Euler's Identity, Leibniz Tables, and the Irrationality of Pi

Timothy Jones

November 5, 2023

Abstract

Using techniques that show show that e and π are transcendental, we give a short, elementary proof that π is irrational based on Euler's formula. The proof involves evaluation of a polynomial using repeated applications of Leibniz formula as organized in a Leibniz table.

Introduction

Paul Nahin's recent book, *Dr. Euler's Fabulous Formula* [9] celebrates the identity $e^{\pi i} + 1 = 0$ and in it he gives an Euler's Identity-based proof of the irrationality of π using techniques of Legendre [7] of 1808. Here we give another, shorter proof using Euler's formula. Ours requires only basic calculus and the evaluation of the sum of the derivatives of a polynomial. The latter task is aided by a device we call a Leibniz table [5].

The Idea

Imagine that we have a complex polynomial $F_p(z)$ of degree a function of p, p a prime, and such that $F_p(0) + F_p(\pi i)$ is a *non-zero* Gaussian integer; that is a number of the form $x + iy$ with x and y integers both divisible by $(p-1)!$. We will show that

$$e^{\pi i} F_p(0) = F_p(\pi i) + \epsilon(p), \tag{1}$$

where

$$\lim_{p \to \infty} \frac{\epsilon(p)}{p!} = 0. \tag{2}$$

Adding $F_p(0)$ to both sides of (1), and using Euler's formula,

$$0 = F_p(0)(e^{\pi i} + 1) = F_p(0) + F_p(\pi i) + \epsilon(p), \tag{3}$$

hence

$$0 = \frac{F_p(0) + F_p(\pi i)}{(p-1)!} + \frac{\epsilon(p)}{(p-1)!}. \tag{4}$$

Now the right hand side of (4) can *not* be zero for sufficiently large p. This contradiction is the idea of the proof.

Some Details

Suppose, contrary to what we want to prove, that $\pi = m/n$ where $m > n > 0$ are natural numbers. We assume, for the time being, that the sum $F_p(z)$ of the derivatives of

$$f_p(z) = z^{p-1}(nz - mi)^p,$$

where p is a prime greater than m, is such that $F_p(0) + F_p(\pi i)$ is a non-zero Gaussian integer. We establish next the right hand side of (3).

Differentiation of $e^{-z}F_p(z)$ gives

$$\frac{d}{dz}(e^{-z}F_p(z)) = -e^{-z}F_p(z) + e^{-z}F_p'(z) = -e^{-z}f_p(z),$$

since $F_p(z) - F_p'(z) = f_p(z)$. Integrating along a suitable path from 0 to πi, we have

$$\int_0^{\pi i} \frac{d}{dz}(e^{-z}F_p(z)) = (e^{-z}F_p(z))\Big|_0^{\pi i} = -\int_0^{\pi i} (e^{-z}f_p(z))dz.$$

This gives

$$e^{-\pi i}F_p(\pi i) - F_p(0) = -\int_0^{\pi i} (e^{-z}f_p(z))dz,$$

and, using Euler's formula,

$$0 = F_p(0) + F(\pi i) - \int_0^{\pi i} (e^{-z}f_p(z))dz,$$

which is (3) with

$$\epsilon(p) = \int_0^{\pi i} (e^{-z} f_p(z)) dz.$$

Now the absolute value of this integral is bounded by the absolute value of its integrand on the path of integration times the length of the path [11]. We can use the product of the upper bounds for the absolute value of each of e^{-z}, z^{p-1} and $(nz-mi)^p$ to construct this upper bound. The first is bounded by a constant R, the second by m^{p-1}, and the third by $(2m)^p$ – recall that $m > n > 0$. An upper bound is given, then, by $R(2m)^{p-1}$, an exponential. The length of the path of integration is a constant. Thus $\epsilon(p)$ has an exponential upper bound. Dividing by $(p-1)!$ and knowing that factorial growth beats exponential, (2) is established.

Leibniz Tables

At this juncture, all that remains is to show that $F_p(0) + F_p(\pi i)$ is a non-zero Gaussian integer divisible by $(p-1)!$. We prove this for $p = 5$ using an argument that applies to the general case.

If you were asked to sum the derivatives of $f_5(z) = z^4(nz - mi)^5$, one approach would be to expand this complex polynomial, calculate each derivative, one after the other, and then add them up. A potentially quicker method is to use Leibniz' formula, which gives the nth derivative of a product: if $f(z) = g(z)h(z)$, then

$$f^{(n)}(z) = \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(z) h^{(n-k)}(z).$$

We need to sum these nth derivatives, so an additional summation is necessary.

$$F_5(z) = \sum_{j=0}^{9} \sum_{k=0}^{j} \binom{j}{k} [z^4]^{(k)} [(nz - mi)^5]^{(n-k)}.$$

Now we only need take derivatives of each of the multiplicands z^k and $(nz-mi)^5$, but still must generate the binomial coefficients and organize the derivatives to reference them several times.

A table provides a way to accomplish both tasks. In Table 1, the derivatives of z^4 are listed along the top row; the derivatives of $(nz - mi)^5$ are along the left column; and the binomial coefficients are the interior cells. Notice how the

	z^4	$4z^3$	$12z^2$	$24z$	$4!$
$(nz-mi)^5$	1	1	1	1	1
$5n(nz-mi)^4$	1	2	3	4	5
$20n^2(nz-mi)^3$	1	3	6	10	15
$60n^3(nz-mi)^2$	1	4	10	20	35
$120n^4(nz-mi)$	1	5	15	35	70
$5!n^5$	1	6	21	56	126

Table 1: The Leibniz table for $z^4(nz-mi)^5$.

derivatives build factorial expressions and taper down to a constant. The binomial coefficients are formed exactly as in Pascal's triangle.

Once the Leibniz table is complete, the zero through the ninth derivatives of the product can be read from the interior diagonals, starting with the upper left zeroth derivative $1z^4(nz-mi)^5$ and finishing with the ninth $126(4!)(5!n^5)$ in the lower right. Further examples are the first derivative

$$1(z^4)(5n(nz-mi)^4)+1(4z^3)((nz-mi)^5)$$

and the eighth

$$56(24z)(5!n^5)+70(4!)(120n^4(nz-mi)).$$

Properties of $F_5(0)+F_5(mi/n)$

The evaluation of $F_5(0)$ and $F_5(mi/n)$ is aided by Table 1. For $F(0)$, set $z=0$ and only the right column is non-zero. This shows $F(0)$ equals

$$4!(-mi)^5+5!x_1(-mi)^4+5!x_2(-mi)^3+5!x_3(-mi)^2+5!x_4(-mi)+5!x_5,$$

where each x_k is an integer. For $F_5(mi/n)$, set $z=mi/n$ and only the last row is non-zero. This shows $F_5(mi/n)$ is

$$5!n^5\left(\frac{(mi)^4}{n^4}+y_1\frac{(mi)^3}{n^3}+y_2\frac{(mi)^2}{n^2}+y_3\frac{(mi)}{n}+y_4\right),$$

where each y_k is again an integer.

Clearly $F_5(0) + F_5(mi/n)$ is of the form $a + bi$ where a and b are integers; the odd powers of $(\pm mi)$ give the complex part of the number, and the even the real. A $4!$ can be factored out, so $(a+bi)/4! = c+di$ with c and d integers. Assuming 5 does not divide m, we can conclude that 5 does not divide d and hence d can't be zero, so $c + di \neq 0$. Thus $(F_5(0) + F_5(mi/n))/4!$ is a non-zero Gaussian integer. The proof of π's irrationality is completed by applying this Leibniz-table-based proof to an arbitrary prime p. Note that for sufficiently large p, p will not divide a given m.

Conclusion

Nahin's treatment of π's irrationality is quite long. The central technique is to use a rational approximation to the exponential function. The techniques used here derive from those used for transcendence proofs of e and π [1, 2, 4, 8]. As an exercise look at the transcendence proofs for e [3, p. 216] and π [10] and construct Leibniz tables at key junctures. They help. It is while investigating these tables I discovered the proof given in this article.

The proof of π's irrationality given here might well usefully find its way into a calculus or advanced calculus course. Leibniz tables are clearly amenable to treatment in such a class. The other element of this proof not currently in typical calculus books is also relatively simple: given a polynomial on a finite interval, find its upper bound. These two techniques, having been introduced and perhaps drilled, the above proof distills down to a few logically evolving steps. Notes on presenting this article to undergraduates are provided are in [6].

References

[1] L. Berggren, J. Borwein, and P. Borwein, *Pi: A Source Book*, 3rd ed., Springer, New York, 2004.

[2] C. Hermite, Sur la fonction exponentialle, *Compt. Rend. Acad. Sci. Paris*, **77** (1873) 18-24, 74-79, 226-233, 285-293.

[3] I.N. Herstein, *Topics in Algebra*, 2nd. ed., John Wiley, New York, 1975.

[4] A. Hurwitz, Beweis der Transcendenz der Zahl *e*, *Mathematische Annalen* **43** (1893) 220-221.

[5] T. W. Jones, Discovering and proving π is irrational. *Amer. Math. Monthly* **117** (2010) 553-557.

[6] T. W. Jones, Euler's Identity, Leibniz tables, and the irrationality of pi with endnotes.

[7] A.M. Legendre, Éléments de géométrie, avec des notes, 7me ed., Didot, Paris, 1808.

[8] F. Lindemann, Über die Zahl π, *Math. Ann.* **20** (1882) 213–225.

[9] P.J. Nahin, *Dr. Euler's Fabulous Formula*, Princeton University Press, Princeton, NJ, 2006.

[10] I. Niven, The transcendence of π. *Amer. Math. Monthly* **46** (1939) 469-471.

[11] M. R. Spiegel, *Complex Variables with an Introduction to Conformal Mapping and Applications*, McGraw-Hill, New York, 1964.