A Truly Easy Proof: Pi is Irrational

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Abstract

Using the sum of the derivatives of an integer polynomial with Euler's formula we prove that π is irrational. We show how the technique can be used to show e and π's transcendence.

Proof

Proofs of the irrationality of π are numerous [1], but none are as easy and direct as the following.

Theorem 1. *π is irrational.*

Proof. A simple case generalizes. Suppose $f_3(x) = x^3$ and consider the sum of its derivatives:

$$F_3(x) = x^3 + 3x^2 + 3!x + 3!.$$

It follows that $F_3(0) = 3!$. Now consider

$$\begin{aligned} F_3(0)e^x &= 3!\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \sum_{k=4}^{\infty}\frac{x^k}{k!}\right) \\ &= F_3(x) + 3!\sum_{k=4}^{\infty}\frac{x^k}{k!} \\ &= F_3(x) + 3!(e^x - s_3(x)), \end{aligned}$$

where $s_3(x)$ is a partial sum of e^x.

Adding $F(0)$ we have

$$(e^x + 1)F_3(0) = F_3(0) + F_3(x) + 3!(e^x - s_3(x)). \tag{1}$$

Now imaging $x = \pi i$ and applying Euler's formula, $e^{\pi i} + 1 = 0$ makes (1)

$$0 = \frac{F_3(0) + F_3(x)}{3!} + (e^x - s_3(x)),$$

after dividing by 3!, the multiplicity of the single root of $f(x)$ factorial.

There is no reason to believe that for a general term of any polynomial this pattern would change. Nor is there any reason that all surviving non-zero coefficients of $F_n(r)$, r a root of $f_n(x)$ would not have factors of the multiplicity of the root factorial (like this easy case), if the coefficients of $f_n(x)$ are integers. Thus assuming $\pi = p/q$, we can use $x^3(qx - pi)^3$, for example, and these conditions are met. So, 0 is an integer plus a something less than 1, a contradiction. □

Of course this is a *forest* only proof. We are definitely not getting into the weeds, the details. The next two, slightly harder ideas, give credence to our evolving forest.

The Mean Value Theorem

Another property of $F(x)$ is

$$F(x) - F'(x) = (x^3 + 3x^2 + 3!x + 3!) - \frac{d}{dx}(x^3 + 3x^2 + 3!x + 3!) \tag{2}$$

$$= x^3 = f(x) \tag{3}$$

and this is clearly the case for any polynomial, $f(x)$. We also notice the product formula for derivatives is of interest: $(fg)' = f'g + g'f$. Consider that $(e^x F(x))' = e^x F(x) + F'(x)e^x$ is close to $e^x(F(x) - F'(x))$. We need subtraction; $-e^{-x}F(x)$ does the trick:

$$(-e^{-x}F(x))' = e^{-x}F(x) + F'(x)(-e^{-x}) = e^{-x}(F(x) - F'(x)). \tag{4}$$

The mean value theorem can be combined with (4). Let $G(x) = -e^{-x}F(x)$, then

$$\frac{G(x) - G(0)}{(x - 0)} = G'(\xi) = e^{-\xi}f(\xi),$$

where $\xi \in (0, x)$. Translating back,

$$-e^{-x}F(x) + e^0 F(0) = xe^{-\xi} f(\xi)$$

and then multiplying by e^x gives

$$-F(x) + e^x F(0) = xe^{x-\xi} f(\xi).$$

This is our pattern: $e^x F(0) = F(x) + xe^{x-\xi} f(\xi)$.

Integration

This pattern $e^x F(0) = F(x) + xe^{x-\xi} f(\xi)$ might be called Hermite's Formula. With it (and other things) he showed e is transcendental and later Lindemann used it again to show π is transcendental too. We can give the essence of their ideas. Before going there, here's another derivation of Hermite's formula using a definite integral.

Just integrating

$$\frac{d(-e^{-x}F(x))}{dx} = e^{-x} f(x)$$

gives

$$\int_0^x \frac{d}{dx}\left(-e^{-x}F(x)\right) = \Big|_0^x \left(-e^{-x}F(x)\right) = -e^{-x}F(x) + F(0)$$

on the left side and

$$\int_0^x e^{-x} f(x) \text{ dx}$$

on the right. Multiplying by $-e^x$ then gives

$$F(x) - e^x F(0) = -e^x \int_0^x e^{-x} f(x) \text{ dx}$$

or

$$e^x F(0) = F(x) + e^x \int_0^x e^{-x} f(x) \text{ dx}.$$

Transcendence

A number is transcendental if it is not the root of an integer polynomial. Naturally, to show a number is transcendental, we suppose that it is a root of polynomial and derive a contradiction. We can use Hermite's polynomial as a starting point. Consider that $\pi i = r_0$ and r_k, $0 < k \leq n$, are other roots an integer polynomial. Supposing we made the right Hermite polynomial, we might hope for something involving

$$F(0)(e^{r_0} + e^{r_1} + \cdots + e^{r_n}) = -F(0) + F(r_1) + \cdots + F(r_n) + \sum_{k=0}^{n} \epsilon_k.$$

But this isn't equal to zero. We need a modification that will make this equal to 0 but also yield a way to sum those large capital F values. We know $e^{\pi i} + 1 = 0$, so we can say

$$0 = F(0)\left((e^{r_0} + 1)(e^{r_1} + 1)\ldots(e^{r_n} + 1)\right)$$

and this will yield sums of F at the various exponents generated. We will have to modify the small case f used for our capitial F to make the roots equal to these exponents. Will the resulting polynomial be the requisite integer polynomial or something close – i.e. with coefficients rationals awaiting a constant multiple to make them all integers.

That's π and it seems complicated. Remember

$$\prod_{k=0}^{n}(x - r_k) = P_0 x^n + P_1 x^{n-1} + P_2 x^{n-2} + \cdots + P_n \tag{5}$$

where P_k is the sum of roots multiplied k at a time. You observe this with

$$(x-1)(x-2)(x-3) = x^3 - (1+2+3)x^2 + (1\cdot 2 + 1\cdot 3 + 2\cdot 3)x^1 - 1\cdot 2\cdot 3.$$

And we can get these coefficients with Maple, Figure 1.

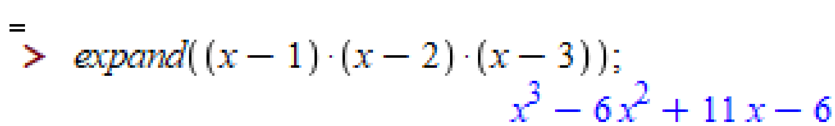

Figure 1: Maple's expand command in action.

This example gives some evidence that integer roots generate integer coefficients; a pretty obvious result. But what about $(x - \sqrt{2})(x + \sqrt{2}) = x^2 - 2$? The roots are not integers, but the coefficients are. So it isn't clear whether or not we can form a polynomial from the roots of an integer polynomial that will for sure also have integer coefficients. More on this in a separate article. Let's try an easier case of transcendence.

Suppose we wanted to prove e is transcendental. As always we assume it isn't and attempt to derive a contradiction. Suppose e is a root of $p(x)$

$$p(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_n,$$

where the coefficients are integers. Then

$$p(e) = c_0 e^n + \cdots + c_n = 0.$$

It seems likely we can form a polynomial $f(x)$ with an $F(0)$ such that

$$0 = F(0)(c_0 e^n + \cdots + c_n) = c_0 F(n) + \cdots + c_n F(0) + \sum_{i=0}^{n} \epsilon_i. \tag{6}$$

We can read the necessary roots off of (6). They are just the integers $0, \ldots, n$; the exponents of e.

Conclusion

This is a *forest* article. With it students might seek to happily learn what's needed to complete our sketches.

References

[1] Eymard, P., Lafon, J.-P. (2004). *The Number* π. Providence, RI: American Mathematical Society.