Several questions about infinity

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【Abstract】 What is infinity? What principles should be adhered to when researching infinity? For a predicate on a natural number system, is it true that finite holds, and infinite holds? This paper reexamines the nature of infinity and proposes two opposite infinite axioms (δ+1=δ or δ+1≠δ). Based on these two infinite axioms, the "infinite induction" of the identity formula is proved; it is found that the infinite axioms in the ***ZF*** system do not satisfy the equality axiom, and there are many contradictions in the reasoning of the Cantor ordinal number. The ordinal theory of set theory ***ZF*** system is not strict. It is hoped that the mathematics community will pay attention to these questions and give a convincing answer.

【Key words】 axiom of infinity, standard infinity, nonstandard infinity, ordinal number, cardinal number, infinite induction.

1. Two types infinity of different natures

Infinity is defined relative to infinity.

Classical infinity definition : let N is a set of all the limited numbers, there is a δ, it is bigger than all the limited number E, δ is an infinity quantity , named infinity ,as : $\delta = \infty$, that is:

$$N = \{0 \cdot 1 \cdot 2 \cdots\cdots n \cdots\},\ \exists \delta\,[\,\forall E \in N\ (\delta > E)]\,.$$

Or define with limit: For arbitrarily large E, there is integral number $K = K(E)$ enabling that for any $n > K$, inequality $|x_n| > E$ is true, and then we named limit of sequence $x_1, x_2, x_3, \cdots, x_n$ as infinity (δ) or the sequence as diverging at infinitely large (δ),namely,

$$\lim_{n \to \delta} x_n = \delta \text{, note as: } \delta = \infty .$$

1.1 Two types infinity of different natures

$0 = \{\ \}$, $1 = \{0\}$, $2 = \{0,1\}$, $3 = \{0,1,2\}$, ……, $n+1 = n \cup \{n\}$,

$$n = \{0,1,2,3,\cdots,n-1\},$$

$$N = \{0,1,2,3,\cdots,n,\cdots\},$$

$$\forall n \in N\ (n+1 \neq n),$$

That is: for any finite number, $n+1 \neq n$, Then, for infinite number δ, $\delta+1=\delta$, or $\delta+1\neq\delta$?

There are only two possibilities: (1) $\delta+1=\delta$, or (2) $\delta+1\neq\delta$.

The type 1 infinity can also be divided into two types $\frac{1}{\delta}=0$ and $\frac{1}{\delta}\neq 0$, namely:

$$\exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1=\delta]\wedge[\frac{1}{\delta}=0]$$

$$\exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1=\delta]\wedge[\frac{1}{\delta}\neq 0]$$

Because the difference of these two kinds of infinity is not very great, we will only discuss the "infinity $\frac{1}{\delta}=0$" for now.

This kind of infinity can also be summarized as "standard infinite axiom", namely:

Standard axiom of infinity: $N=\{0\cdot 1\cdot 2\cdots\cdots n\cdots\}$,

$$\exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1=\delta]\wedge[\frac{1}{\delta}=0].$$

According to the above infinite limit it is easy to know ,

$\lim_{n\to\infty} n=\lim_{n\to\infty} n+1$, notes as $\lim_{n\to\infty} n=\delta$, that is: $\exists\delta\,(\delta+1=\delta)\wedge(\frac{1}{\delta}=0)$,

in general note as,

$$\delta=\lim_{n\to\infty} n=\infty, \text{ that is } \exists\infty\,(\infty+1=\infty)\wedge(\frac{1}{\infty}=0)$$

That is: there is an infinite set larger than all finite sets ∞, and the successor set of this infinite set is equal to it. Relative to the standard axiom of infinity, we propose the following non-standard axiom of infinity,

Non-standard of infinity: $N=\{0\cdot 1\cdot 2\cdots\cdots n\cdots\}$,

$$\exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1\neq\delta];$$

"Standard axiom of infinity" and "non-standard axiom of infinity" which are contradictory are not able to be true in same system.

1.2 Two types of limits

To differentiate operation results of classical axiom of infinity and non-classical axiom of infinity, we introduce following definitions:

Definition 1.2.1 Standard limit and non-standard limit

We rule:

(1) Satisfying standard axiom of infinity $\exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1=\delta]\wedge[\frac{1}{\delta}=0]$, where δ is denoted as $\infty=\lim\limits_{n\to\infty}n$, called $\lim\limits_{n\to\infty}n$ is a standard limit;

(2) Satisfying non-standard axiom of infinity $\exists\delta\,[\,\forall E\in N\,(\delta>E)]\wedge[\delta+1\neq\delta]$, where δ is denoted as $\varpi=\mu\lim\limits_{n\to\infty}n$, called $\mu\lim\limits_{n\to\infty}n$ is a non-standard limit;

Infinite set $n=\{0,1,2,\cdots,(n-1)\}$, when $n\to\infty$,

(1) $\lim\limits_{n\to\infty}n=\{0,1,2,\cdots,\lim\limits_{n\to\infty}(n-2),\lim\limits_{n\to\infty}(n-1)\}$ is a infinite set,

$$\infty=\{0,1,2,\cdots,\infty\};$$

(2) $\mu\lim\limits_{n\to\infty}n=\{0,1,2,\cdots,\mu\lim\limits_{n\to\infty}(n-2),\mu\lim\limits_{n\to\infty}(n-1)\}$ is a infinite set,

$$\varpi=\{0,1,2,\cdots,\varpi-1\};$$

Type 1 infinity, $\delta+1=\delta$, that is $\infty+1=\infty$, according to this infinity, we can easily get,

$$\infty,\infty+n,\cdots n\infty,\cdots\infty^{n},\cdots\ n^{\infty}\cdots\infty^{\infty}\cdots\infty^{\infty^{\infty}}\cdots$$

and so on. All of them are equal infinity, which is often used in standard analysis, and We are familiar with it.

Type 2 infinity, $\delta+1\neq\delta$, $\varpi+1\neq\varpi$, according to this kind of infinity, we can easily get,

$$\cdots\varpi,\varpi+1,\cdots 2\varpi,3\varpi,\cdots\varpi^{2},\varpi^{3},\cdots$$

and so on. All of them are independent infinity.

In type 1 infinity,

$$\lim_{n\to\infty}n-2,\ \lim_{n\to\infty}n-1$$

That is :

$$\infty-2=\infty,\ \infty-1=\infty$$

Generally:

$$\infty-n=\infty,\ n-\infty=-\infty;$$

Obviously, there is subtraction operation in type 1 infinity.

In type 2 infinity,

$$\mu\lim_{n\to\infty}n-2,\ \mu\lim_{n\to\infty}n-1$$

That is;

$\varpi-2$, $\varpi-1$ are independent infinity.

there is subtraction operation in type 2 infinity too.

Generally:

$$\varpi - n \text{ , } k\varpi - n \text{ , } k^{\varpi} - n \text{ , } \varpi^{k} - n \text{ ,......}$$

are independent infinity.

1.4 Axiom of infinity

Based on the above analysis, In the set theory ***ZF*** system,we propose the following two axioms;

Definition 1.4.1 Standard axiom of infinity and non-standard axiom of infinity (Definition of unlimited number and infinite set)

1. Standard axiom of infinity

(1) $\varnothing$ is a set, (2) if n is a set,then $n+1 = n \bigcup \{n\}$ is a set, (3) $\lim\limits_{n\to\infty} n$ also is a set;

$$\text{(denote as: } \infty = \lim_{n\to\infty} n \text{ , } \infty = \infty + 1 \text{ , } \frac{1}{\infty} = 0 \text{)}$$

2. Non-standard axiom of infinity

(1) $\varnothing$ is a set , (2) If n is a set, then $n+1 = n \bigcup \{n\}$ is a set, (3) $\mu \lim\limits_{n\to\infty} n$ also is a set;

$$\text{(denote as: } \varpi = \mu \lim_{n\to\infty} n \text{ , } \varpi \neq \varpi + 1 \text{)}$$

(note: ϖ is different with Cantor's ordinal number ω)

These two axioms of infinity are equivalent to the above two types infinity.

Finite set $n = \{0, 1, 2, \cdots, (n-2), (n-1)\}$, n not only denotes number, but also denotes set;

Take standard limit on finite set $n = \{0, 1, 2, \cdots, (n-2), (n-1)\}$;

$$\lim_{n\to\infty} n = \{0, 1, 2, \cdots, (\lim_{n\to\infty} n - 2), (\lim_{n\to\infty} n - 1)\} \text{ ,}$$

Type 1 infinity

$$\infty = \left\{0, 1, 2, \cdots, \infty\right\} \text{ , } \infty \text{ denotes both number and set;}$$

Take non-standard limit to finite set $n = \{0, 1, 2, \cdots, (n-2), (n-1)\}$

$$\mu \lim_{n\to\infty} n = \{0, 1, 2, \cdots, (\mu \lim_{n\to\infty} n - 2), (\mu \lim_{n\to\infty} n - 1)\}$$

Type 2 infinity

$$\varpi = \left\{0, 1, 2, \cdots, \varpi - 1\right\} \text{ , } \varpi \text{ denotes both number and set.}$$

The two types infinity are contradictory. These two axioms are similar to the axiom of Euclidean geometry and non-Euclidean geometry, and they cannot be established in the same system. They are in two different dimensions of the world, and they cannot be unified. . These two infinities can be regarded as potential infinity and real infinity, respectively.

2.Limitedness can be true, unlimitedness also can be true ?

How should we study infinity? For a formula or a predicate, if the infinite is true, is the infinity still true?

2.1 Calculating identical equation's limit in standard analysis

Example 2.1.1 Calculating identical equation's limit

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$$ is true,

Assume $Q(n)$ is above formula, that is: $\Rightarrow Q(n)$.

$$\lim_{n\to\infty}[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}]=\lim_{n\to\infty}[\frac{n}{n+1}]$$ also is true,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\cdots=1,$$

That is: $Q(\infty)$ is true too. That is : $\Rightarrow Q(\infty)$.

Example 2.1.2 Calculating identical equation's limit

Assume $A_1=1$, $A_{n+1}=\sqrt{1+A_n}$

$$\lim_{n\to\infty}A_{n+1}=\lim_{n\to\infty}\sqrt{1+A_n}\ ,\ A_\infty=\sqrt{1+A_\infty}\ ,\ A_\infty=\frac{\sqrt{5}-1}{2}$$

The above process of calculating limits shows that for identical equation's predicates $Q(n)$, the infinite $Q(n)$ is true and the infinite $Q(\infty)$ is true too. That is $Q(n)\Rightarrow Q(\infty)$. This is not new to us, we often use it in calculus.

2.2 Natural number axiom system and equality axiom system

Natural number axiom system PA :

Initial term 0; successor ' ; plus +; multiplication • ; equal sign =; Variable term $a,b,c,d,\cdots$; predicates $A(x)$.

$PA1$: $\neg a'=0$;

$PA2$: $a'=b'\to a=b$;

$PA3$: $a+0=a$;

$PA4$: $a+b'=(a+b)'$;

$PA5$： $a \cdot 0 = 0$；

$PA6$： $a \cdot b' = a \cdot b + a$；

$PA7$： $A(0) \wedge \forall x(A(x) \to A(x')) \to A(x)$ 。

Equality axiom system:

$E1$： $a = a$；

$E2$： $a = b \to A_i^n(t_1, t_2 \cdots a \cdots t_n) = A_i^n(t_1, t_2 \cdots b \cdots t_n)$ ；

$E3$： $(a = b \to (A_i^n(t_1, t_2 \cdots a \cdots t_n) \to A_i^n(t_1, t_2 \cdots b \cdots t_n)))$

（A_i^n is an any n-ary predicate)

If A_i^n is a 1-ary predicate A, $E2$, $E3$ can be denote as following:

$E2$： $a = b \to A(a) = A(b)$;

$E3$： $(a = b) \to (A(a) \to A(b))$.

According to the equivalent axiom $E2$, for any predicate A, $a = b \to A(a) = A(b)$,

Replace $a = b$ with the identical equation of natural numbers $P(n) = Q(n)$,

$$P(n) = Q(n) \Rightarrow A(P(n)) = A(Q(n)).$$

Regard $\lim_{n \to \infty}$ as predicate A,

$$P(n) = Q(n) \Rightarrow \lim_{n \to \infty}(P(n)) = \lim_{n \to \infty}(Q(n)) \Rightarrow P(\lim_{n \to \infty} n) = Q(\lim_{n \to \infty} n).$$

"Take the limit $\lim_{n \to \infty}$" can be seen as predicate: "let n take infinity" or "n equals infinity".

If using $E(n)$ denotes identical equation of natural predicate $P(n) = Q(n)$, $A[E(n)]$ is $\lim_{n \to \infty}[E(n)]$.

$$E(n) \Rightarrow A[E(n)] \Rightarrow \lim_{n \to \infty}[E(n)] \Rightarrow E(\lim_{n \to \infty} n)$$

Above, example 2.1.2, example 2.1.2 are inference procedure in this way.

Axiom 2.2.1 Limitedness is true, unlimitedness is true too.

$E(n)$ is a predicate formula defined by equality. $E(\infty), E(\varpi)$ also can be defined. If $E(n)$ is true, then $E(\infty), E(\varpi)$ are true too. That is:

$$E(n) \Rightarrow E(\infty),\ E(n) \Rightarrow E(\varpi).$$

Proof:

（1） $E(n)$ ……………hypothesize, $E(n)$ is equivalent to the identical formula $P(n)=Q(n)$,

（2） $A[E(n)]$ ………………………………………………… （1) axiom $E2$,

（3） $\lim_{n\to\infty}[E(n)]$, $\mu\lim_{n\to\infty}[E(n)]$ ⋯（2）regards predicate A as calculating limit $\lim_{n\to\infty}$ or $\mu\lim_{n\to\infty}$,

（4） $E(\lim_{n\to\infty} n)$, $E(\mu\lim_{n\to\infty} n)$ ⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯ （3），

（5） $E(\infty)$, $E(\varpi)$ ⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯ （4），

（6） $E(n)\Rightarrow E(\infty)$, $E(n)\Rightarrow E(\varpi)$ ⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯⋯ （1），（5）.

For any identical property on N , N^* limitednes is true, unlimitedness is true too, $E(n)\Rightarrow E(\infty)$,

2.3 Finite is true, infinite is true too.

We combine mathematical induction, note as,

$$N=\{0,1,2,\cdots,\infty\},$$

$$N^*=\{0,1,2,\cdots,\varpi-1,\varpi\}$$

Assume $n\in N$, E is a identical formula which can be defined on N

$$E(1)\wedge[E(n)\to E(n+1)]\to\forall nE(n).$$

Example 2.3.1 Mathematical induction

$$\frac{1}{3}+\frac{3}{3^2}+\frac{5}{3^3}+\cdots+\frac{2n-1}{3^n}=1-\frac{n+1}{3^n}$$

Above equation，note as $E(n)$

when $n=1$,

$$\frac{1}{3}=1-\frac{2}{3}, \quad E(1) \text{ is true apparently.}$$

Assume when $n=k$, $E(k)$ is true,

$$\frac{1}{3}+\frac{3}{3^2}+\frac{5}{3^3}+\cdots+\frac{2k-1}{3^k}=1-\frac{k+1}{3^k}$$

When $n=k+1$,

$$\frac{1}{3}+\frac{3}{3^2}+\frac{5}{3^3}+\cdots+\frac{2k-1}{3^k}+\frac{2k+1}{3^{k+1}}=1-\frac{k+1}{3^k}+\frac{2k+1}{3^{k+1}}$$

$$=1-\frac{k+1}{3^k}+\frac{2k+1}{3^{k+1}}=1-\frac{3k+3}{3^{k+1}}+\frac{2k+1}{3^{k+1}}=1-(\frac{3k+3-2k-1}{3^{k+1}})=1-\frac{(k+1)+1}{3^{k+1}}$$

$E(k+1)$ is true too.

That is: $E(n)\to E(n+1)$ is true, so, $\forall nE(n)$ is true， that is for any finite natural number, $E(n)$ is true.

According to axiom that "finite is true, infinite is true too", both $E(\infty)$ and $E(\varpi)$ are true.

Combined with the mathematical induction, it holds true for infinity, and it holds for infinity. It can be simply expressed as follows

Theorem 2.3.1 Standard infinite induction

Assume $n\in N$， E is a identical formula which can be defined on N ,

$$E(1)\wedge\forall n\in N[E(n)\to E(n+1)]\to E(\infty)$$

Proof:

（1） $E(1)$ ……………………………………………………………………hypothesize

（2） $\forall n\in N^*[E(n)\to E(n+1)]$ ……………………………………………hypothesize

（3） $E(n)$ ………………………………………………………（1）（2）Standard infinite induction

（4） $E(1)\wedge\forall n\in N[E(n)\to E(n+1)]\to E(n)$ ………（1）（2）(3)Standard infinite induction

（5） $E(n)\to E(\infty)$ ………………………………Theorem2.2.1 finite is true, infinite is true too.

（6） $E(1)\wedge\forall n\in N[E(n)\to E(n+1)]\to E(\infty)$ …………………………（4）（5）

Example 2.3.2 Standard infinite induction

That the following equation is true can be proved,that is $\forall n\in NE(n)$,

$$\sum_{i=1}^{n}\frac{2i-1}{3^i}=1-\frac{n+1}{3^n}$$

So, $A(\infty)$ is true too.

$$\sum_{i=1}^{\infty}\frac{2i-1}{3^i}=1-\lim_{n\to\infty}\frac{n+1}{3^n}=1$$

Theorem 2.3.1 Non-standard infinite induction

Assume $n\in N^*$， E is a identical formula which can be defined on N^*,

$$E(1) \wedge \forall n \in N^*[E(n) \to E(n+1)] \to E(\varpi)$$

Proof:

（1） $E(1)$ ··hypothesize

（2） $\forall n \in N^*[E(n) \to E(n+1)]$ ···hypothesize

（3） $E(n)$ ··· （1）（2）Standard infinite induction

（4） $E(1) \wedge \forall n \in N[E(n) \to E(n+1)] \to E(n)$ ········ （1）（2）(3)Standard infinite induction

（5） $E(n) \to E(\varpi)$ ·······························Theorem2.2.1 finite is true, infinite is true too.

（6） $E(1) \wedge \forall n \in N[E(n) \to E(n+1)] \to E(\varpi)$ ··························· （4）（5）

Example 2.3.3 Non-standard infinite induction

That the following equation is true can be proved, that is $\forall n \in N^* A(n)$

$$\sum_{i=1}^{n} \frac{2i-1}{3^i} = 1 - \frac{n+1}{3^n}$$

So, $A(\varpi)$ is true too.

$$\sum_{i=1}^{\varpi} \frac{2i-1}{3^i} = 1 - \frac{\varpi+1}{3^\varpi}$$

Example 2.3.4 Non-standard infinite induction

Assume $A_1 = 1, A_{n+1} = \sqrt{1+A_n}$

$$\mu \lim_{n \to \infty} A_{n+1} = \mu \lim_{n \to \infty} \sqrt{1+A_n}, A_{\varpi+1} = \sqrt{1+A_\varpi},$$

2.4 Undefinable concept in infinity

For any identical property on N , N^* limitednes is true, unlimitedness is true too. But if it isn't identical property, how is the situation?

We know, in finite numbers, $2n$ denotes even number, $2n-1$ denotes odd number,when $n \to \infty, 2n = 2n-1 = \infty, \infty$ doesn't have parity, so, the concept of parity is indefinable in infinity.

Example 2.4.1 The following identical equations are true

$$1-1+1-1+1-1+\cdots+(-1)^n = \frac{1}{2}[1-(-1)^n]$$

When $n \to \infty$,above identical equation

$$\sum_{i=1}^{\infty}(-1)^i = 1-1+1-1+1-1+\cdots$$

Someone suppose

$$\sum_{i=1}^{\infty}(-1)^{i+1} = (1-1)+(1-1)+(1-1)+\cdots = 0$$

Someone suppose

$$\sum_{i=1}^{\infty}(-1)^i = 1+(-1+1)+(-1+1)+(-1+1)+\cdots = 1$$

Someone suppose

$$s = 1-1+1-1+1-1+1-\cdots = 1-[(1-1)+(1-1)+(1-1)+\cdots] = 1-s$$

$$s = 1-s$$

$$s = \sum_{i=1}^{\infty}(-1)^i = \frac{1}{2}$$

Why do we get a contradictory conclusion in an infinite state? In fact $n \to \infty$, the above identical equations are not true, because ∞ doesn't have parity, or the parity is undefined in ∞, $(-1)^{\infty}$ is uncertain, so the above three solutions are all incorrect.

The correct answer is. Parity is no definition in ∞, the above formula can not promote to infinity.

We have already know, for finite natural number, $n^2+5>n$ is true,when $n \to \infty$, $n^2+5=n=\infty$, There is no difference in size for ∞, therefore, the concepts of "greater than" and "less than" cannot be defined in standard infinity.

Example 2.4.2 Inequality in infinity is wrong.

$$\forall n > 3，\quad n^{n+1} > (n+1)^n$$

$n \to \infty$， above inequalities，$\infty^{(\infty+1)} > (\infty+1)^{\infty}$ is wrong，

Because，the concepts of "greater than" and "less than" cannot be defined in standard infinity.

In fact， we can see from Natural number axiom system ***PA***:The initial concept is only "equal sign" and "successor". Any predicate $A(n)$ on the natural number axiom system ***PA*** can be directly or indirectly defined by the system ***PA***. If $A(\infty)$ $A(\varpi)$ can be defined, any predicate $A(n)$ can be defined by the identical formula. That is :

For any predicate $A(n)$ **, if** $A(n)$ **was defined by system *PA*, and** $A(\infty)$ **,** $A(\varpi)$ **also have definitions, then,**

$$A(n) \Rightarrow A(\infty), A(n) \Rightarrow A(\varpi).$$

3. Some contradictions in Cantor ordinal numbers

3.1 Definition of Cantor ordinal numbers against equality axiom

In the set theory *ZF* system, the recursive definition of Cantor's finite ordinal number is

$$0 = \varnothing, n+1 = n \bigcup \{n\},$$

$$n = \left\{0, 1, 2, \cdots, (n-2), (n-1)\right\}$$

Take $n = 2^k$, then,

$$2^k = \{0, 1, 2, \cdots, (2^k - 2), (2^k - 1)\}.$$

When $n \to \infty, k \to \infty$,

The above are the identity formulas. According to equality axiom $E2$, for any predicate A ,

$$P(n) = Q(n) \Rightarrow A(P(n)) = A(Q(n)) \text{ ,}$$

$$P(n) = Q(n) \Rightarrow \mu \lim_{n\to\infty} P(n) = \mu \lim_{n\to\infty} Q(n)$$

That is:

$$n = \left\{0, 1, 2, \cdots, (n-2), (n-1)\right\} \Rightarrow \omega = \left\{\ 0 \cdot 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (\omega - 2) \cdot (\omega - 1)\ \right\}$$

$$2^k = \{0, 1, 2, \cdots, (2^k - 2), (2^k - 1)\} \Rightarrow 2^\omega = \{0, 1, 2, \cdots, (2^\omega - 2), (2^\omega - 1)\}.$$

That is:the Cantor ordinal numbers and power operation should be:

$\omega = \left\{\ 0 \cdot 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (\omega - 2) \cdot (\omega - 1)\ \right\}$ .... (a),

$2^\omega = \{0, 1, 2, \cdots, (2^\omega - 2), (2^\omega - 1)\}$ .... (b),

However,in the current set theory ZF system, Cantor's infinite ordinal number is derived from the "infinite axiom（***ZF*6**）".

Infinite axiom（*ZF*6）: $\exists\omega[0 \in \omega \wedge (n \in \omega \to n^+ \in \omega)]$,(note: $n^+ = n+1 = n \bigcup \{n\}$),

Definition of Cantor ordinal numbers and power operation is the following form,

$\omega = \{\ 0, 1, 2, 3, \cdots, n, \cdots\}$ .... (c),

$2^\omega = \sup\left\{2^n \mid n < \omega\right\} = \omega$ .... (d), The above (a) and (c), (b) and (d) are contradictory.

The above definitions of Cantor ordinal numbers and power operations all violate the equality axiom.

3.2 What kind of infinite Cantor ordinal numbers belong to?

We have mentioned above that there are two kinds of infinity for numbers, namely:

Type 1 infinity: There is no predecessor $\infty-1=\infty$, and no successor $\infty+1=\infty$, the predecessor and the successor are equal.

Type 2 infinity: There are both predecessors $\varpi-1\neq\varpi$ and successors $\varpi+1\neq\varpi$. Both the precursor and the successor are independent infinite, and they are not equal.

We have to ask, what kind of infinity Cantor ordinal numbers belong to?

(1) From the set point of view, the Cantor ordinal numbers should be satisfied $\omega+1\neq\omega$, and it is easy to obtain:

$$\omega\neq\omega+1\neq\omega+2\neq\cdots\neq\omega+k\neq\cdots \text{ .... (e)}$$

The successors of the Cantor ordinal numbers are not equal, which satisfies the second type of infinity, and the infinite ordinal numbers are independent infinity.

(2) It is generally believed that the Cantor ordinal number does not have a subtraction operation, which is equivalent to no precursor, that is: the precursors are equal,

$$\omega=\omega-1=\omega-2=\cdots=\omega-k=\cdots \text{ .... (f)}$$

If the precursors of the Cantor ordinal numbers are all equal, this satisfies the type 1 infinity .

Why does the Cantor ordinal numbers satisfy two contradictory axioms at the same time?

From the recursive definition of Cantor's finite ordinal number,

$n=\{0,1,2,3,\cdots,n-1\}$, $n-1=\{0,1,2,3,\cdots,n-2\}$, $n-2=\{0,1,2,3,\cdots,n-3\}$

$n+1=\{0,1,2,3,\cdots,n\}$, $n+2=\{0,1,2,3,\cdots,n+1\}$, $n+3=\{0,1,2,3,\cdots,n+2\}$

All of the above are identical formulas, according to equality axiom.

$\omega=\{0,1,2,3,\cdots,\omega-1\}$, $\omega-1=\{0,1,2,3,\cdots,\omega-2\}$, $\omega-2=\{0,1,2,3,\cdots,\omega-3\}$

$\omega+1=\{0,1,2,3,\cdots,\omega\}$, $\omega+2=\{0,1,2,3,\cdots,\omega+1\}$, $\omega+3=\{0,1,2,3,\cdots,\omega+2\}$

The above predecessors and successors are all equivalent. If type 1 infinity is satisfied, both the predecessor and the successor satisfy type 1 infinity; if type 2 infinity is satisfied, the predecessor and the successor both satisfy type 2 infinity.

The Cantor ordinal numbers simultaneously satisfy two contradictory axioms, **(e) and (f) are contradictory：**

The promotion of infinite ordinal numbers as natural numbers must meet the calculation rules of numbers.Cantor infinite ordinal numbers:

$$\omega = \{ 0,1,2,3,\cdots,n,\cdots \}$$

has a lot of speculative ingredients, ω not a strict mathematical concept.

4 .Postscript

Through the above analysis, did you find that Cantor's infinite ordinal number is actually a loose concept of infinity, which caused many contradictions in the ***ZF*** axiom system.

This article is not intended to discuss the errors of the set theory ***ZF*** system. However, for the above problems, the mathematics community should pay enough attention to it. In the set theory ***ZF*** system, a reasonable and convincing explanation must be given.

In another article, I modify the infinite axioms in the ***ZF*** system and establish a new set theory axiom system. Using standard infinity axioms and non-standard infinity axioms, construct two opposite set theory axiom systems, in which infinity will be re-understood.

We know that the set theory ***ZF*** system is composed of 8 axioms, namely:

$$ZF = \{ZF1, ZF2, ZF3, ZF4, ZF5, ZF6, ZF7, ZF8\}$$

Among them, ZF6 is an infinite axiom, that is,

Infinity axiom (***ZF6***) : $\exists\omega[0\in\omega\wedge(n\in\omega\rightarrow n^{+}\in\omega)]$,

It is this infinite axiom "***ZF***6" that confuses the concept of infinity and is not strict.

We note "standard infinity axiom" as ***ZF***6-, and "non-standard infinity axiom" as ***ZF*6+. which is,**

Standard infinity axiom （***ZF6−***）： $\exists\delta\,[\,\forall E\in N\,(\delta > E)]\wedge[\delta+1=\delta]\wedge[\frac{1}{\delta}=0]$,

Non-standard infinity axiom （***ZF6+***）： $\exists\delta\,[\,\forall E\in N\,(\delta > E)]\wedge[\delta+1\neq\delta]$,

Because these two axioms are contradictory, they cannot be established in the same system. We use (***ZF***6-) and (***ZF***6+) to replace ***ZF***6 in the ***ZF*** system respectively and obtain two contradictory axiom systems, namely: ***(SZF-***) And (***SZF***+),

Standard set theory system: $SZF- = \{ZF1, ZF2, ZF3, ZF4, ZF5, ZF6-, ZF7, ZF8\}$

Non-standard set theory system: $SZF+ = \{ZF1, ZF2, ZF3, ZF4, ZF5, ZF6+, ZF7, ZF8\}$

(***SZF***-) and (***SZF***+) are contradictory axiom systems, similar to Euclidean geometry and non-Euclidean geometry axiom systems, which can be true independently of each other, in which we continue to reason to ordinal, cardinal, and infinite induction and arrive at the Cantor set on some different conclusions, it is easy to clarify many plausible conclusions including the "continuum hypothesis", which is very smooth and without contradictions.

We will further discuss this in another article "Reform of Set Theory ***ZF*** System".

Appendix References

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