Density of gravitational energy of curved space

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Abstract: Gravitational energy is localized. It is shown that the massless gravitational energy is enclosed in space, which is curved by matter, according to Einstein's theory. The density of the gravitational energy of curved space is found. This solves the problem of the unambiguous localization of gravitational energy, a problem that Einstein left to the mercy of fate. In the Schwarzschild case, the receipt of the gravitational energy from space by matter increases the inertial mass of the matter. Since the gravitational mass of an isolated system is determined by the curvature of space at infinity, the inertial mass of an isolated system exceeds its gravitational mass by the value of the gravitational energy of space.

Key words: binding energy; mass defect; energy localization

1. Introduction

According to the definition, the *gravitational binding energy* of a system is the minimum energy that must be added to it in order for the system to cease to be in a gravitationally bound state. As is known [1,2][2], the gravitational binding energy of a ball of (inertial) mass M and radius r_1 is equal to

$$U = 3\gamma M^2 / 5r_1 \tag{1}$$

(we define the radius in terms of the length of the equator: $r_1 = l / 2\pi$). This means that the substance of the ball can be removed to infinity during its explosion if the mass-energy of the explosive is $3\gamma M^2 / 5r_1$. During the explosion, this energy is first converted into kinetic mass-energy, and then disappears in the process of expansion as the speed decreases. Thus, the inertial mass of the ball decreases up to a constant m,

$$m = M - 3\gamma M^2 / 5r_1, \tag{2}$$

which defines the invariable outer Schwarzschild space of the ball that is described by the line element [3 (100.14)]

$$dl^2 = g_{rr} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2), \quad g_{rr}(r) = \frac{1}{1 - 2\gamma m / r}. \tag{3}$$

The constant m is the gravitational mass of the ball. It can be expressed in terms of the initial density of the ball substance according to the formula [3 (100.24)]

$$m = 4\pi r_1^3 \rho / 3. \tag{4}$$

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[2] See also https://en.wikipedia.org/wiki/Gravitational_binding_energy

2. Mass of the ball

Formula (4) shows that using $4\pi r_1^3/3$ as the volume of the ball does not give the correct value M for the mass of the ball. This is because, in reality, the volume of the sphere with the equatorial length $2\pi r_1$ is greater than $4\pi r_1^3/3$ due to the curvature of space.

Inside the ball, the metric coefficient $g_{rr}(r)$ is [3 (100.19)]

$$g_{rr} = \frac{1}{1 - r^2/R^2} > 1, \quad R^2 = \frac{3}{8\pi\gamma\rho}. \tag{5}$$

The correct mass of the ball was calculated in [4], taking this circumstance into account:

$$M = \int_0^{r_1} \rho\sqrt{g_{rr}}\, 4\pi r^2 dr = \frac{3}{2\gamma R}\int_0^{r_1} \frac{r^2 dr}{\sqrt{R^2 - r^2}} = \frac{3R}{4\gamma}\left(\sin^{-1}\xi - \xi\sqrt{1-\xi^2}\right), \quad \xi = \frac{r_1}{R}. \tag{6}$$

If the mass of the ball is small, that is $M \approx m$, $\xi^2 = 8\pi\gamma\rho r_1^2/3 = 2\gamma m/r_1 = r_g/r_1 \ll 1$, then

$$(\sin^{-1}\xi - \xi\sqrt{1-\xi^2}) \approx (\frac{2}{3}\xi^3 + \frac{1}{5}\xi^5), \text{ и } M \approx m + \frac{3\gamma m^2}{5r_1}, \tag{7}$$

according to (1) and (2). (We denoted here the gravitational radius $r_g = 2\gamma m$). If the ball is compressed to its gravitational radius, $\xi = 1$, then $M = 3\pi m/4 \approx 2.36m$. For a ball of arbitrary radius, the binding mass-energy is

$$U = M - m = \frac{3R}{4\gamma}\left(\sin^{-1}\xi - \xi\sqrt{1-\xi^2}\right) - m = \frac{3m}{2\gamma\xi^3}\left(\sin^{-1}\xi - \xi\sqrt{1-\xi^2}\right) - m. \tag{8}$$

3. Mass of the shell

The gravitational binding energy cannot be considered as the energy of the "gravitational field", because within the framework of the general relativity, the gravitational field does not exist. Weyl writes: "Gravity is "leading" (Führung) and not a force" [5]. All gravitational phenomena are explained by the curvature of space-time. We consider the binding energy as the energy of space, which is curved by matter, according to the Einstein equation. To determine the energy density of curved space, we calculate the mass M of a spherical shell of small thickness Δr and the gravitational mass m . Let the density ρ be nonzero only between the coordinates r_0 and r_1, $\Delta r = r_1 - r_0$. Then, according to the formula [3 (100.19)], the metric coefficient between these coordinates is

$$g_{rr}(r) = \frac{1}{1 - 8\pi\gamma\rho(r^3 - r_0^3)/3r} = \frac{1}{1 - 8\pi\gamma\rho(r - r_0)(r^2 + rr_0 + r_0^2)/3r}. \tag{9}$$

Considering $r - r_0 \ll r$, we have the mass of the shell:

$$M = \int_{r_0}^{r_1} 4\pi r^2 \rho\sqrt{g_{rr}}\, dr = \int_{r_0}^{r_1} \frac{4\pi r_1^2 \rho dr}{\sqrt{1 - 8\pi\gamma\rho r_1(r - r_0)}}. \tag{10}$$

Using $\int \frac{dx}{\sqrt{a+bx}} = \frac{2}{b}\sqrt{a+bx}$ and $m = 4\pi r_1^2 \rho\Delta r$, in accordance with (4); replacing $r_1 \to r$, we get the mass of the shell

$$M = r(1 - \sqrt{1 - 8\pi\gamma\rho r\Delta r})/\gamma = r(1 - \sqrt{1 - 2\gamma m/r})/\gamma. \tag{11}$$

4. Density of the gravitational energy

Now, using (11), one can find the gravitational energy density u [kg/m^3] contained in the curved space. It should be taken into account that outside of the shell there is a curved Schwarzschild space and inside of the shell the space is Euclidean. So, when the radius of the shell decreases by dr, a new volume of space with the value dV_0 is curved and, in this case, gravitational energy is released: $dM = udV_0 = u4\pi r^2\sqrt{g_{rr}}dr$ where g_{rr} is from (3). So,

$$u = \frac{dM}{dr}\frac{1}{4\pi r^2\sqrt{g_{rr}}} = \frac{1}{4\pi r^2\gamma}\left(\sqrt{1-\frac{2\gamma m}{r}} - 1 + \frac{\gamma m}{r}\right) = \frac{1}{4\pi r^2\gamma}\left(\frac{1}{\sqrt{g_{rr}}} - \frac{1}{2} - \frac{1}{2g_{rr}}\right) < 0. \qquad (12)$$

5. Binding energy of the ball

Using (12), one can calculate the binding energy of the ball (8) by integrating this space energy density:

$$U = -\int_0^\infty u4\pi r^2\sqrt{g_{rr}}dr = M - m. \qquad (13)$$

Only the integral must be divided into two parts, because the metric coefficients g_{rr} inside the ball and outside the ball are different and equal, respectively

$$g_{rr} = \frac{1}{1-r^2/R^2}, \quad g_{rr} = \frac{1}{1-2\gamma m/r}. \qquad (14)$$

We show that, in accordance with formula (8),

$$U = U_{\text{in}} + U_{\text{ext}} = -\int_0^r u4\pi r^2\sqrt{g_{rr}}dr - \int_r^\infty u4\pi r^2\sqrt{g_{rr}}dr = \frac{3R}{4\gamma}\left(\sin^{-1}\xi - \xi\sqrt{1-\xi^2}\right) - m = M - m. \qquad (15)$$

Indeed, the energy coming from the interior space of the ball is equal to

$$U_{\text{in}} = -\frac{1}{\gamma}\int_0^r\left(1 - \frac{1}{2\sqrt{1-r^2/R^2}} - \frac{1}{2}\sqrt{1-r^2/R^2}\right)dr = -\frac{r}{\gamma} + \frac{3R}{4\gamma}\sin^{-1}\frac{r}{R} + \frac{r}{4\gamma}\sqrt{1-r^2/R^2}. \qquad (16)$$

The energy coming from the outer part of the space is equal to

$$U_{\text{ext}} = -\frac{1}{\gamma}\int_r^\infty\left(1 - \frac{1}{2\sqrt{1-2\gamma m/r}} - \frac{1}{2}\sqrt{1-\frac{2\gamma m}{r}}\right)dr = \frac{r}{\gamma} - \frac{r}{\gamma}\sqrt{1-\frac{2\gamma m}{r}} - m = \frac{r}{\gamma} - \frac{r}{\gamma}\sqrt{1-\xi^2} - m. \qquad (17)$$

Summing up (16) and (17), we obtain (15) or (8), which was to be shown.

6. Conclusion

The problem of localization of gravitational energy is solved. The volume density of the gravitational energy of the Schwarzschild space is given by formula (12). This solves the problem of unambiguous localization of gravitational energy, a problem that Einstein left to the mercy of fate [5]. A body that curves space receives the gravitational energy of the curved space during compression. Accordingly, the density of the gravitational energy of the space is negative, since the energy density of the Euclidean space is equal to zero.

The gravitational energy U received by the body from the space increases the inertial mass-energy of the body, according to formula (13). Thus, the inertial mass exceeds the gravitational mass by the amount of energy of the curved space.

7. Discussion. The principle of equivalence

The excess of the inertial mass *M* over the gravitational mass *m* can cause concern about the principle of equivalence. However, there is no principle of equivalence in Einstein's theory. This principle asserts the equality of the gravitational force and the force of inertia. But there is no gravitational force in Einstein's theory. The body lying on a table presses on the table by the force of inertia, since its world line is curved by the substance of Earth. And this force of inertia, by definition, is proportional to its inertial mass. As for the (active) gravitational mass of this body, its operational definition is problematic. The gravitational mass of a body determines an additional curvature of space at infinity, while the inertial mass of a body determines the curvature of space according to Einstein's equation

$$G^{\alpha}_{\beta} \equiv R^{\alpha}_{\beta} - R^{\mu}_{\mu}\delta^{\alpha}_{\beta}/2 = 8\pi\gamma T^{\alpha}_{\beta}, \quad \gamma = 7.4\cdot 10^{-28} \text{ m/kg}. \tag{18}$$

But the gravitational mass is preserved during gravitational contraction, since it includes the gravitational energy of space, and the law of conservation of inertial mass does not exist with respect to gravitational interactions. Passive gravitational mass does not make sense in the theory, since there is no gravitational force.

8. Discussion. The mass defect

The increase in the mass of the shell during compression contradicts the popular belief about the negative gravitational mass defect. Authors of [6] write: “The mass-energy of the Earth-moon system is less than the mass-energy that the system would have if the two objects were at infinite separation. The mass-energy of a neutron star is less than the mass-energy of the same number of baryons at infinite separation”.

But such statements contradict the obvious fact of the increase in the mass-energy of the Earth-apple system when an apple falls and get the kinetic mass-energy $mv^2/2 = mgh$. On the contrary, if the velocity of revolving of Moon around Earth is directed away from Earth, Moon will move away from Earth, and its speed, and, accordingly, its mass will decrease compared to the initial values, and not increase, as the authors claim.

Consider now the removal of baryons of a neutron star to infinity. This removal requires mass-energy. But there is nowhere to take this mass from, except from the star itself. So when you remove baryons, you have to reduce the mass of the star. The mass has to be taken away from the baryons and introduced into the curved Schwarzschild space. This mass is spent on straightening the curved space, on increasing the space energy from a negative value to zero.

9. Discussion. The pseudo-tensor

Einstein's theory, expressed by equation (18), does not contain the concept of gravitational energy. Various versions of the pseudo-tensor have been devised to introduce gravitational energy into the theory. It seemed that success had been achieved. Tolman writes [7, p. 250]: “This satisfactory result can serve to increase our confidence in the practical advantages of Einstein's procedure in introducing the pseudo-tensor densities of potential gravitational energy and momentum $t^{\alpha}_{\beta} = \dfrac{1}{16\pi}\left[-g^{\mu\nu}_{\beta}\dfrac{\partial\mathfrak{L}}{\partial g^{\mu\nu}_{\alpha}} + g^{\alpha}_{\beta}\mathfrak{L}\right]$.”

Unfortunately, this expression is wrong. The pseudo-tensor gives a positive value for the gravitational energy of the liquid sphere, contrary to the our result (13) and to the Tolman equation [7 (97.10)],

$$m = \int \rho dV_0 + \int \frac{1}{2} \rho\psi dV\,, \quad \psi < 0 \qquad (19)$$

where the gravitational energy $\int \frac{1}{2} \rho\psi dV$ is negative due to $\psi < 0$. This is shown in [4].

Declarations
Ethical Approval
Not applicable
Competing interests
There are no interests of a financial or personal nature
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Declaration of interests
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Ниже приведена переписка в обратном хронологическом порядке

Глубокоуважаемый **В.В.Жаринов,**

Ваш рецензент просит меня «вывести формулу (1)». Но вывод был сделан в 1939 году и приведен в Википедии! Формула (1) заключена там в рамку.

Ваш рецензент тяжело не способен читать. Я уже указывал Вам это в связи с рецензией предыдущей статьи. Указание выложено по адресам
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https://www.researchgate.net/publication/374373465_Gravitational_energy_attributed_to_curved_space
http://viXra.org/abs/2310.0007

А редакция снова направила ему на рецензию статью с результатами, которые превосходят достижения Эйнштейна и Ландау. Теперь я просто вкладываю соответствующий кусок Википедии. Пожалуйста, проверьте функциональную грамотность этого рецензента журнала ТМФ, показав ему этот кусок. Очень интересно, кто этот человек!

Gravitational binding energy

Article Talk

From Wikipedia, the free encyclopedia

The **gravitational binding energy** of a system is the minimum energy which must be added to it in order for the system to cease being in a gravitationally bound state. A gravitationally bound system has a lower (*i.e.*, more negative) gravitational potential energy than the sum of the energies of its parts when these are completely separated—this is what keeps the system aggregated in accordance with the minimum total potential energy principle.

For a spherical body of uniform density, the gravitational binding energy U is given by the formula[2][3]

$$U = -\frac{3GM^2}{5R}$$

where G is the gravitational constant, M is the mass of the sphere, and R is its radius.

Derivation for a uniform sphere [edit]

The gravitational binding energy of a sphere with radius R is found by imagining that it is pulled apart by successively moving spherical shells to infinity, the outermost first, and finding the total energy needed for that.

Assuming a constant density ρ, the masses of a shell and the sphere inside it are:

$$m_{\text{shell}} = 4\pi r^2 \rho \, dr$$

and

$$m_{\text{interior}} = \frac{4}{3}\pi r^3 \rho$$

The required energy for a shell is the negative of the gravitational potential energy:

$$dU = -G\frac{m_{\text{shell}} m_{\text{interior}}}{r}$$

Integrating over all shells yields:

$$U = -G\int_0^R \frac{\left(4\pi r^2 \rho\right)\left(\frac{4}{3}\pi r^3 \rho\right)}{r} dr = -G\frac{16}{3}\pi^2\rho^2 \int_0^R r^4 dr = -G\frac{16}{15}\pi^2\rho^2 R^5$$

Since ρ is simply equal to the mass of the whole divided by its volume for objects with uniform density, therefore

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

And finally, plugging this into our result leads to

$$U = -G\frac{16}{15}\pi^2 R^5 \left(\frac{M}{\frac{4}{3}\pi R^3}\right)^2 = -\frac{3GM^2}{5R}$$

Gravitational binding energy

$$U = -\frac{3GM^2}{5R}$$

От tmph@mi-ras.ru 23.10.2023,
Ref_Khrapko.pdf **105 КБ**
Глубокоуважаемый Радий Игоревич,
Редколлегией ТМФ принято решение об отклонении Вашей статьи "Density of gravitational energy of curved space" на основании отзыва рецензента. Отзыв прилагается.
Ответственный секретарь ТМФ В.В.Жаринов

Рецензия для ТМФ

Статья: Density of gravitational energy of curved space
Автор(ы): Khrapko R.I.

Понятие энергии в общей теории относительности до сих пор является предметом дискуссий, поэтому автору необходимо сделать следующее:

1) Ссылок [1,2] недостаточно. Нужно указать или номер формулы, или номер страницы, или номер параграфа, на который автор ссылается.
2) Дать строгое определение энергии.
3) Дать определение “binding energy”.
4) Дать определение плотности энергии.
5) Доказать, что данные определение непротиворечивы, в частности, указать, что энергия в каком-то смысле сохраняется.
6) На основе данных определений вывести формулу (1).

До тех пор, пока это не будет выполнено, выводы автора нельзя считать обоснованными, и обсуждать недостатки не имеет смысла.
Считаю публикацию статьи в ТМФ нецелесообразной.

Статья отклонена без рецензии журналом Письма в ЖЭТФ

Уведомление | Notification

Уважаемый автор! Благодарим вас за направление вашей рукописи "Density of gravitational energy of curved space" **PisJETF2360436** в журнал Письма в ЖЭТФ. После тщательного рассмотрения рукописи нашими редакторами, мы сообщаем, что не можем ее опубликовать в нашем журнале. В случае, если редакторы заключают, что рукопись не соответствует критериям нашего журнала (http://jetpletters.ru/ru/info.shtml) они выносят решение без направления статьи на внешнее рецензирование. В данном случае рассмотрение показало, что рукопись не содержит новых правильных результатов заслуживающих критического рассмотрения.
По поручению Редколлегии Зав. редакцией Ирина Подыниглазова

Апелляция

Уважаемая Ирина Подыниглазова.
Решение Редколлегии неприемлемо, потому что **локализация гравитационной энергии** не заслужила у Редколлегии рассмотрения. Это уже третья статья о локализации гравитационной энергии, на которую редакция не смогла создать *никакую* рецензию.

It is appropriate to quote here from Max Planck:
“An important scientific innovation rarely makes its way by gradually winning over and converting its opponents: it rarely happens that Saul becomes Paul. What does happen is that its opponents gradually die out and that the growing generation is familiarized with the idea from the beginning”

[Planck M., *The Philosophy of Physics* (George Allen & Unwin, London, 1936), p. 90.]

Cordially, Radi Khrapko
khrapkori.wmsite.ru/

PS.

Известно, Эйнштейн отклонил статью Фридмана о расширяющейся Вселенной, поскольку она, по его мнению, не содержала новых правильных результатов заслуживающих критического рассмотрения.

Эйнштейну – простительно. Он защищал полученный им результат. Ныне редакторы защищают свой престиж.