime number theorem for Arithmetic Progressions confirm Modified Polignac's and Twin prime conjectures, and support the generalized and ordinary Riemann hypothesis.*** *Primes from various even Prime gaps are listed in section 2.*

Proof. We use decimal number system (base $b = 10$), and ignore the only single-digit even prime number 2 and odd prime number 5. For $i = 1, 2, 3, 4, 5...$; the last digit of all Gap 2i-Odd Primes can only end in 1, 3, 7 or 9 that are each proportionally and equally distributed as ~25% when $x \to \infty$, whereby this result is consistent with Prime number theorem for Arithmetic Progressions. The 100%-Set of, and its derived four unique 25%-Subsets of, Gap 2i-Odd Primes based on their last digit being 1, 3, 7 or 9 must all be CIS-ALN-decelerating. Since the ALN of Gap 2i are fully represented by all Prime gaps with last digit ending in 0, 2, 4, 6 or 8 that are associated with various permitted combinations of last digit in Gap 2i-Odd Primes being 1, 3, 7 and/or 9 as three or four choices [outlined above in **List** from preceding paragraph that complies with the well-known intrinsic property *"different Prime numbers literally equates to different Prime gaps"*], then these ALN unique subsets of Prime

gaps based on their last digit being 0, 2, 4, 6 or 8 with their correspondingly derived ALN unique subsets constituted by Gap 2i-Odd Primes having last digit 1, 3, 7 or 9 must also all be CIS-ALN-decelerating. With all Gap 2i-Odd Primes having their last digit as 1, 3, 7 or 9; the Probability (any Gap 2i abruptly terminating as $x \rightarrow \infty$) = Probability (any Gap 2i-Odd Primes abruptly terminating as $x \rightarrow \infty$) = 0. Thus Modified Polignac's and Twin prime conjectures is confirmed to be true. With the ordinary Riemann hypothesis being a special case, the generalized Riemann hypothesis formulated for Dirichlet L-function [subsection 2.1] holds once $x > b^2$, or base $b < x^{\frac{1}{2}}$ as $x \rightarrow \infty$. *The proof is now complete for Axiom 1*□.

All primes generated by Sieve-of-Eratosthenes algorithm and all composites generated by Complement-Sieve-of-Eratosthenes algorithm are mutually exclusive and complementary numbers. These two algorithms will act as pseudo-inverse algorithms for each other. Prime gaps P–$g_n = P_{n+1} - P_n$ and Composite gaps C–$g_n = C_{n+1} - C_n$. Prime gaps are constituted by CFS of P–g_n = 1 representing solitary even prime number {2}; and CIS-ALN-decelerating of P–g_n {2, 4, 6, 8, 10...} representing all CIS-ALN-decelerating odd prime numbers {3, 5, 7, 11, 13, 17, 19...}. Composite gaps are constituted by CFS of C–g_n = 1 representing all CIS-IM-accelerating odd composite numbers {9, 15, 21, 25, 27, 33, 35, 39...} and all CIS-IM-accelerating even composite numbers {8, 14, 20, 24, 26, 32, 34, 38, 44...} [that both occur together in between any two odd prime numbers specified by P–$g_n \geq 4$]; and C–g_n = 2 representing all CIS-ALN-decelerating even composite numbers [that precede all odd prime numbers] {4, 6, 10, 12, 16, 18, 22, 28, 30, 36, 40, 42...}.

One notice three *useful* facts: (i) the [only] solitary even prime number 2 do not have a preceding composite number since 1 is neither prime nor composite, (ii) the [recurring] CIS-ALN-decelerating even composite numbers following all twin primes {4, 6, 12, 18, 30, 42, 60, 72...} having C–g_n = 2 always represent the next even composite number that will precede the following odd prime number [with thus complete absence of both even composite numbers having C–g_n = 1 and odd composite numbers having C–g_n = 1 in between the two odd primes that specify the involved twin primes], and (iii) the [only] consecutive twin primes [with both having P–g_n = 2] that occur involves three consecutive odd prime numbers 3, 5 and 7; and are associated with the [only] two existing consecutive even composite numbers 4 and 6 [with both having C–g_n = 2]. Note: (i) Even number 2 is prime [and not a Gap 2-Even composite]. (ii) Each twin prime P_n is always associated with two consecutive Gap 2-Even composites that precede and follow this twin prime.

Combined Completely Predictable Even-Odd formula:
CIS-IM-linear Gap 1-integers {0, 1, 2, 3, 4, 5, 6...} = CIS-IM-linear Gap 2-even numbers {0, 2, 4, 6, 8, 10, 12...} + CIS-IM-linear Gap 2-odd numbers {1, 3, 5, 7, 11, 13...}
We deduce the independent functions (equations) $y = f(x) = 2x$ ***with its inverse function*** $y^{-1} = f^{-1}(x) = \frac{x}{2}$ ***that generate all Gap 2-even numbers and*** $y = f(x) = 2x - 1$ ***with its inverse function*** $y^{-1} = f^{-1}(x) = \frac{x+1}{2} = \frac{x}{2} + \frac{1}{2}$ ***that generate all Gap 2-odd numbers must act as two complementary and balanced functions [whereby these are pseudo-inverse functions of each other since the two inverse functions only differ by the constant*** $\frac{1}{2}$***].*** We compare this to the two dependent functions $y = e(x)$ and $y = ln(x)$ which are complementary and balanced inverse functions of each other.

Combined Incompletely Predictable Prime-Composite formulae:
CIS-IM-linear Gap 1-integers {0, 1, 2, 3, 4, 5, 6...} = CFS integers {0, 1} + CFS even prime number {2} + CIS-ALN-decelerating odd prime numbers {3, 5, 7, 11, 13, 17, 19...} + CIS-IM-accelerating composite numbers {4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20...}.
We deduce the dependent algorithms that generate all Odd Primes and all Composites must act as two complementary and balanced pseudo-inverse algorithms. We further deduce from previous analysis that the dependent paired algorithms that generate (i) {Odd Primes [but which does not include even Prime 2] + Gap 2-Even Composites [but which does not precede solitary Odd Prime 3]} and (ii) {Gap 1-Even Composite + Gap 1-Odd Composites} must also act as two complementary and balanced paired pseudo-inverse algorithms.

Prime-Composite quotient: Let $\mathbb{P}$ = Primes, $\mathbb{C}$ = Composites, $\mathbb{Z}$ = Integers. Based on Prime-$\pi(x)$ and Composite-$\pi(x)$ as $x \to \infty$, the $\mathbb{P}$-$\mathbb{C}$ quotient is derived from Conservation of Set $\mathbb{Z}$ = {0, 1} + Set $\mathbb{P}$ + Set $\mathbb{C}$, and Set Gap 2i-Odd $\mathbb{P}$ = Subset Gap 2-Odd $\mathbb{P}$ + Subset Gap 4-Odd $\mathbb{P}$ + Subset Gap 6-Odd $\mathbb{P}$ +.... Whereas the overall algorithm that generate all Odd $\mathbb{P}$ from even Prime gaps 2, 4, 6, 8, 10... is classified as CIS-ALN-decelerating; so must each and every sub-algorithms that generate Gap 2-Odd $\mathbb{P}$ from even Prime gap 2, Gap 4-Odd $\mathbb{P}$ from even Prime gap 4, Gap 6-Odd $\mathbb{P}$ from even Prime gap 6, etc be also classified as CIS-ALN-decelerating [and not be classified as CIS-IM-accelerating or CFS]. As defined below, the limit of $\mathbb{P}$-$\mathbb{C}$ **quotient** as x increases without bound is 0:

$$\lim_{x\to\infty} \frac{\text{CIS-ALN-decelerating (Gap } 2i\text{-Odd } \mathbb{P}) + \text{CIS-ALN-decelerating (Gap 2-Even } \mathbb{C})}{\text{CIS-IM-accelerating (Gap 1-Even } \mathbb{C}) + \text{CIS-IM-accelerating (Gap 1-Odd } \mathbb{C})} = 0$$

Using asymptotic notation, this [inversely proportional] quotient result can be restated as:

$$\text{CIS-ALN-decelerating (Gap } 2i\text{-Odd } \mathbb{P}) + \text{CIS-ALN-decelerating (Gap 2-Even } \mathbb{C}) \sim \frac{1}{\text{CIS-IM-accelerating (Gap 1-Even } \mathbb{C}) + \text{CIS-IM-accelerating (Gap 1-Odd } \mathbb{C})}$$

We reiterate Gap 1-Even $\mathbb{C}$ and Gap 1-Odd $\mathbb{C}$ are missing between all twin primes. Strictly, CIS-ALN-decelerating (Gap 2i-Odd $\mathbb{P}$) = CIS-ALN-decelerating (Gap 2-Even $\mathbb{C}$) + 1 whereby the even number 2 that precede first odd prime number 3 is prime and thus not a Gap 2-Even $\mathbb{C}$. Since CIS-ALN-decelerating (Gap 2i-Odd $\mathbb{P}$) = CIS-ALN-decelerating (Gap 2-Even $\mathbb{C}$) and CIS-IM-accelerating (Gap 1-Even $\mathbb{C}$) = CIS-IM-accelerating (Gap 1-Odd $\mathbb{C}$) is sufficiently accurate, then the following are also valid statements:

$$\text{CIS-ALN-decelerating (Gap } 2i\text{-Odd } \mathbb{P}) \text{ or CIS-ALN-decelerating (Gap 2-Even } \mathbb{C}) \sim \frac{1}{2 \cdot \text{CIS-IM-accelerating (Gap 1-Even } \mathbb{C}) + 2 \cdot \text{CIS-IM-accelerating (Gap 1-Odd } \mathbb{C})}$$

$$\sim \frac{1}{4 \cdot \text{CIS-IM-accelerating (Gap 1-Even } \mathbb{C})}$$

$$\sim \frac{1}{4 \cdot \text{CIS-IM-accelerating (Gap 1-Odd } \mathbb{C})}$$

Finally, two randomly selected consecutive Odd $\mathbb{P}$ can be systematically classified [non-overlappingly] according to P–g_n = 2, 4, 6, 8, 10... as previously shown in section 2.

Prime-$\pi(x)$ is prime-counting function = number of primes $\leq x$, for any real number x. Composite-$\pi(x)$ is composite-counting function = number of composites $\leq x$, for any real number x. Prime number theorem and our derived Composite number theorem describe the asymptotic distribution of primes and composites among positive integers. Respectively, they

formalize the intuitive idea that primes (and composites) become deceleratingly less (and acceleratingly more) common as they become larger by precisely quantifying the rate at which this occurs. Prime number theorem is concluded to be heuristically true and was proved independently by Jacques Hadamard[8] and Charles Jean de la Vallee Poussin[31] in 1896 using ideas introduced by Bernhard Riemann (in particular, Riemann zeta function). This theorem has also been rigorously proven as the elementary proofs of Atle Selberg[22] and Paul Erdos[2] in 1949, and as the non-elementary proof in 1980 by Donald J. Newman[14] in the sense that he used Cauchy's integral theorem from complex analysis in his proof.

The asymptotic law of distribution for prime numbers [that involves natural logarithm function and **confirms the CIS-ALN-decelerating property**] is given as $\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)}{\left[\frac{x}{\ln(x)}\right]} = 1$. Using asymptotic notation, this result from Prime number theorem can be restated as $\text{Prime-}\pi(x) \sim \frac{x}{\ln(x)}$. This theorem is also equivalent to the statement that the n^{th} prime number P_n satisfies $P_n \sim n\ln(n)$.

The [pseudo-inverse] asymptotic law of distribution for composite numbers [that involves natural exponential function and **confirms the CIS-IM-accelerating property**] is given as $\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)}{\left[\frac{x}{e(x)}\right]} = 1$. Using asymptotic notation, this result can be heuristically restated as $\text{Composite-}\pi(x) \sim \frac{x}{e(x)}$ to represent our Composite number theorem. This theorem is also equivalent to the statement that the n^{th} composite number C_n satisfies $C_n \sim ne(n)$.

The following asymptotic relations are logically equivalent:

$\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)\ln(x)}{x} = 1$, and $\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)\ln(\text{Prime-}\pi(x))}{x} = 1$.

$\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)e(x)}{x} = 1$, and $\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)e(\text{Composite-}\pi(x))}{x} = 1$.

Prime number theorem is also equivalent to $\lim_{x\to\infty} \frac{\vartheta(x)}{x} = \lim_{x\to\infty} \frac{\psi(x)}{x} = 1$, where ϑ and ψ are the first and the second Chebyshev functions respectively, and to $\lim_{x\to\infty} \frac{M(x)}{x} = 0$, where $M(x) = \sum_{n\le x} \mu(n)$ is the Mertens function. Here, the most common generalized counting function is Chebyshev function $\psi(x)$ defined by $\psi(x) = \sum_{\substack{p^k\le x,\\ p \text{ is prime}}} \ln p$. This is sometimes written as $\psi(x) = \sum_{n\le x} \Lambda(n)$, where Λ(n) is the von Mangoldt function, namely $\Lambda(n) = \begin{cases} \ln p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \ge 1, \\ 0 & \text{otherwise.} \end{cases}$.

The logarithmic integral function li(x) is defined by $\text{li}(x) = \int_0^x \frac{dt}{\ln t}$. An even better approximation to Prime-$\pi(x)$ is given by the offset logarithmic integral function Li(x) which is defined by $\text{Li}(x) = \int_2^x \frac{dt}{\ln t} = \text{li}(x) - \text{li}(2)$; or equivalently, $\text{li}(x) = \int_0^x \frac{dt}{\ln t} = \text{Li}(x) + \text{li}(2)$. Also, $\text{Li}(x) = Ei(\ln(x)) - Ei(\ln(2))$ since logarithmic integral function li(x) is related to inverse

exponential function Ei(x) via equation li(x) = $Ei(\ln(x))$, valid for x > 0. Both li(x) and Li(x) strongly support the notion that density of prime numbers around t should be $\frac{1}{\ln(t)}$; and is related to natural logarithm by the asymptotic expansion $\mathrm{Li}(x) \sim \frac{x}{\ln x}\sum_{k=0}^{\infty}\frac{k!}{(\ln x)^k} = \frac{x}{\ln x} + \frac{x}{(\ln x)^2} + \frac{2x}{(\ln x)^3} + \cdots$. With the inverse Ei(x) = Li ($e(x)$) – Li ($e(2)$), we conclude $Ei(x) - Ei(2)$ or $Ei(x)$ will both strongly support the notion that density of composite numbers around t should be $\frac{1}{e(t)}$.

The asymptotic law of distribution for prime numbers [and composite numbers] can also be given as $\lim_{x\to\infty}\frac{\text{Prime-}\pi(x)}{\mathrm{Li}(x)} = 1$ or $\lim_{x\to\infty}\frac{\text{Prime-}\pi(x)}{\mathrm{li}(x)} = 1$ [and $\lim_{x\to\infty}\frac{\text{Composite-}\pi(x)}{\mathrm{Ei}(x) - \mathrm{Ei}(2)} = 1$ or $\lim_{x\to\infty}\frac{\text{Composite-}\pi(x)}{\mathrm{Ei}(x)} = 1$]. Using asymptotic notation, this result is correspondingly restated as Prime-$\pi(x) \sim \mathrm{Li}(x)$ or Prime-$\pi(x) \sim \mathrm{li}(x)$ [and Composite-$\pi(x) \sim \mathrm{Ei}(x) - \mathrm{Ei}(2)$ or Composite-$\pi(x) \sim \mathrm{Ei}(x)$], thus equivalently representing Prime number theorem [and Composite number theorem]. In 1899, de la Vallee Poussin proved the estimate Prime-$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\ln x}}\right)$ as $x \to \infty$ is valid for some positive constant a, where $O(...)$ is the big O notation. The statements Prime-$\pi(x) = \mathrm{Li}(x) + O\left(\sqrt{x}\ln x\right)$ and $|\mathrm{li}(x) - \text{Prime-}\pi(x)| = O(x^{1/2+a})$ for any $a > 0$ are equivalent to Riemann hypothesis.

Riemann's prime-power counting function, usually denoted as $\Pi_0(x)$ or $J_0(x)$, has jumps of $\frac{1}{n}$ at prime powers p^n and takes a value halfway between the two sides at the discontinuities of $\pi(x)$. That added detail is used because the function may then be defined by an inverse Mellin transform. We formally define $\Pi_0(x)$ by $\Pi_0(x) = \frac{1}{2}\left(\sum_{p^n<x}\frac{1}{n} + \sum_{p^n\le x}\frac{1}{n}\right)$ where the variable p in each sum ranges over all primes within the specified limits. We may also write $\Pi_0(x) = \sum_{n=2}^{x}\frac{\Lambda(n)}{\ln n} - \frac{\Lambda(x)}{2\ln x} = \sum_{n=1}^{\infty}\frac{1}{n}\pi_0(x^{1/n})$ where $\Lambda(n)$ is the von Mangoldt function and $\pi_0(x) = \lim_{\varepsilon\to 0}\frac{\pi(x-\varepsilon) + \pi(x+\varepsilon)}{2}$. The Mobius inversion formula then gives $\pi_0(x) = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\Pi_0(x^{1/n})$, where $\mu(n)$ is the Mobius function. Using Perron formula and the relationship between logarithm of Riemann zeta function and von Mangoldt function Λ; we have $\ln\zeta(s) = s\int_0^{\infty}\Pi_0(x)x^{-s-1}\,\mathrm{d}x$. The exact form of Prime-$\pi(x)$ was provided by Bernhard Riemann (1826 – 1866). For $x > 1$, let $\pi_0(x) = \pi(x) - \frac{1}{2}$ when x is a prime number, and $\pi_0(x) = \pi(x)$ otherwise. It is proved that $\pi_0(x) = \mathrm{R}(x) - \sum_{\rho}\mathrm{R}(x^{\rho})$, where R(x) $= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\mathrm{li}(x^{1/n})$, μ(n) is the Mobius function, $li(x)$ is the logarithmic integral function, ρ indexes every zero of Riemann zeta function, and li($x^{\rho/n}$) is not evaluated with a branch cut

k	p_k	k#
1	2	2
2	3	6
3	5	30
4	7	210
5	11	2310
6	13	30,030
7	17	510,510
8	19	9,699,690
9	23	223,092,870
10	29	6,469,693,230

Table 1 Tabulated data of k primorial for k = 1 to 10.

but instead considered as Ei($\frac{\rho}{n} \ln x$) where Ei(x) is the exponential integral.

Remark 7. ***Asymptotic Law of Distribution for Prevalences of Primes and Composites:***

As two continuous functions valid for $x \to \infty$ range that share a common mathematical constant $e \approx 2.71828$, the natural logarithm function $log_e x$ or $ln(x)$ is used in Asymptotic Law of Distribution for prime numbers and the natural exponential function exp(x) or e^x is used in Asymptotic Law of Distribution for composite numbers. Since we can validly "project" $log_e x$ onto Prevalence of Primes and e^x onto Prevalence of Composites; then these functions are consequently acting as unique two allowable complementary-reciprocal functions that can also be individually "projected" onto (i) Prevalence of Odd Primes derived separately from even Prime gaps 2, 4, 6, 8, 10... and Prevalence of Gap 2-Even Composites [w.r.t. $log_e x$], and (ii) Prevalence of Gap 1-Even Composites and Gap 1-Odd Composites [w.r.t. e^x]. Thus the full $x \to \infty$ range applicability of $log_e x$ to each and every even Prime gaps will support Modified Polignac's and Twin prime conjectures to be true.

6.1 Admissible Prime *k*-tuplets, Inadmissible Prime (*k*+1)-tuples, Dirichlet Sigma-Power Law and Principle of Equidistant for Multiplicative Inverse

For k $\geq$ 2, a Prime k-tuple [that can be subdivided into available subtuples for sufficiently large k values] is a repeatable pattern of finite k consecutive primes {p_1, p_2,..., p_k} [viz, a finite collection with $p_1 < p_2 <...< p_k$] having diameter d defined as difference between its largest and smallest elements [viz, diameter $d = p_k - p_1$]. There are two main types of Prime k-tuples: [repeating] Admissible Prime k-tuples and [non-repeating] Inadmissible Prime k-tuples. These are further classified into various subtypes and varieties[27] with this aspect not discussed in this paper. An Admissible Prime k-tuplet is a sequence of finite k consecutive primes such that the distance between first prime and last prime is in some well-defined sense as small as possible. Then an Admissible Prime k-tuple is a sequence of finite k consecutive primes such that this same distance is in some well-defined sense *not* as small as possible.

The k primorial mentioned in caption of Table 1 is for k = 1 to 10 with its underlying principles given here: Let p_k be the k^{th} prime with k = 1, 2, 3, 4, 5.... Then k primorial (k#) is product of first k primes whereby [even] numbers in third column are product of

primes in second column. It is a well-defined Incompletely Predictable function acceleratingly reaching an infinity value. As part of group theory with notation that read as (Z/nZ)*, concepts behind multiplicative group of integers modulo n are important for theory of prime k-tuples or constellations. It contains a subset of integers from 1 to n-1. The elements of (Z/nZ)* are integers from 1 to n-1 that are relatively prime to n. If n is a prime number, then (Z/nZ)* contains all integers from 1 to n-1. If n has many divisors, then (Z/nZ)* will contain fewer elements. To find Admissible Prime k-tuplets, we need to consider the multiplicative group of integers mod k primorial. This group contains set of integers less than k primorial that are relatively prime to k primorial.

The multiplicative group mod 6 [2#] has two elements; viz, (Z/6Z)* = {1, 5}. Then all primes greater than 3 have the form 6*n ± 1. To search for the smaller of twin prime pairs [Admissible Prime 2-tuplets], one should look at [odd] numbers of the form 6*n + 5. The multiplicative group mod 30 [3#] has 8 elements; viz, (Z/30Z)* = {1, 7, 11, 13, 17, 19, 23, 29}. By looking at the differences between adjacent elements in this set, we see Admissible Prime 3-tuplets as pattern (p, p+2, p+6) is found only in the expressions 30*n + 11 and 30*n + 17. The ordered set (Z/30Z)* = {1, 7, 11, 13, 17, 19, 23, 29} can be manipulated by taking the differences between adjacent elements; viz, d30 = [6, 4, 2, 4, 2, 4, 6] $\implies$ the particular pattern (p, p+2, p+6, p+8) which has differences [2, 4, 2] is found inside ordered set d30. Thus we see Admissible Prime 4-tuplets having pattern (p, p+2, p+6, p+8) must have the form 30*n + 11.

Proposition 1. *Let finite k consecutive primes {p_1, p_2,..., p_k} represent Admissible Prime k-tuplets that are computed using p_1 commencing values 2, 3, 5, 7, 11, 13.... Then except for p_1 commencing value 2 [having the empty set of Admissible Prime k-tuple], we can uniquely generate a finite number of Admissible Prime k-tuplets / k-tuples that are specified by the k values and an associated arbitrarily large number of Inadmissible Prime (k+1)-tuples that are specified by the larger k+1 values whereby both types of tuplets / tuples will comply with corresponding admissibility and inadmissibility criteria.*

Proof. Suppose one is given a k_0-tuple $\mathcal{H} = (h_1, \ldots, h_{k_0})$ of k_0 distinct integers for some $k_0 \geq 1$, arranged in increasing order. We often anticipate finding an arbitrarily large number of translates $n + \mathcal{H} = (n+h_1, \ldots, n+h_{k_0})$ of $\mathcal{H}$ which consist entirely of consecutive primes will prove (Modified) Polignac's and Twin prime conjectures to be true. The case $k_0 = 1$ is just Euclid's theorem on the infinitude of primes. The case $k_0 = 2$ [as subset of $k_0 \geq 2$] with $\mathcal{H} = (0, 2)$ correspond to twin prime conjecture that non-overlappingly deals with prime gap = 2. The arbitrarily large number of cases $k_0 \geq 2$ [as full set] in their entirety correspond to Polignac's conjecture that [additionally] involve all other remaining cases such as $k_0 = 3$ with $\mathcal{H} = (0, 2, 6)$ as pattern-1 or (0, 4, 6) as pattern-2, $k_0 = 4$ with $\mathcal{H} = (0, 2, 6, 8)$ as solitary pattern, etc. Thus we have [overlappingly] dealt with all even Prime gaps = 2, 4, 6, 8, 10....

More generally, if there is a prime p_1 such that $\mathcal{H}$ meets each of the p_1 residue classes 0 mod p_1, 1 mod p_1, 2 mod p_1, ..., p_1-1 mod p_1, then every translate of $\mathcal{H}$ contains at least one multiple of p_1. Since p_1 is the only multiple of p_1 that is prime, this shows that there are only finitely many translates of $\mathcal{H}$ that consist entirely of consecutive primes.

A k_0-tuple $\mathcal{H}$ is admissible if it avoids at least one residue class mod p for each prime p. It is easy to check for admissibility in practice, since a k_0-tuple is automatically admissible in every prime p larger than k_0, so one only needs to check a finite number of primes in order

to decide on admissibility of a given tuple. Being a likely unprovable conjecture according to Godel's incompleteness theorem, we can succinctly state **first Hardy-Littlewood conjecture** or **Prime k-tuple conjecture** in its qualitative form: If $\mathcal{H}$ is an admissible k_0-tuple, then there exists an arbitrarily large number of translates of $\mathcal{H}$ that consist entirely of consecutive primes. We then deduce neither proving nor disproving the first Hardy-Littlewood conjecture will definitively prove or disprove (Modified) Polignac's and Twin prime conjectures.

The statement "Probability (Odd Primes that are [discriminatorily] derived from any of the Arbitrarily Large Number of even Prime gaps 2, 4, 6, 8, 10... will abruptly terminate) = 0" $\implies$ (Modified) Polignac's and Twin prime conjectures must be true. Apart from the only countably finite even prime number 2, all the countably arbitrarily large number of odd prime numbers 3, 5, 7, 11, 13... can be fully represented by solitary Admissible Prime 2-tuplet that represent even Prime gap 2 and arbitrarily large number of Admissible Prime 2-tuples that represent even Prime gaps 4, 6, 8, 10, 12....

Constituted from entire CIS-ALN-decelerating prime numbers 2, 3, 5, 7, 11, 13..., our p_1 commencing values act as reference points to orderly include all possible Admissible Prime k-tuplets / k-tuples and Inadmissible Prime k-tuples whereby these k-tuplets and k-tuples are constituted by k consecutive prime numbers starting from p_1. We invoke multiplicative group of integers modulo p_1 that, via brute force algorithm, must result in a subset of consecutive integers as residues from 0 to p_1-2 and p_1-1 whereby some of these integers that represent corresponding residues will inevitably repeat more than once. For instance at p_1 commencing value = 11, the sequence of integers that mechanically represent corresponding residues from mod prime 11 as iteratively computed using all available prime gaps are 0, 2, 6, 8, 1, 7, 9, 4, 8, 10, 3, 9, 4, 6, 1 [Admissible] and 5 [Inadmissible] whereby five [non-comprehensive] integers 1, 4, 6, 8 and 9 are overlappingly depicted more than once and two [uniquely nominated] integers 0 and 5 must always be non-overlappingly depicted just once but with the [solitary] integer 5 being (firstly) absent when the involved k-tuple is admissible and (secondly) present when the involved (k+1)-tuple is inadmissible. The p_1 commencing value = 11 has thus provided us with (i) Admissible Prime 15-tuplet as consecutive primes (11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67) that is mechanically $\equiv$ progressive prime gaps (0, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6) $\equiv$ cummulative prime gaps (0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56) and (ii) Inadmissible Prime 16-tuple as consecutive primes (11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71) that is mechanically $\equiv$ progressive prime gaps (0, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4) $\equiv$ cummulative prime gaps (0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56, 60). We note all the involved [consecutive] prime gaps of 2, 4 and 6 are each overlappingly depicted more than once for both the involved Admissible Prime 15-tuplet and Inadmissible Prime 16-tuple.

We deduce there must be at least p_1-1 consecutive integers representing residues 0, 1, 2, 3,..., p_1-2 that cater for longest possible Admissible Prime k-tuplet [and at least p_1 consecutive integers representing residues 0, 1, 2, 3,..., p_1-1 that cater for shortest possible Inadmissible Prime (k+1)-tuple]. Apart from first four cardinality that are smaller than or equal to p_1-1, all subsequent cardinality must not be smaller than their corresponding p_1-1 with the all-important implication that we can always derive arbitrarily long Admissible Prime k-tuples with maximal k values that must be at least equal to [but are usually always larger than] p_1-1. In general, we recognize that ever larger p_1 commencing values are [overall] associated with ever larger k-valued Admissible Prime k-tuplets / k-tuples that characteristically

have ever larger zenith diameter d and zenith average gaps. Reproduced with permission below, these properties were confirmed using computations on initial 15 p_1 commencing values (out of an arbitrarily large number of other commencing values)[27] whereby it is insightful to regard **absolutely** Inadmissible Prime k-tuples as those Prime k-tuples that begin with p_1 commencing value = 2 for all $k \geq 2$ values and p_1 commencing value = 3 for all $k \geq 3$ values, and **relatively** Inadmissible Prime k-tuples as those Prime k-tuples with p_1 commencing value = 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47... for corresponding k $\geq 6, 7, 16, 22, 21, 36, 40, 60, 96, 74, 95, 78, 79$... cyclical values. We notice the p_1 commencing value = 3 with k = 2 value will simply represent an Admissible Prime 2-tuplet.

Admissible Prime k-tuplets / k-tuples and Inadmissible Prime k-tuples for initial 15 p_1 commencing values:

p_1 commencing value = 2. Set Admissible Prime k-tuples as k-value = 0 [empty set] with its cardinality = ||CFS|| = 0. Set Inadmissible Prime k-tuples as k-value = 2 [having nadir diameter d = 1 and nadir average gap = 1/2 = 0.5], 3, 4, 5, 6... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At $p_1 = 2$, failure at mod 2 (term 3) first occur at k = 2 with minimum diameter d = 1.

p_1 commencing value = 3. Set Admissible Prime k-tuplets as k-value = 2 [having zenith diameter d = 2 and zenith average gap = 2/1 = 2] with its cardinality = ||CFS|| = 1. Set Inadmissible Prime k-tuples as k-value = 3 [having nadir diameter d = 4 and nadir average gap = 4/3 = 1.33], 4, 5, 6, 7... with its cardinality = ||CIS-ALN-decelerating|| = symbolically $\aleph_0$. At $p_1 = 3$, failure at mod 3 (term 7) first occur at k = 3 with minimum diameter d = 4.

p_1 commencing value = 5. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5 [having zenith diameter d = 12 and zenith average gap = 12/5 = 2.4] with its cardinality = ||CFS|| = 4. Set Inadmissible Prime k-tuples as k-value = 6 [having nadir diameter d = 14 and nadir average gap = 14/6 = 2.33], 7, 8, 9, 10... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At $p_1 = 5$, failure at mod 5 (term 19) first occur at k = 6 with minimum diameter d = 14.

p_1 commencing value = 7. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5, 6 [having zenith diameter d = 16 and zenith average gap = 16/6 = 2.67] with its cardinality = ||CFS|| = 5. Set Inadmissible Prime k-tuples as k-value = 7 [having nadir diameter d = 22 and nadir average gap = 22/7 = 3.14], 8, 9, 10, 11... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At $p_1 = 7$, failure at mod 7 (term 29) first occur at k = 7 with minimum diameter d = 22.

p_1 commencing value = 11. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 15 [having zenith diameter d = 56 and zenith average gap = 56/15 = 3.73] with its cardinality = ||CFS|| = 14. Set Inadmissible Prime k-tuples as k-value = 16 [having nadir diameter d = 60 and nadir average gap = 60/16 = 3.75], 17, 18, 19, 20... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At $p_1 = 11$, failure at mod 11 (term 71) first occur at k = 16 with minimum diameter d = 60.

p_1 commencing value = 13. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 21 [having zenith diameter d = 88 and zenith average gap = 88/21 = 4.19] with its cardinality = ||CFS|| = 20. Set Inadmissible Prime k-tuples as k-value = 22 [having nadir diameter d = 90 and nadir average gap = 90/22 = 4.09], 23, 24, 25, 26... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At $p_1 = 13$, failure at mod 13 (term 103) first occur at k = 22 with minimum diameter d = 90.

p_1 commencing value = 17. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 20 [having zenith diameter d = 84 and zenith average gap = 84/20 = 4.2] with its cardinality = ||CFS|| = 19. Set Inadmissible Prime k-tuples as k-value = 21 [having nadir diameter d = 86 and nadir average gap = 86/21 = 4.10], 22, 23, 24, 25... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 17, failure at mod 17 (term 103) first occur at k = 21 with minimum diameter d = 86.
p_1 commencing value = 19. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 35 [having zenith diameter d = 162 and zenith average gap = 162/35 = 4.63] with its cardinality = ||CFS|| = 34. Set Inadmissible Prime k-tuples as k-value = 36 [having nadir diameter d = 172 and nadir average gap = 172/36 = 4.78], 37, 38, 39, 40... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 19, failure at mod 19 (term 191) first occur at k = 36 with minimum diameter d = 172.
p_1 commencing value = 23. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 39 [having zenith diameter d = 188 and zenith average gap = 188/39 = 4.82] with its cardinality = ||CFS|| = 38. Set Inadmissible Prime k-tuples as k-value = 40 [having nadir diameter d = 200 and nadir average gap = 200/40 = 5], 41, 42, 43, 44... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 23, failure at mod 23 (term 223) first occur at k = 40 with minimum diameter d = 200.
p_1 commencing value = 29. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 59 [having zenith diameter d = 308 and zenith average gap = 308/59 = 5.22] with its cardinality = ||CFS|| = 58. Set Inadmissible Prime k-tuples as k-value = 60 [having nadir diameter d = 318 and nadir average gap = 318/60 = 5.3], 61, 62, 63, 64... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 29, failure at mod 29 (term 347) first occur at k = 60 with minimum diameter d = 318.
p_1 commencing value = 31. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 95 [having zenith diameter d = 540 and zenith average gap = 540/95 = 5.68] with its cardinality = ||CFS|| = 94. Set Inadmissible Prime k-tuples as k-value = 96 [having nadir diameter d = 546 and nadir average gap = 546/96 = 5.69], 97, 98, 99, 100... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 31, failure at mod 31 (term 577) first occur at k = 96 with minimum diameter d = 546.
p_1 commencing value = 37. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 73 [having zenith diameter d = 396 and zenith average gap = 396/73 = 5.42] with its cardinality = ||CFS|| = 72. Set Inadmissible Prime k-tuples as k-value = 74 [having nadir diameter d = 402 and nadir average gap = 402/74 = 5.43], 75, 76, 77, 78... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 37, failure at mod 37 (term 439) first occur at k = 74 with minimum diameter d = 402.
p_1 commencing value = 41. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 94 [having zenith diameter d = 536 and zenith average gap = 536/94 = 5.70] with its cardinality = ||CFS|| = 93. Set Inadmissible Prime k-tuples as k-value = 95 [having nadir diameter d = 546 and nadir average gap = 546/95 = 5.75], 96, 97, 98, 99... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 41, failure at mod 41 (term 587) first occur at k = 95 with minimum diameter d = 546.
p_1 commencing value = 43. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 77 [having zenith diameter d = 420 and zenith average gap = 420/77 = 5.45] with its cardinality = ||CFS|| = 76. Set Inadmissible Prime k-tuples as k-value = 78 [having nadir

diameter d = 424 and nadir average gap = 424/78 = 5.44], 79, 80, 81, 82... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 43, failure at mod 43 (term 467) first occur at k = 78 with minimum diameter d = 424.
p_1 commencing value = 47. Set Admissible Prime k-tuplets/k-tuples as k-value = 2, 3, 4, 5,..., 78 [having zenith diameter d = 432 and zenith average gap = 432/78 = 5.54] with its cardinality = ||CFS|| = 77. Set Inadmissible Prime k-tuples as k-value = 79 [having nadir diameter d = 440 and nadir average gap = 440/79 = 5.57], 80, 81, 82, 83... with its cardinality = ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating. At p_1 = 47, failure at mod 47 (term 487) first occur at k = 79 with minimum diameter d = 440.
The proof is now complete for Proposition 1□.

All Admissible Prime k-tuplets will usually have their unique allowable patterns. A083409 *Number of prime k-tuplet constellations, i.e., patterns with minimal diameter A008407*[1] is relevant. Computed for k = 2, 3, 4, 5, 6...; the number of possible patterns are 1, 2, 1, 2, 1, 2, 3, 4, 2, 2, 2, 6, 2, 4, 2, 4, 2, 4, 2, 2, 4, 2, 4, 18, 2, 8, 10, 2, 2, 2, 4, 14, 20, 2, 2, 2, 6, 26, 26, 8, 2, 6, 18, 4, 4, 4, 2, 2, 22, 22, 2, 2, 26, 6, 6, 2, 2, 4, 2, 2, 6, 2, 2, 2, 2, 18, 2, 20, 2, 2, 2, 10, 2, 14, 14, 40, 8, 2, 14, 14, 16, 4, 2, 2, 60, 50, 2, 2, 2, 16, 2, 18, 12.... The 18 patterns of Admissible Prime 25-tuplets, as a random example, are given in Appendix B whereby we also depict useful calculations behind frequency of the involved patterns.

For every appropriately paired Admissible Prime k-tuplet patterns endowed with same modulo number, there exists a counterpart. For instance, Admissible Prime 7-tuplet pattern-1 (0, 2, 6, 8, 12, 18, 20) has its p_1 congruent to 11 (modulo 210) and Admissible Prime 7-tuplet pattern-2 (0, 2, 8, 12, 14, 18, 20) has its p_1 congruent to 179 (modulo 210). We see that 11 + 179 (viz, the counterpart) + 20 (viz, the diameter d) = 210 (viz, the modulo number). The *offset and multiplier* containing variable n is related to *p_1 congruent to p (modular q)* for Admissible Prime k-tuplets as explained using below examples.
Example 1: For Admissible Prime 7-tuplet with pattern-1 given as cummulative prime gaps (0, 2, 6, 8, 12, 18, 20) ≡ consecutive prime numbers (11, 13, 17, 19, 23, 29, 31) [as based on first-occurring p_1 = 11]; the p_1 congruent to 11 (modulo 210) is equivalent to offset and multiplier 11 + 210*n. This is given by A022009 *Initial members of prime septuplets (p, p+2, p+6, p+8, p+12, p+18, p+20).*[17] having values 11, 165701, 1068701, 11900501, 15760091, 18504371, 21036131, 25658441, 39431921, 45002591, 67816361, 86818211, 93625991, 124716071, 136261241, 140117051, 154635191, 162189101, 182403491, 186484211, 187029371, 190514321, 198453371... which is cross linked to A182387 *Numbers n such that 210*n+11, 13, 17, 19, 23, 29, 31 are 7 consecutive primes.*[21] having values 0, 789, 5089, 56669, 75048, 88116, 100172, 122183, 187771, 214298, 322935, 413420, 445838, 593886, 648863, 667224, 736358, 772329, 868588, 888020, 890616, 907211, 945016, 1052954, 1078331, 1106177, 1146724, 1223888, 1432230, 1452437, 1458355, 1509878, 1535216....
Example 2: For Admissible Prime 7-tuplet with pattern-2 given as cummulative prime gaps (0, 2, 8, 12, 14, 18, 20) ≡ consecutive prime numbers (5639, 5641, 5647, 5651, 5653, 5657, 5659) [as based on first-occurring p_1 = 5639]; the p_1 congruent to 179 (modulo 210) is equivalent to offset and multiplier 179 + 210*n. This is given by A022010 *Initial members of prime septuplets (p, p+2, p+8, p+12, p+14, p+18, p+20).*[18] having values 5639, 88799, 284729, 626609, 855719, 1146779, 6560999, 7540439, 8573429, 17843459, 19089599, 24001709, 42981929, 43534019, 69156539, 74266259, 79208399, 80427029, 84104549, 87988709,

124066079, 128469149, 144214319, 157131419, 208729049, 218033729... which is cross linked to A357889 *a(n) = (A022010(n) – 179)/210.*[16] having values 26, 422, 1355, 2983, 4074, 5460, 31242, 35906, 40825, 84968, 90902, 114293, 204675, 207304, 329316, 353648, 377182, 382985, 400497, 418993, 590790, 611757, 686734, 748244, 993947, 1038255, 1181931, 1246060, 1310026, 1347976, 1354707, 1440679, 1477788, 1559980, 1720425, 1915719, 1989590....

Example 3: For Admissible Prime 38-tuplet there are six possible patterns with pattern-4 given as cummulative prime gaps (0, 6, 8, 14, 18, 20, 24, 30, 36, 38, 44, 48, 50, 56, 60, 66, 74, 78, 80, 84, 86, 90, 104, 108, 114, 116, 126, 128, 134, 140, 144, 150, 156, 158, 168, 170, 174, 176) ≡ consecutive prime numbers (23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199) [as based on first-occurring $p_1 = 23$]; the p_1 congruent to 2541318803 (modulo 6469693230) which is equivalent to offset and multiplier 2541318803 + 6469693230*n is also applicable in a similar manner to previous two examples.

A prime k-tuple is *admissible* in its sequence of consecutive primes $\{p_1, p_2,..., p_k\}$ such that for every prime $q \leq k$, not all the residues modulo q are represented by $p_1, p_2,..., p_k$. Simpliest Admissible Prime k-tuplets and k-tuples using $k = 2$ value include all twin primes as 2-tuplet with smallest possible diameter d (prime gap) = 2; all cousin primes as 2-tuple with larger diameter d (prime gap) = 4; all sexy primes as 2-tuple with larger diameter d (prime gap) = 6; etc. An example of Admissible Prime 3-tuple pattern-1 (p+0, p+2, p+8) is given by consecutive prime numbers (5639, 5641, 5647). We note this particular Prime 3-tuple is also a subtuple forming part of Admissible Prime 7-tuplet pattern-2 (p+0, p+2, p+8, p+12, p+14, p+18, p+20) given by first occurrence consecutive prime numbers (5639, 5641, 5647, 5651, 5653, 5657, 5659).

Both the Admissible Prime 2-tuplets as two consecutive primes (p_1, p_k) with diameter d or prime gap $= p_k - p_1 = 2$ and Admissible Prime 2-tuples as two consecutive primes (p_1, p_k) with diameter d or prime gap $= p_k - p_1 \geq 4$ can match an arbitrarily large number of positions in the sequence of prime numbers. For n = 1, 2, 3, 4, 5...; there are the rarely occurring but nevertheless arbitrarily large number of Admissible Prime 2-tuples conforming to criterion $p_k - p_1 = p_1 - p_{k-2}$ and manifesting as two identical consecutive prime gaps (6n, 6n) = (6, 6), (12, 12), (18, 18), etc [and also manifesting as three or more much rarer identical consecutive prime gaps (6n, 6n, 6n...) = (6, 6, 6...), (12, 12, 12...), (18, 18, 18...), etc]. They could in principle also form [bridging] smaller subtuples of steady primes in Admissible Prime k-tuplets / k-tuples or Inadmissible Prime k-tuples when $k \geq 3$. The criterion $p_k - p_1 < p_1 - p_{k-2}$ will be conformed to by an arbitrarily large number of Admissible Prime 2-tuples whereby they could in principle also form [bridging] smaller subtuples of decelerating primes in Admissible Prime k-tuplets / k-tuples or Inadmissible Prime k-tuples when $k \geq 3$. The three subtuples [that also includes smaller subtuples of accelerating primes] were further elaborated upon in section 2 whereby they essentially form eternal repeated groupings of small and/or large prime numbers and gaps.

Remark 8. *At ever larger $x \geq 4$ integer range manifesting progressively less Odd Primes [with associated prime gaps], we intuitively expect an overall slowly increasing prevalence of Admissible Prime k-tuples that cater for large(r) Odd Primes which is reciprocally and simultaneously associated with an overall slowly decreasing prevalence of Admissible Prime*

k-tuples that cater for small(er) Odd Primes. When Admissible Prime 2-tuplets as two consecutive primes (p_1, p_k) with diameter d or prime gap = $p_k - p_1 = 2$ are combined with Admissible Prime 2-tuples as two consecutive primes (p_1, p_k) with diameter d or prime gap = $p_k - p_1 \geq 4$, they are able to [uniquely] represent every known Odd Primes in an non-overlapping manner.

A Prime k-tuple is *inadmissible* in its sequence of consecutive primes {p_1, p_2,..., p_k} such that for some of the prime $q \leq k$ [example, for one of the prime $q \leq k$ when $k \geq 3$ or for two of the prime $q \leq k$ if $p_1 = 2$ forms part of a Prime k-tuple when $k \geq 4$]; all the residues modulo q are represented by p_1, p_2,..., p_k. All [non-repeating] Inadmissible Prime k-tuples only match one finite position in the sequence of prime numbers and are defined by their diameter d being the shortest. An arbitrarily large number of examples with one all-prime solution for this subtype include Prime 2-tuple (p+0, p+1) as primes (2, 3) with d = 1; Prime 3-tuple (p+0, p+1, p+3) as primes (2, 3, 5) with d = 3; Prime 3-tuple (p+0, p+2, p+4) as primes (3, 5, 7) with d = 4; Prime 4-tuple (p+0, p+1, p+3, p+5) as primes (2, 3, 5, 7) with d = 5; Prime 4-tuple (p+0, p+2, p+4, p+8) as primes (3, 5, 7, 11) with d = 8; etc.

Modular arithmetic: a (mod n) is a/n ≡ r whereby a = dividend, n = divisor and r = remainder [round up to the next integer]. Therefore, a (mod n) ≡ a – (r ∗ n). With abbreviation *n* denoting numbers, we analyze the Completely Predictable even *n* and odd *n*. For i = 0, 1, 2, 3, 4, 5...; congruence $n \equiv 0 \pmod 2$ holds for even $n = E_i = 2*i = 0, 2, 4, 6, 8, 10...$ and for i = 1, 2, 3, 4, 5, 6...; congruence $n \equiv 1 \pmod 2$ holds for odd $n = O_i = (2*i)–1 = 1, 3, 5, 7, 9, 11...$ We note 0 is then the zeroth even *n* when we only consider all (non-negative) positive even *n* and odd *n*. We analyze the Incompletely Predictable prime numbers collectively grouped as k-tuples. For the worked example of modular arithmetic applied to test for admissibility on Inadmissible Prime 4-tuple (p+0, p+1, p+3, p+5) ≡ cummulative prime gaps (0, 1, 3, 5) with earliest and only candidate as consecutive prime numbers (2, 3, 5, 7) having progressive prime gaps (0, 1, 2, 2); we can use either [I] cummulative prime gaps: congruence 0, 1, 3, 5 ≡ 0, 1, 1, 1 (mod prime 2) and congruence 0, 1, 3, 5 ≡ 0, 1, 0, 2 (mod prime 3) or [II] consecutive prime numbers: congruence 2, 3, 5, 7 ≡ 0, 1, 1, 1 (mod prime 2) and congruence 2, 3, 5, 7 ≡ 2, 0, 2, 1 (mod prime 3). There are two failures at [firstly] mod prime 2 on second term = 1 (as prime gap) or 3 (as prime number) and [secondly] mod prime 3 on last term = 5 (as prime gap) or 7 (as prime number) $\implies$ this Inadmissible Prime 4-tuple is now truly confirmed to be inadmissible. Since twin prime (3, 5) is a Admissible Prime 2-tuplet when first element p = 3, we can redundantly generate a complete all-inclusive countably arbitrarily large number of [non-repeating] Inadmissible Prime k-tuples using progressively longer $k \geq 3$ values that should have the shortest diameter when first element p = 3. We can also redundantly generate a complete all-inclusive countably arbitrarily large number of [non-repeating] Inadmissible Prime k-tuples using progressively longer $k \geq 2$ values that should have the shortest diameter when first element p = 2.

We hereby explain an example of [non-existing] Inadmissible Prime k-tuple which is linked to Admissible Prime 3-tuplet (p+0, p+2, p+6) pattern-1 having diameter d = 6 that first appear as consecutive primes (5, 7, 11). This Admissible Prime 3-tuplet is associated with Inadmissible Prime 3-tuples with failure at mod prime 3 (last term = 10, 16, 22, 28...) and must fully conform with the forbidden condition as stated here: Just as two consecutive twin primes given by Prime 3-tuple (p+0, p+2, p+4+6n) cannot exist at all apart from the solitary Inadmissible Prime 3-tuple occurring as consecutive primes (3, 5, 7) when n = 0, then so must all two consecutive *twin-related* primes given by Prime 3-tuple (p+0, p+2, p+4+6n)

cannot exist at all when n = 1, 2, 3, 4... [since at least one of the three primes is divisible by 3]. Two other forbidden conditions that must be conformed to by all Prime k-tuplets and Prime k-tuples including Inadmissible Prime k-tuples are:
(1) Apart from the solitary [single-digit] odd prime number 5 with its last and only digit also ending in odd number 5, all other larger [multiple-digit] odd prime numbers cannot have their last digit ending in odd number 5 and, consequently, these forbidden numbers can never belong to any Prime k-tuplets and Prime k-tuples. Thus, apart from the solitary odd prime number 5, it is an established mathematical fact that all odd prime numbers must have their last digit ending in odd numbers 1, 3, 7 or 9.
(2) The arbitrarily large number of Admissible Prime 4-tuplets (p+0, p+2, p+6, p+8) with smallest possible diameter d = 8 is first given by consecutive primes (5, 7, 11, 13) whereby this must be differentiated from the totally different [solitary] Inadmissible Prime 4-tuple (p+0, p+2, p+4, p+8) given by consecutive primes (3, 5, 7, 11) with [same-valued] smallest diameter d = 8. All the arbitrarily large number of ≥ 2-digit primes in Admissible Prime 4-tuplets commencing sequentially as (11, 13, 17, 19), (101, 103, 107, 109), (191, 193, 197, 199), (821, 823, 827, 829)... must always occur in the same ten-block. Hence it is an established mathematical fact that there must be exactly one with each of these unit digits 1, 3, 7 and 9 in all ≥ 2-digit primes from Admissible Prime 4-tuplets. Except for the first term p_1 = 5 in Admissible Prime 4-tuplet (5, 7, 11, 13), all other terms are congruent to 11 (mod 30). Thus all Admissible Prime 4-tuplets except when first term p_1 = 5 are of the form (15k-4, 15k-2, 15k+2, 15k+4) with k ≥ 1, and so are centered on 15k.

With needing to include diameter d = 2 when k = 2 [viz, s(2) = 2]; Admissible Prime k-tuplets for k ≥ 3 can be computed recursively using the following algorithm ([4], p. 1740) whereby the diameter d is denoted by s(k), gcd is abbreviation for greatest common divisor, and for *p* prime, the notation *p*# is product of all primes up to and including *p*.
Procedure s(k): Do S(s,3,1) for s = s(k-1)+2, s(k-1)+4,... until an admissible set B is found.
Procedure S(s,q,H): *Step 1.* Set U = q#, the product of all the primes q. Set D = $\frac{U}{q}$ and h = H. *Step 2.* Set B = {i: i = 0, 2,..., s, gcd(h+i, U) =1}. *Step 3.* If B does not contain both 0 and s, go to step 8. *Step 4.* If B has less than k elements, go to step 8. *Step 5.* If B has more than k elements, do S(s,q',h), where q' is the next prime after q. Then go to step 8. *Step 6.* If B has exactly k elements and if for each prime p, q < p ≤ k, all residues modulo p are represented by B, go to step 8. *Step 7.* Indicate that B is an admissible set and report s(k) = s. *Step 8.* Add D to h. If h < H + U, go to step 2. Otherwise return.

The above algorithm is related to A008407 *Minimal difference s(n) between beginning and end of n consecutive large primes (n-tuplet) permitted by divisibility considerations.*[5] having values 0 [symbolizing the nonexisting 1-tuple], 2, 6, 8, 12, 16, 20, 26, 30, 32, 36, 42, 48, 50, 56, 60, 66, 70, 76, 80, 84, 90, 94, 100, 110, 114, 120, 126, 130, 136, 140, 146, 152, 156, 158, 162, 168, 176, 182, 186, 188, 196, 200, 210, 212, 216, 226, 236, 240, 246, 252, 254, 264, 270, 272, 278....
Remark 9. *As opposed to Admissible Prime k-tuplets [with diameter d as small as possible], Admissible Prime k-tuples [with diameter d not as small as possible] will cater more for existence of prime numbers with large(r) prime gaps that tend to occur at large(r) range of x integer values. We deduce when these Prime k-tuplets and Prime k-tuples are combined together, they should in principle be able to represent every known odd prime numbers albeit in an overlapping manner.*

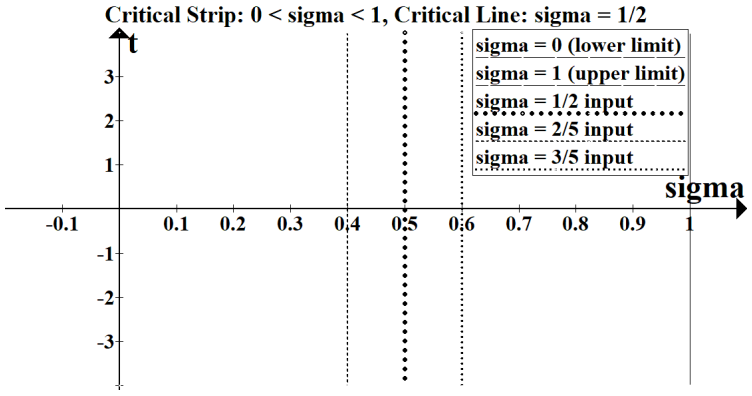

Fig. 5 INPUT for $\sigma = \frac{1}{2}$ (for Figure 6), $\frac{2}{5}$ (for Figure 7), and $\frac{3}{5}$ (for Figure 8). Riemann zeta function, $\zeta(s)$, has countable infinite set of Completely Predictable trivial zeros located at s = all negative even numbers and countable infinite set of Incompletely Predictable nontrivial zeros located at $\sigma = \frac{1}{2}$ as various t-valued transcendental numbers.

Proposition 2. *Both $f(n)$ simplified Dirichlet eta function and $F(n)$ Dirichlet Sigma-Power Law will manifest Principle of Equidistant for Multiplicative Inverse.*

Proof. Dirichlet eta function η(s) is the proxy function for Riemann zeta function ζ(s). We use sim-η(s) to denote $f(n)$ simplified Dirichlet eta function. With also containing variable n, and parameters t and σ; sim-η(s) is essentially obtained by applying Euler formula to η(s) and the $F(n)$ Dirichlet Sigma-Power Law, denoted by DSPL, refers to $\int$sim-η(s)dn.

$$\frac{1}{2^{\frac{1}{2}}}\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}\left[(2n)^{\frac{1}{2}}\cos(t\ln(2n)-\frac{1}{4}\pi)-(2n-1)^{\frac{1}{2}}\cos(t\ln(2n-1)-\frac{1}{4}\pi)+C\right]_1^{\infty}=0 \quad (6)$$

With exact Dimensional analysis homogeneity, Eq. (6) is F(n) DSPL at $\sigma = \frac{1}{2}$ that will incorporate all nontrivial zeros [as Pseudo-zeroes to Zeroes conversion]. There is total absence of (non-existent) virtual nontrivial zeros [as virtual Pseudo-zeroes to virtual Zeroes conversion]. If alternatively using infinitesimal small number below [by letting variable $\delta = \frac{1}{\infty}$], we immediately recognize this action also confirms Proposition 2 to be true [and provide further definitive evidence to support rigorous Equation-type proof for Riemann hypothesis].

Let variable $\delta = \frac{1}{10}$. This will consistently generate in Figure 7 and Figure 8 the δ induced shift of [infinitely many] Varying Loops in reference to Origin; viz, the simple relationship of [more negative] left-shift given by $\zeta(\frac{1}{2} - \delta + \imath t)$ [Figure 7] < [neutral] nil-shift given by $\zeta(\frac{1}{2} + \imath t)$ [Figure 6] < [more positive] right-shift given by $\zeta(\frac{1}{2} + \delta + \imath t)$ [Figure 8].

Given $\delta = \frac{1}{10}$, the $\sigma = \frac{1}{2} - \delta = \frac{2}{5}$-non-critical line (represented by Figure 7) and $\sigma = \frac{1}{2} + \delta = \frac{3}{5}$-non-critical line (represented by Figure 8) are **equidistant** from $\sigma = \frac{1}{2}$-critical line (represented by Figure 6). The additive inverse operation of sin(δ) + sin(-δ) = 0 indicating symmetry with respect to Origin [or cos(δ) - cos(-δ) = 0 indicating symmetry with respect to y-axis] is not applicable to our complex single sine wave [or single cosine wave] since **(2n)-complex or (2n-1)-complex term with transcendental functions** consisting of sine, cosine, single sine wave, single cosine wave, natural logarithm are **independent of parameter** σ.

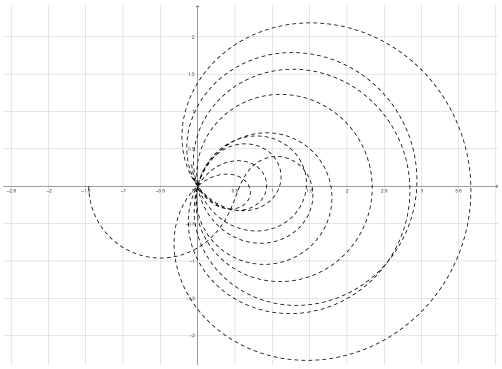

Fig. 6 OUTPUT for $\sigma = \frac{1}{2}$ as Gram points. Polar graph of $\zeta(\frac{1}{2} + \imath t)$ plotted along critical line for real values of t running from 0 to 34. Horizontal axis: $Re\{\zeta(\frac{1}{2} + \imath t)\}$. Vertical axis: $Im\{\zeta(\frac{1}{2} + \imath t)\}$. Presence of Origin intercept points.

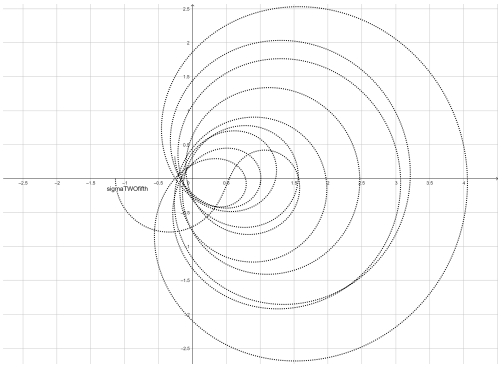

Fig. 7 OUTPUT for $\sigma = \frac{2}{5}$ as virtual Gram points. Varying Loops are shifted to left of Origin with horizontal axis: $Re\{\zeta(\frac{2}{5} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{2}{5} + \imath t)\}$. Nil Origin intercept points.

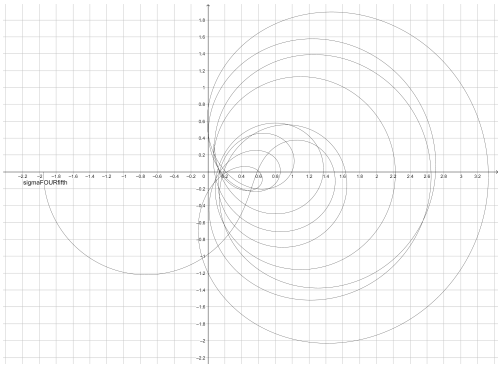

Fig. 8 OUTPUT for $\sigma = \frac{3}{5}$ as virtual Gram points. Varying Loops are shifted to right of Origin with horizontal axis: $Re\{\zeta(\frac{3}{5} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{3}{5} + \imath t)\}$. Nil Origin intercept points.

However, **(2n)-complex or (2n-1)-complex term with algebraic functions** consisting of powers, fractional powers, root extraction [and scaled amplitude R on its (in)dependency on parameter t] are **dependent on parameter** σ. Let x = $(2n)$ or $\frac{1}{(2n)}$ or $(2n-1)$ or $\frac{1}{(2n-1)}$. With multiplicative inverse operation of $x^{\delta} \cdot x^{-\delta} = 1$ or $\frac{1}{x^{\delta}} \cdot \frac{1}{x^{-\delta}} = 1$ that is applicable, this imply intrinsic presence of **Multiplicative Inverse** in sim-$\eta(s)$ or DSPL for all σ values with this function or law rigidly obeying relevant trigonometric identity. We call this phenomenon **Principle of Equidistant for Multiplicative Inverse**. By letting $\delta = 0$, one can always generate Figure 6 representing $\sigma = \frac{1}{2}$-critical line. *The proof is now complete for Proposition 2*□.

7 Infinitesimal numbers applied to Prime numbers and Nontrivial zeros

The following is considered under real-valued functions of a positive real variable: The asymptotic law of distribution of prime numbers states the limit of quotient of two functions Prime-$\pi(x)$ and $\frac{x}{\log_e x}$ as x increases without bound is 1; viz, $\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)}{\left[\frac{x}{\log_e x}\right]} = 1$. Whereby the deceleratingly distributed prime numbers mathematically involves $\log_e x$, then the acceleratingly distributed composite numbers must mathematically involves e^x since these two set of numbers are [complementary] mutually exclusive entities and natural logarithm function is the inverse function of natural exponential function. Then the corresponding asymptotic law of distribution of composite numbers is $\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)}{\left[\frac{x}{e^x}\right]} = 1$. A direct consequence of Prime number theorem [discussed under li(x) in section 6] is average gap between primes [arising from arbitrarily large number of all even prime gaps 2, 4, 6, 8...] increases as natural logarithm of these primes, and therefore the ratio of average prime gap to all primes involved decreases (and is asymptotically zero). Heuristically, we expect the probability that the ratio of the length of prime gap to the natural logarithm is ≥ a fixed positive number k to be e^{-k}. The average prime gap between all primes, and the ratio of prime gap to number of digits in the integers involved, will both increase without bound as we go out on the number line.

Manifesting perpetual asymptotically zero behavior as the graphical distance gets closer to 0 or ∞; the natural logarithm of a number is its logarithm to base of mathematical constant e – an irrational (transcendental) number ≈ 2.718281828459. With $\log_e e = 1$ and $\log_e 1 = 0$, the $\log_e x$ function will deceleratingly grow to +∞ [conceptually a "zero"] as x increases without bound, and deceleratingly grow to –∞ [conceptually a "zero"] as x approaches 0. Let $f(x) = g(x) + h(x) + i(x)...$ be a mathematically well-defined component function constituted by sum of its sub-component functions that all contain natural logarithm [and are all Infinite-Length functions]. Then $f(x)$ together with $g(x)$, $h(x)$, $i(x)...$ must all manifest the asymptotically zero behavior of natural logairthm. Conceptually, $f(x)$ can represent average prime gap between all odd primes [from arbitrarily large number of all even Prime gaps 2, 4, 6, 8, 10...], $g(x)$ can represent average prime gap between all twin primes [from even Prime gap 2], $h(x)$ can represent average prime gap between all cousin primes [from even Prime gap 4], $i(x)$ can represent average prime gap between all sexy primes [from even Prime gap 6], etc whereby

relevant computations from Appendix C confirm average prime gap between all odd primes manifesting this asymptotically zero behavior of natural logarithm.

For Modified Polignac's and Twin prime conjectures to be true, none of the arbitrarily large number of Subsets Odd Primes generated by corresponding even Prime gaps should ever become countably finite subsets. There are two somewhat anomalous situations.
(A) Prime numbers tend to be clustered around large or larger prime gaps occurring as multiples of 6; viz, prime gaps 6, 12, 18.... We deduce this observation do not prove or disprove Modified Polignac's and Twin prime conjectures, and can be logically explained as follow. Excepting the first two prime gaps, all prime gaps are between numbers that are either 1 or 5 modulo 6. Under the assumption that both cases are equally likely, half the prime gaps will be between numbers in the same class, and therefore of size 0 modulo 6, and the other half will be between numbers in different classes, which split up into sizes that are 2 and 4 modulo 6. Since each of the latter cases only gets one quarter of the total, it is clear that ignoring all other factors, gaps that are 2 or 4 modulo 6 are about half as likely to occur as gaps of the same approximate magnitude that are 0 modulo 6.
(B) Here is a simple proof for two consecutive prime gaps that are equal must be of the form (6n, 6n) for n = 1, 2, 3, 4, 5...: Suppose there were two consecutive gaps between 3 consecutive prime numbers that were equal, but not divisible by 6. Then the difference is 2k where k is not divisible by 3. Therefore the (supposed) prime numbers will be p, p+2k, p+4k. But then p+4k is congruent modulo 3 to p+k. That makes the three numbers congruent modulo 3 to p, p+k, p+2k. One of those is divisible by 3 and so cannot be prime. So two consecutive gaps must be divisible by 3 and therefore (as they have to be even) by 6. However this observation neither prove nor disprove Modified Polignac's and Twin prime conjectures.

Riemann hypothesis propose all nontrivial zeros to be located on $\sigma = \frac{1}{2}$-critical line of Riemann zeta function. Previous confirmation of first 10,000,000,000,000 nontrivial zeros location on this critical line implies but does not prove Riemann hypothesis to be true. Hardy initially[9], and then with Littlewood[10], showed there are *infinitely many nontrivial zeros lying on critical line* or, equivalently, there are *infinitely many Origin intercept points lying on Origin point* by considering moments of certain functions related to Riemann zeta function. This discovery cannot constitute rigorous proof for Riemann hypothesis because they have not exclude theoretical existence of nontrivial zeros located away from critical line when $\sigma \neq \frac{1}{2}$. Furthermore, it is literally a mathematical impossibility (*mathematical impasse*) to computationally check [in a complete and successful manner] the locations of all infinitely many nontrivial zeros to correctly be the critical line. There must be infinitely many $\pm t$-valued Origin intercept points lying on Origin point [and hence infinitely many $\pm t$-valued nontrivial zeros] since variable t has full range of values given by $-\infty < t < +\infty$ that involves $\pm\infty$.

An infinitesimal number is a quantity that is closer to zero than any standard real number, but that is not zero. The mathematical concept infinity is represented by symbol ∞. The reciprocal or inverse symbol $\frac{1}{\infty}$ is the representation of the mathematical concept infinitesimal.

Proposition 3. *With the prevalence of various selected odd prime numbers as endpoints never becoming zero [which are conceptually defined as the nonexisting zero in this instance], we can apply infinitesimal numbers to rigorously show both the prevalence of total odd prime numbers having all even Prime gaps and the prevalence of subtotal odd prime numbers having corresponding even Prime gaps will never become zero.*

Proof. We recall that all CIS-ALN-decelerating computed prime numbers are extrapolated out over a wide range of x ≥ 2 integer values; the prime counting function Prime-π(x) = number of primes ≤ x with x [conveniently] assigned to having odd number values of the form $10^n - 1$ whereby n = 1, 2, 3, 4, 5...; and the **Prevalence of prime numbers** = Prime-π(x) / x = Prime-π(x) / (10^n − 2) when x = 2 to 10^n − 1. Note: The probability theory applied to n-digit primes and n-digit composites are given in subsection 7.1.

Apart from even Prime gaps of the form 6n with n = 1, 2, 3, 4, 5... and the [solitary] consecutive prime gaps (2, 2) present in Inadmissible Prime 3-tuple with consecutive primes (3, 5, 7), no other types of two consecutive prime gaps that are identical is possible. In reality, one could then rigorously argue from first principle alone there must be at least three even Prime gaps that will perpetually reappear over the entire sequence of prime numbers because the alternatingly appearance of just two different even Prime gaps at extremely large x integer values simply cannot occur.

We recall from Proposition 1 concerning a given k_0-tuple $\mathcal{H} = (h_1, \ldots, h_{k_0})$ of k_0 distinct integers for some $k_0 \geq 1$, arranged in increasing order whereby one can, in principle, find an arbitrarily large number of translates $n + \mathcal{H} = (n+h_1, \ldots, n+h_{k_0})$ of $\mathcal{H}$ which consists entirely of consecutive primes. The case $k_0 = 1$ is just Euclid's theorem on the infinitude of primes. From this simple theorem, we provide following mathematical arguments:
The cardinality of all prime numbers [or all odd prime numbers when we validly ignore the only even prime number 2] is given by ||CIS-ALN-decelerating|| = $\aleph_0$-decelerating when $n \to \infty$ in x = 2 to $10^n - 1$. The cardinality of all integer numbers is given by ||CIS-IM-linear|| = $\aleph_0$-linear when $n \to \infty$ in x = 0 to $10^n - 1$. As $n \to \infty$, there are an arbitrarily large number (ALN) of deceleratingly-occurring prime numbers amongst the infinitely many linearly-occurring x integer numbers; viz, x integer numbers ≫ prime numbers. Then **Prevalence of prime numbers** = Prime-π(x) / x = ALN / ∞ = an infinitesimal number symbolized by $\frac{1}{\infty}$ when denominator x represents the range 0 to ∞. Since Euclid's theorem holds for x = 2 to ∞, then **Prevalence of prime numbers** is constituted by an infinitesimal number but can never become zero; viz, **Prevalence of prime numbers** conceptually have a nonexisting zero.

A substantial amount of previous materials refer to the proposal on subsets of odd prime numbers uniquely derived from corresponding arbitrarily large number of even Prime gaps 2i with i = 1, 2, 3, 4, 5... in that all these subsets [which equates to Admissible Prime 2-tuplets with diameter or prime gap 2 + Admissible Prime 2-tuples with diameter or prime gap 4, 6, 8, 10...] must also be arbitrarily large in number. Remark 8, in particular, support this proposal. There must be full compliance with (i) Dimensional analysis homogeneity on relevant cardinality, and (ii) even Prime gaps will never terminate. All odd prime numbers having all even Prime gaps 2, 4, 6, 8, 10... = odd prime numbers having even Prime gap 2 + odd prime numbers having even Prime gap 4 + odd prime numbers having even Prime gap 6 +... + odd prime numbers having even Prime gap 2i $\implies$ $\aleph_0$-decelerating [all odd prime numbers] = $\aleph_0$-decelerating [odd prime numbers having even Prime gap 2] + $\aleph_0$-decelerating [odd prime numbers having even Prime gap 4] + $\aleph_0$-decelerating [odd prime numbers having even Prime gap 6] +... + $\aleph_0$-decelerating [odd prime numbers having even Prime gap 2i]. Based on similar reasoning from previous paragraph, we logically deduce that for x = 2 to ∞, **Prevalence of various odd prime numbers as specified by their corresponding even Prime gaps 2i** can similarly all be constituted by infinitesimal numbers symbolized by $\frac{1}{\infty}$ but

never become zero; viz, **Prevalence of various odd prime numbers as specified by their corresponding even Prime gaps 2i** conceptually have a nonexisting zero.

The proof is now complete for Proposition 3□.

Proposition 4. *With $\sigma = \frac{1}{2}$-Origin point or $\sigma = \frac{1}{2}$-critical line of Riemann zeta function (via proxy Dirichlet eta function) regarded as the zero endpoint [which is conceptually defined as the existing zero in this instance], we can apply infinitesimal numbers to rigorously show the equivalent [geometrical] Origin intercept points located at the zero-dimensional $\sigma = \frac{1}{2}$-Origin point and [mathematical] nontrivial zeros located at the one-dimensional $\sigma = \frac{1}{2}$-critical line will uniquely appear only when parameter $\sigma = \frac{1}{2}$.*

Proof. For simplicity, we use Riemann zeta function to also indicate Dirichlet eta function, simplified Dirichlet eta function and Dirichlet Sigma-Power Law [and note the relevant *Zeros* = *Pseudo-zeros* – $\frac{\pi}{2}$ relationship]. We recall that all CIS-IM-linear computed nontrivial zeros are extrapolated out over a wide range of t ≥ 0 real number values; and the Nontrivial zeros gaps, Nontrivial zeros counting function and Prevalence of nontrivial zeros can be defined. Although inevitably fluctuating, the initial **Prevalence of nontrivial zeros** is approximated by, for instance, using t = 0 to 100 range as 29/100 = 0.29 = 29% since there are precisely 29 nontrivial zeros in this range. As noted in section 3, the *rolling* **Prevalence of nontrivial zeros** seems to overall increase by around 0.366 in a "linear" manner over t = 0 to ∞.

We recall variable $\delta = \frac{1}{10}$ when applied to Riemann zeta function in Proposition 2 to confirm **Principle of Equidistant for Multiplicative Inverse** refers to Figure 7 representing $\sigma = \frac{2}{5}$-non-critical line and Figure 8 representing $\sigma = \frac{3}{5}$-non-critical line. We recognize zero-dimensional $\sigma = \frac{1}{2}$-Origin point in Figure 6 is synonymous with one-dimensional $\sigma = \frac{1}{2}$-critical line, and this particular point or line is conceptually regarded as the existing zero. Then Varying Loop trajectory in Figure 6 will only depict CIS-linear [geometrical] Origin intercept points that is precisely equivalent to CIS-linear [mathematical] nontrivial zeros when $\delta = 0$ since Origin point is a zero-dimensional point that can only be touched by the trajectory when $\delta = 0$ and $\sigma = \frac{1}{2}$. We logically deduce variable δ = infinitesimal number value $\frac{1}{\infty}$ will never become the existing zero since this equates to $\sigma \approxeq \frac{1}{2}$ [or the trajectory is extremely close to zero-dimensional Origin point] but this is categorically still not the same as $\sigma = \frac{1}{2}$ [or the trajectory touching zero-dimensional Origin point]. Thus variable δ will instead only become the existing zero when both $\sigma = \frac{1}{2}$ and $\delta = 0$ conditions are simultaneously fully satisfied.

Manifestation of Dimensional analysis (DA) homogeneity by parameter σ in Riemann zeta function: The *exact DA homogeneity* indicate calculated values of [exact] integer −1 and 1 as derived from $\sum$(all fractional exponents) = $2(-\sigma)$ and $2(1-\sigma)$. Respectively, these act as surrogate markers in simplified Dirichlet eta function and Dirichlet Sigma-Power Law on the solitary unique $\sigma = \frac{1}{2}$ "Proposition" situation. Otherwise, for the infinitely many non-unique $\sigma \neq \frac{1}{2}$ "Corollary" situations, calculated values of [inexact] fractional numbers ≠ integer –1 and ≠ integer 1 are derived from $\sum$(all fractional exponents) = $2(-\sigma)$ and $2(1-\sigma)$ to indicate *inexact DA homogeneity*. *The proof is now complete for Proposition 4*□.

Let $\mathbb{P}_i$, $\mathbb{P}_{i+1}$, $\mathbb{P}_{i+2}$ and $\mathbb{P}_{i+3}$ = four randomly selected consecutive prime numbers whereby $\mathbb{P}_{i+3} > \mathbb{P}_{i+2} > \mathbb{P}_{i+1} > \mathbb{P}_i$. If this four primes are considered in total isolation, then there are only three possible prime gaps able to be computed: Prime $\text{gap}_i = \mathbb{P}_{i+1} - \mathbb{P}_i$, Prime gap_{i+1}

$= \mathbb{P}_{i+2} - \mathbb{P}_{i+1}$ and Prime $\text{gap}_{i+2} = \mathbb{P}_{i+3} - \mathbb{P}_{i+2}$. In principle, we recognize these three prime gaps can be constituted by all possible combinations of small prime gaps 2 and 4 and/or large prime gaps ≥ 6; viz, all three prime gaps are constituted by small prime gaps, all three prime gaps are constituted by large prime gaps, and the three prime gaps are constituted by a mixture of small and large prime gaps. Intuitively, every even Prime gap 2, 4, 6, 8, 10... and its correspondingly associated odd prime numbers must exist at least once; viz, occurring only one time, occurring a finite number of times, or occurring an arbitrarily large number of times. Proving the only correct possibility of both *even Prime gaps 2, 4, 6, 8, 10... and their correspondingly associated odd prime numbers occurring an arbitrarily large number of times* is equivalent to proving Modified Polignac and Twin prime conjectures to be true.

There is one and only one $\sigma = \frac{1}{2}$ critical line that is mutually exclusive and independent from the infinitely many $\sigma \neq \frac{1}{2}$ noncritical lines. Since there is zero probability that any particular parameter $\sigma \neq \frac{1}{2}$ values that do occur in Dirichlet eta function will mathematically represent the $\sigma = \frac{1}{2}$ critical line [or will geometrically represent the analogous $\sigma = \frac{1}{2}$ Origin point], we consequently deduce all countably infinitely large number of nontrivial zeros that linearly reach an infinity value as generated from Dirichlet eta function when parameter $\sigma = \frac{1}{2}$ will, by default, also have to be located on the $\sigma = \frac{1}{2}$ critical line. Proving the only correct possibility of unique $\sigma = \frac{1}{2}$ critical line location for all nontrivial zeros is equivalent to proving Riemann hypothesis to be true.

7.1 Probability theory applied to n-digit Primes and n-digit Composites

With Probability = 100% X Proportion, Probability and Proportion are literally equivalent to each other for analysis on prime and composite numbers (and nontrivial zeros). If the probability [range between 0 or 0% and 1 or 100%] of an event occurring is Y, then the probability [range between 0 or 0% and 1 or 100%] of the event not occurring is 1–Y. The odds of an event represent the ratio *Probability that the event will occur* : *Probability that the event will not occur*. This can be succinctly expressed as Odds of event $= \dfrac{Y}{1-Y}$.

Based on cardinality of (sub)sets of primes and composites used in Prime-Composite quotient from section 6, we interpret their Probability or Proportion will satisfy

$$\text{P(odd primes)} \approx \frac{1}{2 \cdot \text{P(Gap 1-Even composites)} + 2 \cdot \text{P(Gap 1-Odd composites)}}.$$

P(any number is divisible by a prime p, or in fact any integer) $= 1/p$. Let there be k randomly chosen integers. When k = 2, P(two numbers are both divisible by p) $= 1/p^2$, and P(at least one of the two numbers is not divisible by p) $= 1 - 1/p^2$. Any finite collection of divisibility events associated to distinct primes is mutually independent. For example, in the case of two events, a number is divisible by primes p and q *iff* it is divisible by pq; the latter event has probability 1/pq. We make the heuristic assumption that such reasoning can be extended to infinitely many divisibility events. Then, P(two numbers are coprime) $= \prod_{\text{prime } p} \left(1 - \dfrac{1}{p^2}\right) = \left(\prod_{\text{prime } p} \dfrac{1}{1-p^{-2}}\right)^{-1} = \dfrac{1}{\zeta(2)} = \dfrac{6}{\pi^2} \approx 0.607927102 \approx 61\%$ – a product over all primes. More generally, P(k randomly chosen integers being coprime) $= \dfrac{1}{\zeta(k)}$.

The fundamental theorem of arithmetic asserts that every nonzero integer can be written as a product of primes in a unique way, up to ordering and multiplication by units. Prime numbers are defined as *All integers apart from 0 and 1 that are evenly divisible by itself and by 1*. Composite numbers are defined as *All integers apart from 0 and 1 that are evenly divisible by numbers other than itself and 1*. The integer numbers ($\mathbb{Z}$) = {0, 1, 2, 3, 4...}, prime numbers ($\mathbb{P}$) = {2, 3, 5, 7, 11...} and composite numbers ($\mathbb{C}$) = {4, 6, 8, 9, 10...} can all be analyzed in terms of their corresponding unique n-digit numbers.

n	0	1	2	3	4	5	6	7	8	9	10	...
A006879$_n$	0	4	21	143	1061	8363	68906	586081	5096876	45086079	404204977	...
A006880$_n$	0	4	25	168	1229	9592	78498	664579	5761455	50847534	455052511	...

A006879 Number of primes with n digits. Number of primes between $10^{(n-1)}$ and 10^n[23]. Using our unique n-digit $\mathbb{P}$ grouping, this statement is mathematically equivalent to Number of primes between $10^{(n-1)}$ and $10^n - 1$ since the integer 10^n itself can never be prime.

A006880 Number of primes < 10^n. Number of primes with at most n digits or Prime counting function $\mathbb{P}$-$\pi(< 10^n)$ defined as $\|\mathbb{P} < 10^n\|$[24]. Using our unique n-digit $\mathbb{P}$ and n-digit $\mathbb{C}$ groupings, Prime counting function $\mathbb{P}$-$\pi(\leq 10^n - 1)$ is defined as $\|\mathbb{P} \leq 10^n - 1\|$; and Composite counting function $\mathbb{C}$-$\pi(\leq 10^n - 1)$ as $\|\mathbb{C} \leq 10^n - 1\|$.

The above two integer sequences A006879 and A006880 are directly related to our unique n-digit $\mathbb{P}$ and n-digit $\mathbb{C}$ groupings whereby n = 0, 1, 2, 3, 4... [to an arbitrarily large number]. A006880 forms the partial sums of A006879. Using n-digit $\mathbb{P}$ grouping, A006879 can be alternatively defined as *The number of primes between* $10^{(n-1)}$ *and* $10^n - 1$ *which supply precisely the original and identical A006879$_n$ as n-digit prime number values.* By employing similar crucial step of using n-digit $\mathbb{C}$ grouping *The number of composites between* $10^{(n-1)}$ *and* $10^n - 1$*, we obtain the complementary-A006879$_n$ as n-digit composite number values.* There are precisely $10^n - 1$ minus $10^{(n-1)}$ plus $1 = 10^n - 10^{(n-1)}$ integer numbers between $10^{(n-1)}$ and $10^n - 1$. The important implication is that we are now always dealing with the same n-digit integer, prime and composite numbers whereby the relationship n-digit $\mathbb{Z}$ = n-digit $\mathbb{P}$ + n-digit $\mathbb{C}$ will always hold [except for when n = 1 because 0 and 1 are neither prime nor composite]. We note from A006879 and A006880 the number of primes that are still constituted by very large number values will proportionately decline rapidly with progressively larger n values assigned to $10^{(n-1)}$ and $10^n - 1$. One can aesthetically speculate there will always be many allocated primes to theoretically represent all possible small or large Prime gaps.

For i = 1, 2, 3, 4, 5..., Set of $\mathbb{Z}_i$ {0, 1, 2, 3, 4...} as **CIS-IM-linear** = Set of neither $\mathbb{P}$ nor $\mathbb{C}$ {0, 1} as **CFS** + Set of $\mathbb{P}_i$ {2, 3, 5, 7, 11...} as **CIS-ALN-decelerating** + Set of $\mathbb{C}_i$ {4, 6, 8, 9, 10...} as **CIS-IM-accelerating**. All $\mathbb{P}$ are odd except for the first and only even $\mathbb{P}$ 2. There is only one solitary even $\mathbb{P}$ 2 and one solitary odd $\mathbb{P}$ 5 that are not $\mathbb{C}$. Otherwise, all $\mathbb{Z}$ with their last digit ending as even numbers 0, 2, 4, 6 or 8, or odd number 5 must always be $\mathbb{C}$. Apart from $\mathbb{P}$ 2 and $\mathbb{P}$ 5, all $\mathbb{P}$ have their last digit ending as odd numbers 1, 3, 7 or 9. But not all $\mathbb{Z}$ with their last digit ending as odd numbers 1, 3, 7 or 9 are $\mathbb{P}$ – in fact, these numbers are more likely to be $\mathbb{C}$ than $\mathbb{P}$. We deduce that for ≥2-digit numbers, (i) $\mathbb{C}$ can have their last digit ending in 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 but (ii) $\mathbb{P}$ can only have their last digit ending in 1, 3, 7 or 9; and thus (iii) all $\mathbb{Z}$ with their last digit ending in 0, 2, 4, 5, 6 or 8 must be $\mathbb{C}$.

For n = 1, 2, 3, 4, 5...[to an arbitrarily large number]; we apply probability theory to the generated subsets of n-digit $\mathbb{P}$ as **CIS-ALN-decelerating** and n-digit $\mathbb{C}$ as **CIS-IM-accelerating**. With probability 1, all randomly selected $\mathbb{Z}$ that has its last digit ending with 0,

2, 4, 5, 6 or 8 must be *almost surely* $\mathbb{C}$. This is equivalently stated as: With probability 0, all randomly selected $\mathbb{Z}$ that has its last digit ending with 0, 2, 4, 5, 6 or 8 must be *almost never* $\mathbb{P}$. Thus, P(randomly selected $\mathbb{Z}$ is $\mathbb{C}$ with 100% certainty) = 0.6 [except for the isolated 1-digit $\mathbb{Z}$ 0 and 1-digit $\mathbb{P}$ 2 and 5 which are not $\mathbb{C}$]. The terms *almost surely* and *almost never* can now be replaced with *surely* and *never* when we disregard the 1-digit $\mathbb{P}$ and 1-digit $\mathbb{C}$. Since the condition "randomly selected $\mathbb{Z}$ that has its last digit ending with 0, 2, 4, 5, 6 or 8 must be *surely* $\mathbb{C}$" will always apply to any chosen subsets of ≥2-digit $\mathbb{Z}$, the consequently derived equivalent condition "60% of all $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty" will also always apply to these same subsets. Then, with 60% of all $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty, there are always more $\mathbb{C}$ than $\mathbb{P}$ for any chosen corresponding subsets of ≥2-digit $\mathbb{P}$ and ≥2-digit $\mathbb{C}$.

Constraints on Prime numbers and Prime gaps: We define Prime $\text{gap}_i = \mathbb{P}_{i+1} - \mathbb{P}_i$. We ignore $\mathbb{P}_1 = 2$ and $\mathbb{P}_3 = 5$. We convey the paired list of (last digit for $\mathbb{P}_i$, last digit for $\mathbb{P}_{i+1}$) as full range of choices permissible for corresponding specified groupings of prime gaps:
CIS-ALN-decelerating $\mathbb{P}_i$ selected from Prime gap_i = 2, 12, 22, 32...[to an arbitrarily large number as **CIS-ALN-decelerating**] → (1, 3), (7, 9), (9, 1). The last digit of $\mathbb{P}_i$ with prime gap having last digit ending in 2 cannot end in 3 or 5 but can end in 1, 7 or 9.
CIS-ALN-decelerating $\mathbb{P}_i$ selected from Prime gap_i = 4, 14, 24, 34...[to an arbitrarily large number as **CIS-ALN-decelerating**] → (3, 7), (7, 1), (9, 3). The last digit of $\mathbb{P}_i$ with prime gap having last digit ending in 4 cannot end in 1 or 5 but can end in 3, 7 or 9.
CIS-ALN-decelerating $\mathbb{P}_i$ selected from Prime gap_i = 6, 16, 26, 36...[to an arbitrarily large number as **CIS-ALN-decelerating**] → (1, 7), (3, 9), (7, 3). The last digit of $\mathbb{P}_i$ with prime gap having last digit ending in 6 cannot end in 5 or 9 but can end in 1, 3 or 7.
CIS-ALN-decelerating $\mathbb{P}_i$ selected from Prime gap_i = 8, 18, 28, 38...[to an arbitrarily large number as **CIS-ALN-decelerating**] → (1, 9), (3, 1), (9, 7). The last digit of $\mathbb{P}_i$ with prime gap having last digit ending in 8 cannot end in 5 or 7 but can end in 1, 3 or 9.
CIS-ALN-decelerating $\mathbb{P}_i$ selected from Prime gap_i = 10, 20, 30, 40...[to an arbitrarily large number as **CIS-ALN-decelerating**] → (1, 1), (3, 3), (7, 7), (9, 9). The last digit of $\mathbb{P}_i$ with prime gap having last digit ending in 0 cannot end in 5 but can end in 1, 3, 7 or 9.

The coprime numbers [and all prime numbers] are numbers whose HCF is 1. The difference between any two Odd Primes is always equal to 2, 4, 6, 8, 10... whereas the difference between two coprime numbers can be any number. Odd Primes are always prime numbers, whereas coprime numbers can also be composite numbers. As per the List above [which was previously discussed under Axiom 1], it is a mathematical impossibility that, for instance, twin primes $\mathbb{P}_i$ can be constituted by a random integer with last digit ending in 3 or 5 since only the paired last digit combinations of $(\mathbb{P}_{i+1}, \mathbb{P}_i)$ = (1, 3), (7, 9), (9, 1) are possible.

CIS-ALN-decelerating $\mathbb{P}_i$ having Prime gap_i [given as multiples of 10] with last digit ending in 0 is associated with four choices that are available arbitrarily often. Otherwise, **CIS-ALN-decelerating** $\mathbb{P}_i$ having Prime gap_i with last digit ending in 2, 4, 6 or 8 is each associated with three choices that are available arbitrarily often. Statistically, the last digit of $\mathbb{Z}_i$ ending in 1, 3, 7 or 9 are more likely to be just $\mathbb{O}_i$ than [odd] $\mathbb{P}_i$. At ever larger range of numbers for the paired list of (last digit for $\mathbb{P}_i$, last digit for $\mathbb{P}_{i+1}$), we can intuitively surmise that $\mathbb{P}$ associated with progressively larger prime gaps moving from left to right and from top to bottom should occur relatively more often than $\mathbb{P}$ associated with comparatively smaller prime gaps. However, both $\mathbb{P}$ associated with progressively larger prime gaps and $\mathbb{P}$ associated with comparatively smaller prime gaps should generally occur less often at ever larger range

of numbers. Thus, although prime gap having last digit ending in 0 can be associated with last digit of $\mathbb{P}_i$ ending in 1, 3, 7 or 9 as four choices [instead of just three choices]; these prime gaps as a unique group will still always constitute larger prime gaps that will overall intrinsically occur less often at ever larger range of numbers.

The crucial overall inference here is that all known last digit of $\mathbb{P}_i$ ending in 1, 3, 7 or 9 that literally represent all existing even Prime gaps must do so on in an eternal manner thus confirming Modified Polignac's and Twin prime conjectures to be true.

Definition 1. *Our Modified Polignac's and Twin prime conjectures can now be explicitly defined here as* ***||Set all even Prime gaps|| = ||Subset odd prime numbers associated with each even Prime gap|| = CIS-ALN-decelerating****. Previously, Polignac's and Twin prime conjectures were traditionally stated in a less informative manner as* ***||Set all even Prime gaps|| = ||Subset odd prime numbers associated with each even Prime gap|| = CIS****.*

Constraints on Composite numbers and Composite gaps: We define Composite $\text{gap}_i = \mathbb{C}_{i+1} - \mathbb{C}_i$. For 1-digit $\mathbb{P}$ and 1-digit $\mathbb{C}$ that are members of 1-digit $\mathbb{Z}$, there are always more $\mathbb{P}$ than $\mathbb{C}$ except at $\mathbb{Z} = 9$ which is $\mathbb{C}$ and whereby now $||\mathbb{P}|| = ||\mathbb{C}|| = 4$. There will always be more $\mathbb{C}$ as **CIS-IM-accelerating** than $\mathbb{P}$ as **CIS-ALN-decelerating** when $\mathbb{Z} \geq 10$ with $||\mathbb{C}|| = 2\,||\mathbb{P}||$ at $\mathbb{Z} = 14$ and $||\mathbb{C}|| > 2\,||\mathbb{P}||$ at $\mathbb{Z} > 14$.

Let P(certain $\mathbb{C}$) denote P(randomly selected $\mathbb{Z}$ is $\mathbb{C}$ with 100% certainty). Then, P(certain even $\mathbb{C}$) = 0.5 for all subsets ≥2-digit $\mathbb{C}$ having elements with their last digit ending in 0, 2, 4, 6 or 8 [as **CIS-IM-linear**] and P(certain odd $\mathbb{C}$) = 0.1 for subset ≥2-digit $\mathbb{C}$ having elements with their last digit ending in 5 [as **CIS-IM-linear**].

One can more closely analyze $\mathbb{P}$-$\mathbb{C}$ **identifier grouping** which was defined in section 5. Event 1: P(uncertain even $\mathbb{C}$ with last digit ending in 0, 2, 6 or 8) [as **CIS-ALN-decelerating**] can represent Gap-2-E-$\mathbb{C}_1$ that always occur before every O-$\mathbb{P}_i$ with last digit ending in 1, 3, 7 or 9. Event 2: P(uncertain even $\mathbb{C}$ with last digit ending in 0, 2, 4 or 8) [as **CIS-IM-accelerating**] can represent Gap-1-E-$\mathbb{C}_2$ that always occur after every O-$\mathbb{P}_i$ with last digit ending in 1, 3, 7 or 9. Event 3: P(uncertain odd $\mathbb{C}$ with last digit ending in 1, 3, 5, 7 or 9) [as **CIS-IM-accelerating**] can represent Gap-1-O-$\mathbb{C}_3$ that always occur after Gap-1-E-$\mathbb{C}_2$. Event 4: P(uncertain even $\mathbb{C}$ with last digit ending in 0, 2, 4, 6 or 8) [as **CIS-IM-accelerating**] can represent Gap-1-E-$\mathbb{C}_4$ that always occur after Gap-1-O-$\mathbb{C}_3$. Event 5: P(uncertain odd $\mathbb{C}$ with last digit ending in 1, 3, 5, 7 or 9) [as **CIS-IM-accelerating**] can represent Gap-1-O-$\mathbb{C}_5$ that always occur after Gap-1-E-$\mathbb{C}_4$... until Event 6: P(uncertain even $\mathbb{C}$ with last digit ending in 0, 2, 6 or 8) [as **CIS-ALN-decelerating**] can represent Gap-2-E-$\mathbb{C}_n$ that always occur before every O-$\mathbb{P}_{i+1}$ with last digit ending in 1, 3, 7 or 9.

Only Events 1 and 6 [but not Events 2, 3, 4 and 5] can occur for twin primes. In Event 2, there are four choices for Gap-1-E-$\mathbb{C}$ [because O-$\mathbb{P}_i$ generally cannot have their last digit ending in 5] as opposed to Event 4 whereby there are, instead, five choices for Gap-1-E-$\mathbb{C}$. With only the solitary 1-digit Odd $\mathbb{P}$ 5 existing, we deduce odd numbers with last digit ending in 5 are almost always Gap-1-O-$\mathbb{C}$ [or almost never Odd $\mathbb{P}$].

Computed data on n-digit prime numbers including their average prime gaps are supplied in Appendix C. **CIS-ALN-decelerating** Subsets Gap 2i-Odd $\mathbb{P}$, **CIS-ALN-decelerating** Gap 2-Even $\mathbb{C}$, **CIS-IM-accelerating** Gap 1-Even $\mathbb{C}$ and **CIS-IM-accelerating** Gap 1-Odd $\mathbb{C}$ present in $\mathbb{P}$-$\mathbb{C}$ **identifier grouping** must obey *Constraints on Prime numbers & Prime gaps* and *Constraints on Composite numbers & Composite gaps*.

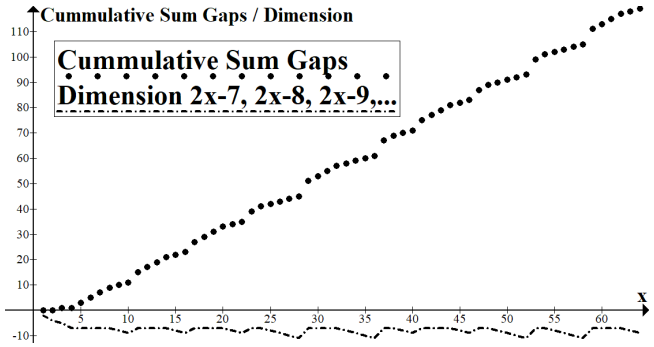

Fig. 9 Prime-Composite mathematical (graphed) landscape for x = 2 to 64.

Remark 10. ***Summary of major statistical deductions on Primes and Composites based on last digit of the chosen number ending in even or odd numbers.***

· (I) P(certain Even $\mathbb{C}$ with last digit ending in 0, 2, 4, 6, 8) = 0.5 when contextually based on "Apart from the even Prime number 2, all even numbers cannot be prime numbers [or must be either Gap 1-Even composite numbers or Gap 2-Even composite numbers] with 100% certainty". P(certain Odd $\mathbb{C}$ with last digit ending in 5) = 0.1 when contextually based on "Apart from the 1-digit Odd prime number 5, all other odd numbers with last digit ending in 5 cannot be prime numbers [or must be Gap 1-Odd composite numbers] with 100% certainty". **Then P(certain $\mathbb{C}$ with 100% certainty) = 0.5 + 0.1 = 0.6 total**.

· (II) P(uncertain Gap 1-Odd $\mathbb{C}$ with last digit ending in 1, 3, 7, 9) + P(uncertain Gap 1-Even $\mathbb{C}$ with last digit ending in 0, 2, 4, 6, 8) + P(uncertain Gap 2-Even $\mathbb{C}$ with last digit ending in 0, 2, 6, 8) + P(uncertain Gap 2i-Odd $\mathbb{P}$ with last digit ending in 1, 3, 7, 9) = 0.4 when contextually based on **P(uncertain $\mathbb{C}$ with < 100% certainty) + P(uncertain $\mathbb{P}$ with < 100% certainty) = 0.4 total**, whereby

$$\text{P(uncertain Gap 1-Even } \mathbb{C}\text{) + P(uncertain Gap 1-Odd } \mathbb{C}\text{)} \propto \frac{1}{\text{P(uncertain Gap 2-Even } \mathbb{C}\text{) + P(uncertain Gap 2i-Odd } \mathbb{P}\text{)}}.$$

As $x \to \infty$, P(uncertain Gap 1-Even $\mathbb{C}$) + P(uncertain Gap 1-Odd $\mathbb{C}$) approaches 1 [but never becomes 1] and P(uncertain Gap 2-Even $\mathbb{C}$) + P(uncertain Gap 2i-Odd $\mathbb{P}$) approaches 0 [but never becomes 0].

8 Anatomy of Prime-Composite Varying Loop

The tabulated and graphed Prime-Composite finite scale mathematical landscape are provided in Table 2 and Figure 9. Figure 10 geometrically depict Incompletely Predictable Prime-Composite Varying Loops. This allows visual representation of two algorithms in action; viz, Sieve-of-Eratosthenes algorithm that generate all primes and Complement-Sieve-of-Eratosthenes algorithm that generate all composites.

Let **N** = natural numbers, **P** = prime numbers, and **C** = composite numbers. Based on the innovative Dimension (2x - N) system with N = 2x - ΣPC_x-Gap and x = all integers commencing from 1; Dimension (2x - N) when expanded is numerically just equal to ΣPC_x-Gap since Dimension (2x - N) = 2x - 2x + ΣPC_x-Gap = ΣPC_x-Gap. Definition for this system is explained using position x = 31 and 32. For i and x ∈ **N** [in Table 2]; ΣPC_x-Gap = ΣPC_{x-1}-Gap + Gap value at $\mathbf{P}_{i-1}$ or Gap value at $\mathbf{C}_{i-1}$ whereby (i) $\mathbf{P}_i$ or $\mathbf{C}_i$ at position x is determined

x	P_i/C_i Gaps	ΣPC_x-Gaps	Dim	x	P_i/C_i Gaps	ΣPC_x-Gaps	Dim
1	N/A	0	2x-2	33	C21, 1	58	2x-8
2	P1, 1	0	2x-4	34	C22, 1	59	2x-9
3	P2, 2	1	2x-5	35	C23, 1	60	2x-10
4	C1, 2	1	Y	36	C24, 2	61	2x-11
5	P3, 2	3	Y	37	P12, 4	67	Y
6	C2, 2	5	Y	38	C25, 1	69	Y
7	P4, 4	7	Y	39	C26, 1	70	2x-8
8	C3, 1	9	Y	40	C27, 1	71	2x-9
9	C4, 1	10	2x-8	41	P13, 2	75	Y
10	C5, 2	11	2x-9	42	C28, 2	77	Y
11	P5, 2	15	Y	43	P14, 4	79	Y
12	C6, 2	17	Y	44	C29, 1	81	Y
13	P6, 4	19	Y	45	C30, 1	82	2x-8
14	C7, 1	21	Y	46	C31, 2	83	2x-9
15	C8, 1	22	2x-8	47	P15, 6	87	Y
16	C9, 1	23	2x-9	48	C32, 1	89	Y
17	P7, 2	27	Y	49	C33, 1	90	2x-8
18	C10, 2	29	Y	50	C34, 1	91	2x-9
19	P8, 4	31	Y	51	C35, 1	92	2x-10
20	C11, 1	33	Y	52	C36, 1	93	2x-11
21	C12, 1	34	2x-8	53	P16, 6	99	Y
22	C13, 2	35	2x-9	54	C37, 1	101	Y
23	P9, 6	39	Y	55	C38, 1	102	2x-8
24	C14, 1	41	Y	56	C39, 1	103	2x-9
25	C15, 1	42	2x-8	57	C40, 1	104	2x-10
26	C16, 1	43	2x-9	58	C41, 1	105	2x-11
27	C17, 1	44	2x-10	59	P17, 2	111	Y
28	C18, 2	45	2x-11	60	C42, 2	113	Y
29	P10, 2	51	Y	61	P18, 6	115	Y
30	C19, 2	53	Y	62	C43, 1	117	Y
31	P11, 6	55	Y	63	C44, 1	118	2x-8
32	C20, 1	57	Y	64	C45, 1	119	2x-9

Table 2 Prime-Composite mathematical (tabulated) landscape for x = 2 to 64. **Legend: C** = composite, **P** = prime, Dim = Dimension, Y = 2x - 7, N/A = Not Applicable.

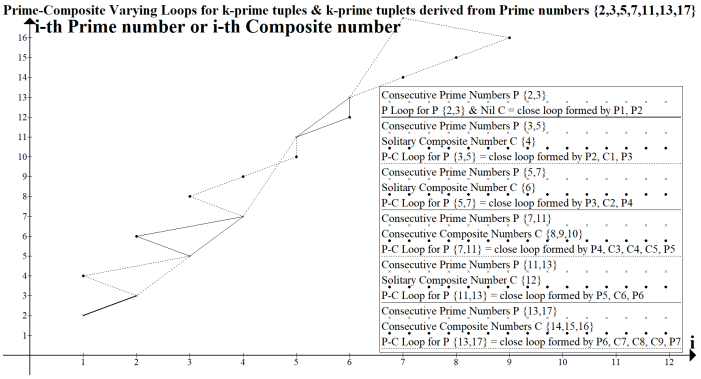

Fig. 10 Prime-Composite Varying Loops. This figure is a geometric representation of prime and composite numbers computed for prime numbers {2, 3, 5, 7, 11, 13 and 17}.

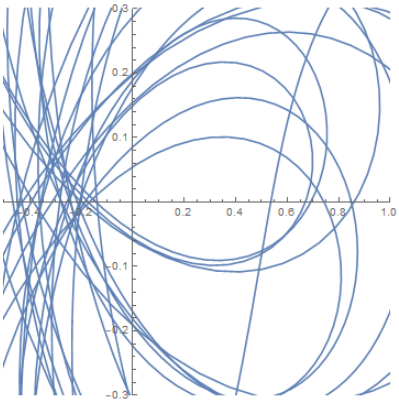

Fig. 11 Close-up view of virtual Origin points when $\sigma = \frac{1}{3}$. OUTPUT for $\sigma = \frac{1}{3}$ [$\sigma < \frac{1}{2}$ situation] as virtual Gram points. Polar graph of $\zeta(\frac{1}{3} + \imath t)$ plotted along non-critical line for real values of t running between 0 and 100, horizontal axis: $Re\{\zeta(\frac{1}{2} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{1}{2} + \imath t)\}$. Total absence of all Origin intercept points at "static" Origin point. Total presence of all virtual Origin intercept points (as additional negative virtual Gram[y=0] points on x-axis) at "varying" [infinitely many] virtual Origin points.

by whether relevant x value belongs to a **P** or **C**, and (ii) both ΣPC_1-Gap and ΣPC_2-Gap = 0. Examples: For position x = 31: 31 is **P** (**P11**). Desired Gap value at **P10** = 2. ΣPC_{31}-Gap (55) = ΣPC_{30}-Gap (53) + Gap value at **P10** (2). For position x = 32: 32 is **C** (**C20**). Desired Gap value at **C19** = 2. ΣPC_{32}-Gap (57) = ΣPC_{31}-Gap (55) + Gap value at **C20** (2).

Plus-Minus Gap 2 Composite Number Alternating Law refers to *rhythmic patterns of alternating presence and absence* for relevant Gap 2 Composite Numbers. Mathematically, it has built-in intrinsic mechanism to automatically generate all prime numbers from prime gaps ≥ 4 appearances in a consistent *ad infinitum* manner. Plus Gap 2 Composite Number Continuous Law refers to *(non-)rhythmic patterns with continual presence* for relevant Gap 2 Composite Numbers. Mathematically, it has built-in intrinsic mechanism to automatically generate all prime numbers from prime gap = 2 appearances in a consistent *ad infinitum* manner. These two deduced Laws ***that crucially involve both prime and composite numbers being dependently and algorithmically tabulated together with subsequent analysis on their [combined] corresponding gaps*** will qualitatively confirm Modified Polignac's and Twin prime conjectures to be true.

9 Anatomy of Nontrivial Zeros-Gram Points Varying Loop

Let Origin intercept point = nontrivial zero (or NTZ) = Gram[x=0,y=0] point (or G[x=0,y=0]P); x-axis intercept point = Gram[y=0] point (or G[y=0]P aka the *'usual'/'traditional'* Gram point); and y-axis intercept point = Gram[x=0] point (or G[x=0]P). We follow the peculiar choice of the index n used for Gram points and NTZ [depicted in order of their initial appearances for $\sigma = \frac{1}{2}$ and positive t values]: n = –3 for 1^{st} –ve G[y=0]P, n = –1 for 1^{st} –ve G[x=0]P, n = –2 for 2^{nd} +ve G[y=0]P, n = –1 for 3^{rd} +ve G[y=0]P, n = 1 for 1^{st} NTZ, n = 0 for 2^{nd} +ve G[x=0]P, n = 0 for 4^{th} +ve G[y=0]P, n = 1 for 3^{rd} –ve G[x=0]P, n = 2 for 2^{nd} NTZ, n = 1 for 5^{th} +ve G[y=0]P, n = 3 for 3^{rd} NTZ, n = 2 for 4^{th} +ve G[x=0]P, n = 2 for 6^{th} +ve G[y=0]P, n = 3 for 5^{th} –ve G[x=0]P, and so on. Thus, we observe different varieties

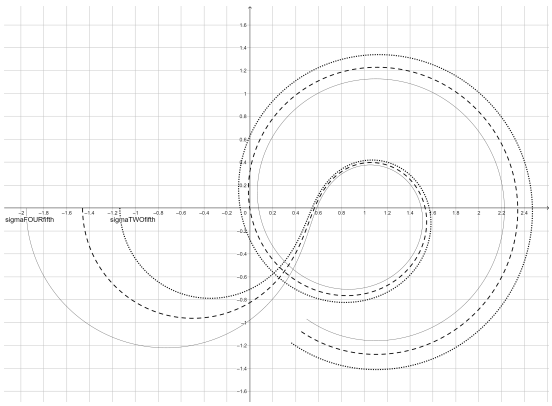

Fig. 12 Simulated dynamic trajectories showing Origin intercept points when $\sigma = \frac{1}{2}$ and virtual Origin intercept points when $\sigma = \frac{2}{5}$ and $\sigma = \frac{4}{5}$. Horizontal axis: $Re\{\zeta(\sigma + \imath t)\}$, and vertical axis: $Im\{\zeta(\sigma + \imath t)\}$. Total presence of all Origin intercept points at the [static] Origin point. Total presence of all virtual Origin intercept points as additional negative virtual Gram[y=0] points on the x-axis (e.g. when using $\sigma = \frac{2}{5}$ value) at the [infinitely many varying] virtual Origin points; viz, these negative virtual Gram[y=0] points on the x-axis cannot exist at the solitary Origin point since the two trajectories form two co-lines.

of Nontrivial Zeros-Gram Points Varying Loops commencing from 1^{st} NTZ: (A) NTZ, +ve G[x=0]P, +ve G[y=0]P, –ve G[x=0]P, NTZ; (B) NTZ, +ve G[y=0]P, NTZ; (C) NTZ, +ve G[x=0]P, +ve G[y=0]P, –ve G[x=0]P, NTZ; (D) NTZ, +ve G[y=0]P, NTZ; (E) NTZ, +ve G[x=0]P, +ve G[y=0]P, –ve G[x=0]P, NTZ; (F) NTZ, +ve G[y=0]P, –ve G[x=0]P, NTZ; (G) NTZ, +ve G[y=0]P, (H) NTZ; (I) NTZ, +ve G[x=0]P, +ve G[y=0]P, –ve G[x=0]P, NTZ; etc.

We geometrically depict $\sigma = \frac{1}{2}$ as Gram points in Figure 3, Close-up view of virtual Origin points when $\sigma = \frac{1}{3}$ in Figure 11, and Simulated dynamic trajectories showing Origin intercept points when $\sigma = \frac{1}{2}$ and virtual Origin intercept points when $\sigma = \frac{2}{5}$ and $\sigma = \frac{4}{5}$ in Figure 12. As demonstrated in Figure 12, two different trajectories as specified by two different σ values will always form two colinear lines (colines) [**which is conveniently defined as two parallel curved lines that will never cross over**]. We crucially note the unique trajectory formed by solitary $\sigma = \frac{1}{2}$ value will also always form colines with other trajectories formed by any arbitrarily chosen $\sigma \neq \frac{1}{2}$ values. Since only the trajectory formed by $\sigma = \frac{1}{2}$ value will intersect with Origin point thus giving rise to Origin intercept points [nontrivial zeros], all other trajectories formed by $\sigma \neq \frac{1}{2}$ values will never intersect with Origin point.

In Figure 11 for $\sigma = \frac{1}{3}$ [$\sigma < \frac{1}{2}$ situation], there are relatively more virtual Gram[x=0] points existing as y-axis intercept points. On the contrary $\sigma > \frac{1}{2}$ situation e.g. $\sigma = \frac{2}{3}$, there will instead be virtual Origin intercept points (as additional positive virtual Gram[y=0] points on x-axis) at the "varying" [infinitely many] virtual Origin points with relatively less virtual Gram[x=0] points existing as y-axis intercept points. Then proof for Riemann hypothesis can be stated as fulfilling two conditions: The position of Origin point when $\sigma = \frac{1}{2}$ is uniquely a solitary point, and the positions of virtual Origin points for any σ values when $\sigma \neq \frac{1}{2}$ are non-uniquely infinitely many points but these cannot include the position of Origin point.

The Incompletely Predictable Nontrivial zeros-Gram points Varying Loops (NTZ-GP VL), indicating NTZ gaps as geometrically depicted in Figure 6, are dynamically defined by the line tracing joining n^{th} NTZ to $(n+1)^{th}$ NTZ with the [solitary] Origin point acting as the

unique $\sigma = \frac{1}{2}$-Attractor. The four boundaries in a usual NTZ-GP VL on the short range scale will typically consist of the two sequential patterns n^{th} NTZ, then a [alternatingly] positive and negative G[x=0]P (or *vice versa*), then a positive G[y=0]P, and finally $(n+1)^{th}$ NTZ. The area enclosed by each NTZ-GP VL can be obtained by integrating the relevant equation for each Varying Loop in interval from 0π to 2π.

10 Conclusions

Godel's incompleteness theorems are two theorems of mathematical logic concerning the limits of provability in formal axiomatic theories. The first incompleteness theorem states that no consistent system of axioms whose theorems can be listed by an effective procedure (i.e., an algorithm) is capable of proving all truths about the arithmetic of natural numbers. For any such consistent formal system, there will always be statements about natural numbers that are true, but that are unprovable within the system. The second incompleteness theorem, an extension of the first, shows that the system cannot demonstrate its own consistency. Thus Godel's incompleteness theorems may apply to the first Hardy-Littlewood conjecture.

Side Note: The **second Hardy-Littlewood conjecture** states that Prime-$\pi(x + y) \leq$ Prime-$\pi(x)$ + Prime-$\pi(y)$ for all $x, y \geq 2$ whereby Prime-$\pi(x)$ is the prime counting function; viz, the number of primes from $x + 1$ to $x + y$ is always less than or equal to the number of primes from 1 to y. These two Hardy-Littlewood conjectures[11] were subsequently proven to be incompatible with each other[12] with an arbitrarily large number of violations. The first such violation is expected to likely occur for very large values of x; for example, an Admissible Prime k-tuplet of 447 primes [with smallest possible diameter = 3158] can be found in an interval of $y = 3159$ integers, while Prime-$\pi(3159) = 446$. Although unproven, the first Hardy-Littlewood conjecture is generally considered by most people to likely be true. If that is the case, it implies that the second Hardy-Littlewood conjecture, in contrast, is false.

Gap 1-Composites as Even and Odd Composites belong to CIS-IM-accelerating. Gap 2-Composites as Even Composites [and Gap 2i-Odd Primes as All Odd Primes (with i = 1, 2, 3, 4, 5...)] belong to CIS-ALN-decelerating. Therefore, not least to preserve homogeneity in all cardinality to be "CIS" [and not be partially or fully "CFS"]; **Total** Gap 2i-Odd Primes as ALN of **Subtotals** Gap 2-Odd Primes, Gap 4-Odd Primes, Gap 6-Odd Primes, Gap 8-Odd Primes, Gap 10-Odd Primes... must logically all belong to CIS-ALN-decelerating.

In 2013, Yitang Zhang proved a landmark result showing some unknown even number N < 70 million such that this condition holds: There are CIS-ALN-decelerating Odd Primes that differ by N between each other[32]. Aesthetically, this solitary N < 70 million value as an even Prime gap is insufficient since its generated CIS-ALN-decelerating Odd Primes simply cannot exist alone in the large range of prime numbers. Hence there must be at least two, if not three, existing even Prime gaps that generate their corresponding CIS-ALN-decelerating Odd Primes. Modified Polignac's and Twin prime conjectures equates to all even Prime gaps 2, 4, 6, 8, 10... will generate their corresponding CIS-ALN-decelerating Odd Primes.

Treated as *Incompletely Predictable problems*, we provide a comparatively elementary algorithm-type proof for Modified Polignac's and Twin prime conjectures [whereby this could now be dubbed *Modified Polignac's and Twin prime Theorem*]. This statement can also be phrased as *Plus-Minus Gap 2 Composite Number Alternating Law and Plus Gap 2 Composite Number Continuous Law that are applicable on the finite (small) scale, are also applicable on*

the infinite (large) scale. There is zero probability that any particular prime gaps from eternal repeated groupings of small and/or large prime gaps that faithfully generate all the countably arbitrarily large number of Odd Primes will abruptly terminate or disappear.

Treated as *Incompletely Predictable problems*, we provide a comparatively elementary equation-type proof on Riemann hypothesis [whereby this could now be dubbed *Riemann Theorem*] while explaining the existence of mutually exclusive three types of Gram points and two types of virtual Gram points. There is zero probability that any of the countably infinitely many nontrivial zeros can be located away from [geometrical] Origin point, which is equivalent to [mathematical] critical line.

With methods historically based on the blends of computational, analytic, algebraic and geometric number theory; the geometrical-mathematical unified approach used in all our proofs is analogically similar to the algebra-geometry unified approach of geometric Langlands program that was formalized by Professor Peter Scholze and Professor Laurent Fargues[3]. Our Algebra and Number Theory achievements represent solving the overall complex (meta-) properties on Incompletely Predictable problems.

Author's Personal Note In perspective, there are also Completely Unpredictable entities (Completely Random entities) [e.g. obtained from true Random Number Generator with maximum entropy] as well as Completely Predictable entities and Incompletely Predictable entities [→ Mathematics for Incompletely Predictable Problems]. In increasing order of complexity, we have the following previously derived Laws as outlined on p. 49 of [26]:

Law I: Simple Elementary Fundamental Law for "simple" Nonliving Things with simple properties
Law II: Complex Elementary Fundamental Law for "complex" Nonliving Things with complex properties
Law III: Simple Emergent Fundamental Law for "simple" Living Things with simple properties
Law IV: Complex Emergent Fundamental Law for "complex" Living Things with complex properties

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A Gram's Law and Rosser's Rule

Named after Danish mathematician Jørgen Pedersen Gram (June 27, 1850 – April 29, 1916), ['traditional' / 'usual'] Gram points or (mathematical) Gram[y=0] points or (geometrical) x-axis intercept points are other conjugate pairs values in Riemann zeta function $\zeta(s)$ on $\sigma = \frac{1}{2}$ critical line. Then s = $\frac{1}{2} + \imath t$ gives rise to $\zeta(\frac{1}{2} + \imath t)$ on critical line; and Gram points when defined in terms of $\zeta(s)$ is given by $\sum ReIm\{\zeta(s)\} = Re\{\zeta(s)\} + 0$, or simply $\mathrm{Im}\{\zeta(s)\} = 0$. Alternatively defined using expression denoting $\zeta(s)$ on critical line $\zeta(\frac{1}{2} + \imath t) = Z(t)e^{-\imath\theta(t)}$ whereby Hardy's function, Z, is real for real t, and θ is Riemann-Siegel theta function given in terms of gamma function as $\theta(t) = \arg\left(\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right) - \frac{\ln\pi}{2}t$ for real values of t; we note that $\zeta(s)$ is real when $\sin(\theta(\mathrm{t})) = 0$. This implies that θ(t) is an integer multiple of π which allows for location of Gram points to be calculated easily by inverting the formula for θ. As already alluded to in section 3, Gram points are historically [crudely] numbered as g_n for n = 0, 1, 2, 3,..., whereby g_n is the unique solution of $\theta(\mathrm{t}) = n\pi$. Here, n = 0 is the [first] g_0 value of 17.8455995405... which is larger than the smallest [first] positive nontrivial zeros (NTZ) value of 14.13472515.... Thus, n = -3 correspond to g_{-3} = 0, n = -2 correspond to g_{-2} = 3.4362182261..., and n = -1 correspond to g_{-1} = 9.6669080561....

Paired [infinite-length] integer sequences with prestigious connections:
A100967+0, which is A100967[15], is precisely defined as "Least k such that binomial(2k+1, k-n-1) ≥ binomial(2k, k) viz. (2k+1)!k!k! ≥ (2k)!(k-n-1)!(k+n+2)!". The terms commencing from Position 0, 1, 2, 3,... of A100967+0 are 3, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 225, 263, 303, 347, 393, 442, 494, 549, 606, 667, 730, 797, 866, 938, 1013, 1091, 1172, 1255, 1342, 1431, 1524, 1619, 1717, 1818, 1922, 2029, 2138, 2251, 2366, 2485, 2606, 2730, 2857, 2987, 3119, 3255, 3394, 3535,....

A100967+1 is precisely defined as "Add 1 to each and every terms from A100967+0". The terms commencing from Position 0, 1, 2, 3,... of A100967+1 are 4, 10, 19, 30, 45, 62, 82, 105, 131, 160, 192, 226, 264, 304, 348, 394, 443, 495, 550, 607, 668, 731, 798, 867, 939, 1014, 1092, 1173, 1256, 1343, 1432, 1525, 1620, 1718, 1819, 1923, 2030, 2139, 2252, 2367, 2486, 2607, 2731, 2858, 2988, 3120, 3256, 3395, 3536,....

A228186[25] is defined as "Greatest natural number k > n such that calculated peak values for ratio R = $\dfrac{\text{Combinations With Repetition}}{\text{Combinations Without Repetition}} = \dfrac{(k+n-1)!(n-k)!}{n!(n-1)!}$ belong to maximal rational numbers < 2". It is also defined as "Smallest natural number k > n such that (k+n+1)!(k-n-2)! < 2k!(k-1)!". The terms commencing from Position 0, 1, 2, 3,... of A228186 are 4, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 226, 263, 304, 347, 393, 442, 494, 549, 607, 667, 731, 797, 866, 938, 1013, 1091, 1172, 1256, 1342, 1432, 1524, 1619, 1717, 1818, 1922, 2029, 2139, 2251, 2367, 2485, 2606, 2730, 2857, 2987, 3120, 3255, 3394, 3535,....

Unexpected connection [and unrelated to NTZ and Gram points]: A228186 can be considered an innovative [infinite-length] *"Hybrid integer sequence"* identical to *"non-Hybrid integer sequence"* A100967+0 except for the interspersed [finite] 21 'exceptional' terms located at Position 0, 11, 13, 19, 21, 28, 30, 37, 39, 45, 50, 51, 52, 55, 57, 62, 66, 70, 73, 77, and 81 with their corresponding 21 values exactly specified by [infinite-length] *"non-Hybrid integer sequence"* A100967+1.

A114856-"bad"-Gram-points, which is A114856[30], is precisely defined as "Indices n of Gram points g_n for which $(-1)^n Z(g_n) < 0$ with Z(t) being Riemann-Siegel Z-function [and full given range of values n = 0, 1, 2, 3,...]". The terms of A114856-"bad"-Gram-points are: 126, 134, 195, 211, 232, 254, 288, 367, 377, 379, 397, 400, 461, 507, 518, 529, 567, 578, 595, 618, 626, 637, 654, 668, 692, 694, 703, 715, 728, 766, 777, 793, 795, 807, 819, 848, 857, 869, 887, 964, 992, 995, 1016, 1028, 1034, 1043, 1046, 1071, 1086,....

A114856-"good"-Gram-points, given by "total"-Gram points minus A114856-"bad"-Gram-points, is precisely defined as "Indices n of Gram points g_n for which $(-1)^n Z(g_n) > 0$ with Z(t) being Riemann-Siegel Z-function [and full given range of values n = 0, 1, 2, 3,...]". The derived terms of A114856-"good"-Gram-points: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,....

A216700[7] is precisely defined as "Violations of Rosser's rule: numbers n such that the Gram block $[g_n, g_{n+k}]$ contains fewer than k points t such that Z(t) = 0 with Z(t) being Riemann-Siegel Z-function [and full given range of values n = 0, 1, 2, 3,...]". The terms of A216700 are 13999525, 30783329, 30930927, 37592215, 40870156, 43628107, 46082042, 46875667, 49624541, 50799238, 55221454, 56948780, 60515663, 61331766, 69784844, 75052114, 79545241, 79652248, 83088043, 83689523, 85348958, 86513820, 87947597,....

Note: All NTZ (as conjectured by Riemann hypothesis) and Gram points (by definition) are located on the same critical line of Riemann zeta function. Counting NTZ can be validly reduced to counting all Gram points where Gram's Law is satisfied and adding count of NTZ inside each Gram block. With this process, we need not locate NTZ but just have to accurately compute Z(t) to show that it changes sign.

Gram's Law: there is [usually] exactly one NTZ (Gram[x=0,y=0] points or Origin intercept points) between any two "good" Gram points. Examples of closely related statements equivalent to Gram's law are: $(-1)^n Z(g_n)$ is [usually] positive or Z(t) [usually] has opposite sign at consecutive Gram points. Thus, a t-valued Gram point is called a "good" Gram point if $\zeta(s)$ is positive at $\frac{1}{2} + \imath t$ with $(-1)^n Z(g_n) > 0$ and a "bad" Gram point if $\zeta(s)$ is negative at $\frac{1}{2} + \imath t$ with $(-1)^n Z(g_n) < 0$. The indices of "bad" Gram points where Z has the 'wrong' sign are given by A114856 in OEIS. A Gram block $[g_n, g_{n+k}]$ is a half-open interval bounded by two "good" Gram points g_n and g_{n+k} such that all Gram points g_{n+1},..., g_{n+k-1} between them are "bad" Gram points. A refinement of Gram's Law is known as Rosser's Rule[19] which stated that Gram blocks [usually] have the expected number of NTZ in them (identical to number of Gram intervals), even though some of the individual Gram intervals in the block may not have exactly one NTZ in them. Example, the interval bounded by g_{125} and g_{127} is a Gram block containing a unique "bad" Gram point g_{126} and expected number 2 of NTZ although neither of its two Gram intervals contains a unique NTZ.

Gram's Law and Rosser's Rule both imply that in some sense NTZ do not stray too far from their expected positions, and that they hold most of the time but are violated infinitely

often (in an Incompletely Predictable manner)[28], [29]. Professor Timothy Trudgian in 2011 explicitly showed that both Gram's Law and Rosser's Rule fail in a positive proportion of cases. In particular, it is expected that in about 73% [$\approx\frac{3}{4}$] one NTZ is enclosed by two successive Gram points [and thus Gram's Law fails for about 27% $\approx\frac{1}{4}$ of all Gram intervals to contain exactly one NTZ], but in about 14% no NTZ and in about 13% two NTZ are in such a Gram interval on the long run.

B The 18 patterns of Admissible Prime 25-tuplets

The 18 patterns for the [randomly-selected] Prime 25-tuplets are depicted as cummulative prime gaps and progressive prime gaps. Frequency of patterns [that are progressively decreasing by 8, 4 and 2 as related by 2^{-1}] containing prime gap 8 = 16/18, prime gap 10 = 8/18, prime gap 12 = 4/18, and prime gap 14 = 2/18. Frequency of patterns [that are progressively increasing by 8, 4 and 2 as related by 2^{-1}] NOT containing prime gap 8 = 2/18, prime gap 10 = 10/18, prime gap 12 = 14/18, and prime gap 14 = 16/18.
Pat-1 (0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 48, 50, 56, 62, 68, 72, 78, 86, 90, 96, 98, 102, 110) ≡ (0, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 6, 6, 4, 6, 8, 4, 6, 2, 4, 8); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 0
Pat-2 (0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 48, 56, 66, 68, 72, 78, 80, 86, 90, 92, 96, 98, 108, 110) ≡ (0, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 6, 8, 10, 2, 4, 6, 2, 6, 4, 2, 4, 2, 10, 2); #Gap 8 = 2, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-3 (0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 48, 50, 56, 66, 68, 72, 78, 80, 86, 90, 92, 98, 108, 110) ≡ (0, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 6, 2, 6, 10, 2, 4, 6, 2, 6, 4, 2, 6, 10, 2); #Gap 8 = 1, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-4 (0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 50, 56, 66, 68, 72, 78, 80, 86, 90, 92, 96, 98, 108, 110) ≡ (0, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 8, 6, 10, 2, 4, 6, 2, 6, 4, 2, 4, 2, 10, 2); #Gap 8 = 2, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-5 (0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 50, 56, 62, 66, 68, 72, 78, 80, 86, 90, 92, 96, 108, 110) ≡ (0, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 8, 6, 6, 4, 2, 4, 6, 2, 6, 4, 2, 4, 12, 2); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 1, #Gap 14 = 0
Pat-6 (0, 2, 6, 8, 12, 20, 26, 30, 36, 38, 42, 48, 56, 62, 66, 68, 72, 78, 80, 86, 90, 92, 96, 108, 110) ≡ (0, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 6, 8, 6, 4, 2, 4, 6, 2, 6, 4, 2, 4, 12, 2); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 1, #Gap 14 = 0
Pat-7 (0, 2, 6, 8, 12, 18, 20, 26, 30, 32, 36, 42, 50, 56, 62, 68, 72, 78, 86, 90, 92, 96, 98, 102, 110) ≡ (0, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 8, 6, 6, 6, 4, 6, 8, 4, 2, 4, 2, 4, 8); #Gap 8 = 3, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 0
Pat-8 (0, 2, 6, 12, 14, 20, 24, 26, 30, 32, 42, 44, 54, 56, 60, 66, 72, 74, 80, 86, 90, 96, 102, 104, 110) ≡ (0, 2, 4, 6, 2, 6, 4, 2, 4, 2, 10, 2, 10, 2, 4, 6, 6, 2, 6, 6, 4, 6, 6, 2, 6); #Gap 8 = 0, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-9 (0, 6, 8, 14, 20, 24, 30, 36, 38, 44, 50, 54, 56, 66, 68, 78, 80, 84, 86, 90, 96, 98, 104, 108, 110) ≡ (0, 6, 2, 6, 6, 4, 6, 6, 2, 6, 6, 4, 2, 10, 2, 10, 2, 4, 2, 4, 6, 2, 6, 4, 2); #Gap 8 = 0, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-10 (0, 2, 8, 12, 14, 18, 24, 30, 32, 38, 42, 44, 50, 54, 60, 68, 72, 74, 78, 80, 84, 98, 102, 108, 110) ≡ (0, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2, 4, 14, 4, 6, 2); #Gap 8 = 1, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 1

Pat-11 (0, 2, 8, 12, 26, 30, 32, 36, 38, 42, 50, 56, 60, 66, 68, 72, 78, 80, 86, 92, 96, 98, 102, 108, 110) ≡ (0, 2, 6, 4, 14, 4, 2, 4, 2, 4, 8, 6, 4, 6, 2, 4, 6, 2, 6, 6, 4, 2, 4, 6, 2); #Gap 8 = 1, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 1
Pat-12 (0, 8, 12, 14, 18, 20, 24, 32, 38, 42, 48, 54, 60, 68, 74, 78, 80, 84, 90, 92, 98, 102, 104, 108, 110) ≡ (0, 8, 4, 2, 4, 2, 4, 8, 6, 4, 6, 6, 6, 8, 6, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2); #Gap 8 = 3, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 0
Pat-13 (0, 8, 12, 14, 20, 24, 32, 38, 42, 48, 54, 60, 62, 68, 74, 78, 80, 84, 90, 92, 98, 102, 104, 108, 110) ≡ (0, 8, 4, 2, 6, 4, 8, 6, 4, 6, 6, 6, 2, 6, 6, 4, 2, 4, 6, 2, 6, 4, 2, 4, 2); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 0, #Gap 14 = 0
Pat-14 (0, 2, 12, 14, 18, 20, 24, 30, 32, 38, 42, 44, 54, 60, 68, 72, 74, 80, 84, 90, 98, 102, 104, 108, 110) ≡ (0, 2, 10, 2, 4, 2, 4, 6, 2, 6, 4, 2, 10, 6, 8, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2); #Gap 8 = 2, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-15 (0, 2, 12, 18, 20, 24, 30, 32, 38, 42, 44, 54, 60, 62, 68, 72, 74, 80, 84, 90, 98, 102, 104, 108, 110) ≡ (0, 2, 10, 6, 2, 4, 6, 2, 6, 4, 2, 10, 6, 2, 6, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2); #Gap 8 = 1, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-16 (0, 2, 12, 14, 18, 20, 24, 30, 32, 38, 42, 44, 54, 62, 68, 72, 74, 80, 84, 90, 98, 102, 104, 108, 110) ≡ (0, 2, 10, 2, 4, 2, 4, 6, 2, 6, 4, 2, 10, 8, 6, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2); #Gap 8 = 2, #Gap 10 = 2, #Gap 12 = 0, #Gap 14 = 0
Pat-17 (0, 2, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 54, 62, 68, 72, 74, 80, 84, 90, 98, 102, 104, 108, 110) ≡ (0, 2, 12, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 8, 6, 4, 2, 6, 4, 6, 8, 4, 2, 6, 2); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 1, #Gap 14 = 0
Pat-18 (0, 2, 14, 18, 20, 24, 30, 32, 38, 42, 44, 48, 54, 60, 68, 72, 74, 80, 84, 90, 98, 102, 104, 108, 110) ≡ (0, 2, 12, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 8, 4, 2, 6, 4, 6, 8, 4, 2, 4, 2); #Gap 8 = 2, #Gap 10 = 0, #Gap 12 = 1, #Gap 14 = 0

C Computed data on n-digit primes with average prime gaps

Examples of computed data on n-digit prime numbers that include their average prime gaps:
Corresponding subsets 1-digit $\mathbb{P}$ {2, 3, 5 and 7} and $\mathbb{C}$ {4, 6, 8 and 9} that are derived from subset 1-digit $\mathbb{Z}$ {0, 1, 2, 3, 4, 5, 6, 7, 8 and 9} with cardinality of 10 have both equal cardinality of 4. First 1-digit $\mathbb{P}_i$ occurs at i = 1 (odd position) and last 1-digit $\mathbb{P}_i$ ends at i = 4 (even position). Average $\mathbb{P}$ gap for 1-digit $\mathbb{P}$ = 10/4 = 2.5.
Corresponding subsets 2-digit $\mathbb{P}$ {11, 13, 17, 19, 23...} with cardinality of 21 and $\mathbb{C}$ {10, 12, 14, 15, 16...} with cardinality of 69 together form subset 2-digit $\mathbb{Z}$ {10, 11, 12, 13, 14,..., 99} with cardinality of 90. There are 60% of 90 $\mathbb{Z}$ = 54 $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty. Consequently, there are 21 $\mathbb{P}$ and 69 - 54 = 15 $\mathbb{C}$ that together constitute the P(uncertain $\mathbb{P}$ + uncertain $\mathbb{C}$) = 0.4 whereby we note that there are more uncertain $\mathbb{P}$ [21/36 = 58.3%] than uncertain $\mathbb{C}$ [15/36 = 41.7%]. First 2-digit $\mathbb{P}_i$ starts at i = 5 (odd position) and last 2-digit $\mathbb{P}_i$ ends at i = 25 (odd position). Average $\mathbb{P}$ gap for 2-digit $\mathbb{P}$ = 90/21 = 4.29.
Corresponding subsets 3-digit $\mathbb{P}$ {101, 103, 107, 109, 113...} with cardinality of 143 and $\mathbb{C}$ {100, 102, 104, 105, 106...} with cardinality of 757 together form subset 3-digit $\mathbb{Z}$ {100, 101, 102, 103, 104..., 999} with cardinality of 900. There are 60% of 900 $\mathbb{Z}$ = 540 $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty. Consequently, there are 143 $\mathbb{P}$ and 757 - 540 = 217 $\mathbb{C}$ that together constitute the P(uncertain $\mathbb{P}$ + uncertain $\mathbb{C}$) = 0.4 whereby we note that there are less uncertain $\mathbb{P}$ [143/360 = 39.7%] than uncertain $\mathbb{C}$ [217/360 = 60.3%]. First 3-digit $\mathbb{P}_i$ starts at i = 26 (even

position) and last 3-digit $\mathbb{P}_i$ ends at i = 168 (even position). Average $\mathbb{P}$ gap for 3-digit $\mathbb{P}$ = 900/143 = 6.29.
Corresponding subsets 4-digit $\mathbb{P}$ {1009, 1013, 1019, 1021, 1031...} with cardinality of 1061 and $\mathbb{C}$ {1000, 1001, 1002, 1003, 1004...} with cardinality of 7939 together form subset 4-digit $\mathbb{Z}$ {1000, 1001, 1002, 1003, 1004..., 9999} with cardinality of 9000. There are 60% of 9000 $\mathbb{Z}$ = 5400 $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty. Consequently, there are 1061 $\mathbb{P}$ and 7939 - 5400 = 2539 $\mathbb{C}$ that together constitute the P(uncertain $\mathbb{P}$ + uncertain $\mathbb{C}$) = 0.4 whereby we note that there are less uncertain $\mathbb{P}$ [1061/3600 = 29.5%] than uncertain $\mathbb{C}$ [2539/3600 = 70.5%]. First 4-digit $\mathbb{P}_i$ starts at i = 169 (odd position) and last 4-digit $\mathbb{P}_i$ ends at i = 1229 (odd position). Average $\mathbb{P}$ gap for 4-digit $\mathbb{P}$ = 9000/1061 = 8.48.
Corresponding subsets 5-digit $\mathbb{P}$ {10007, 10009, 10037, 10039, 10061...} with cardinality of 8363 and $\mathbb{C}$ {10000, 10001, 10002, 10003, 10004...} with cardinality of 81637 together form subset 5-digit $\mathbb{Z}$ {10000, 10001, 10002, 10003, 10004..., 99999} with cardinality of 90000. There are 60% of 90000 $\mathbb{Z}$ = 54000 $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty. Consequently, there are 8363 $\mathbb{P}$ and 81637 - 54000 = 27637 $\mathbb{C}$ that together constitute the P(uncertain $\mathbb{P}$ + uncertain $\mathbb{C}$) = 0.4 whereby we note that there are less uncertain $\mathbb{P}$ [8363/36000= 23.2%] than uncertain $\mathbb{C}$ [27637/36000 = 76.8%]. First 5-digit $\mathbb{P}_i$ starts at i = 1230 (even position) and last 5-digit $\mathbb{P}_i$ ends at i = 9592 (even position). Average $\mathbb{P}$ gap for 5-digit $\mathbb{P}$ = 90000/8363 = 10.76.
Corresponding subsets 6-digit $\mathbb{P}$ {100003, 100019, 100043, 100049, 100057...} with cardinality of 68906 and $\mathbb{C}$ {100000, 100001, 100002, 100004, 100005...} with cardinality of 831094 together form subset 6-digit $\mathbb{Z}$ {100000, 100001, 100002, 100003, 100004..., 999999} with cardinality of 900000. There are 60% of 900000 $\mathbb{Z}$ = 540000 $\mathbb{Z}$ being $\mathbb{C}$ with 100% certainty. Consequently, there are 68906 $\mathbb{P}$ and 831094 - 540000 = 291094 $\mathbb{C}$ that together constitute the P(uncertain $\mathbb{P}$ + uncertain $\mathbb{C}$) = 0.4 whereby we note that there are less uncertain $\mathbb{P}$ [68906/360000 = 19.1%] than uncertain $\mathbb{C}$ [291094/360000 = 80.9%]. First 6-digit $\mathbb{P}_i$ starts at i = 9593 (0dd position) and last 6-digit $\mathbb{P}_i$ ends at i = 78498 (even position). Average $\mathbb{P}$ gap for 6-digit $\mathbb{P}$ = 900000/68906 = 13.06.