Proof of ABC Conjecture

LiXiaohui

Abstract

This paper utilizes the fact that the prime factor among all factors in the root number rad (c) can only be a power of 1. Then, analyze all combinations of c that satisfy rad (c)=c, calculate the value of the combination, and find the maximum and minimum values of the root number rad, as well as the maximum exponent between them. Using this maximum exponent then an equivalent inequality is constructed to prove the ABC conjecture.

Key words: *Root/Prime factor/ Constant C*

The positive integers *a* , *b* , and *c* , satisfying the following conditions: *a* + *b* = *c* , and (*a*, *b*) = 1 (*a* , *b* are mutually prime).

It is not difficult to find that when all factors in rad (c) are prime numbers and the powers of prime numbers are all 1, then rad (c) = c.

eg: $rad(165) = rad(3^1 \cdot 5^1 \cdot 11^1) = 3 \times 5 \times 11 = 165$

Through the prime number theorem, we know that given a positive integer x , the number of prime numbers that do not exceed x is approximately: $\pi(x) \sim x/\ln(x)$

Now let's set the value range of the positive integer c to: $1 < c \le x$

We set the number of prime numbers not exceeding x to be a positive integer h , so the value of h is:

$$h = [\pi(x) \sim x/\ln(x)], h \in N^+$$

We use the set $X = \{p_1, p_2 \cdots p_n\}$ to represent the set of all prime numbers that do not exceed the integer x .

Easy to detect: when $c = p_1^{\ 1}$ or $c = p_1^{\ 1} \cdot p_h^{\ 1}$ or $c = p_1^{\ 1} \cdot p_2^{\ 1} \cdot p_3^{\ 1} \cdots p_h^{\ 1}$, etc,The value of rad(c) is exactly equal to c , that is:

$$c = rad(c)$$

We can calculate the maximum number of combinations in the set of prime numbers where $rad(c) = c$ is:

$$C_h^1 + C_h^2 + C_h^3 + \cdots + C_h^h$$

Because (a,b) =1, then (a,b, c) =1
Proof:
If a and c are not prime each other, there must be a common divisor k, and because b=c-a, then b and a must also have a common divisor k, which contradicts the prime of a and b, so a, b, and c are also prime each other

If the power of all prime factors in the radical $rad(c)$ is 1.
Then $c = rad(c)$
Then $rad(a \cdot b \cdot c) = rad(a \cdot b) \cdot rad(c)$

Now let's return to $rad(a \cdot b \cdot c)$ for analysis:

We know that the minimum value of prime factors in $rad(a \cdot b \cdot c)$ is 2, and the minimum number of these prime factors is 1. Therefore, the minimum value of $rad(a \cdot b \cdot c)$ is:

$$rad(a \cdot b \cdot c)_{\min} = 2^1$$

Similarly, when the power of the prime factor in $rad(a \cdot b \cdot c)$ is equal to 1 and the maximum number of these prime factors is the integer $h = [\pi(x) \sim x/\ln(x)]$, then the maximum value of $rad(a \cdot b \cdot c)$ is:

$$rad(a \cdot b.c)_{\max} = \prod_{i=1}^{h} p_i = P, \left(p_i \in X, P \in N^+\right)$$

So we can immediately launch:

$2 \leq rad(a \cdot b \cdot c) \leq P$ (1)

Now let's set $(rad(\mathrm{a} \cdot \mathrm{b} \cdot c)_{\min})^m = rad(\mathrm{a} \cdot \mathrm{b} \cdot c)_{\max}, m \in R,$ i.e. $2^m = P$, to find the maximum exponent between the minimum and maximum values. by taking the logarithm of both sides of the equation, we can obtain the value of m as:

$m = \dfrac{\log P}{\log 2}$ (2)

Let's analyze the value of c:

We know that the value range of c is: $1 < c \leq x$

We know that the set $X = \{p_1, p_2 \dots p_n\}$ is a set of all prime numbers that does not exceed the integer x, so the construction of the value of the integer c must be:

$$c = \prod p_i^{\ n} \quad \left(p_i \in X, i \in N^+, n \in N^+, c \leq x\right)$$

We know that in the integer interval $[3, n]$, when $n > 3$, according to the prime number theorem: $\pi\ (n) \sim n/\ln(n) > 2$, there must be an odd prime number in the interval $[3, n]$, we can set it as:

$$pr_1 = n - k_1 \quad \left(k_1 \in N^+, 0 < k_1 < n\right)$$

Meanwhile, according to the Bertrand Chebyshev theorem, when $n>3$, in the interval $(n,2n-2)$, there is at least one odd prime number, we can set it as:

$$pr_2 = n + k_2 \quad \left(k_2 \in N^+, 0 < k_2 < n-2\right)$$

There must be three different scenarios for the value of x.

The first scenario:

If x is an even number, then we can set $x = 2n, n \in N^+$

There must be an odd prime number $pr_1 = n - k_1$ and an odd prime number $pr_2 = n + k_2$ .and $2, pr_1, pr_2 \in X$

So the following two inequalities always hold:

1. $P \ge 2 \cdot pr_1 \cdot pr_2$
2. $2pr_1 \cdot pr_2 - x = 2(n-k_1)(n+k_2) - 2n > 0$

Immediately available: $c \le x \le P$

Second scenario:

Similarly,if x is an odd number, then we can set $x = 2n-1, n \in N^+$

There must be an odd prime number $pr_1 = n - k_1$ and an odd prime number $pr_2 = n + k_2$. and $pr_1, pr_2 \in X$

1. $P \ge pr_1 \cdot pr_2$
2. $$\begin{aligned} & pr_1 \cdot pr_2 - x = (n-k_1)(n+k_2) - (2n-1) \\ & = n^2 - 2n + 1 + k_2 n - k_1 n - k_1 k_2 \\ & = (n-1)^2 + n(k_2 - k_1) - k_1 k_2 \\ & \ge (n-1)^2 + (k_2+2)(k_2-k_1) - k_1 k_2 \\ & = (n-1)^2 + k_2^2 + 2k_2 - k_1 k_2 - 2k_1 - k_1 k_2 \\ & = (n-1)^2 + k_2^2 + 2k_2 + 1 - 2k_1 k_2 - 2k_1 - 1 \\ & = (n-1)^2 + (k_2+1)^2 - 2k_1(k_2+1) - 1 \\ & = (n-1)^2 + (k_2+1)(k_2+1-2k_1) - 1 \\ & \ge (n-1)^2 + (k_2+1)(k_2-1) - 1 \\ & \ge (n-1)^2 + k_2^2 - 2 \ge 0 \end{aligned}$$

Immediately available: $c \le x \le P$

The third scenario:

If x is an odd number, then we can set $x = 2n+1, n \in N^+$

There must be an odd prime number $pr_1 = n - k_1$ and an odd prime number $pr_2 = n + k_2$ .and $2, pr_1, pr_2 \in X$

1. $P \ge 2 \cdot pr_1 \cdot pr_2$
2. $2pr_1 \cdot pr_2 - x = 2(n-k_1)(n+k_2) - (2n+1) \ge 2(n+k_2) - 2n - 1 > 0$

Similarly,Immediately available: $c \le x \le P$

So whether x is odd or even,we can obtain: $c \le x \le P$

And because $P = \prod_{i=1}^{h} p_i = rad\ (\mathrm{a} \cdot \mathrm{b} \cdot c)_{\max} = 2^m$, we can immediately obtain:

$c \le P = 2^m$ (3)

Because $2 \le rad\,(a \cdot b \cdot c) \le P$, then inequality (3) can be transformed as follows:

$$c \le 2^{m-1} \cdot 2^{1} \le 2^{m-1}(rad\,(a \cdot b \cdot c))^{1} < 2^{m-1}(rad\,(a \cdot b \cdot c))^{1+\varepsilon} \quad \forall \varepsilon > 0$$

$$\Rightarrow c < 2^{m-1}(rad\,(a \cdot b \cdot c))^{1+\varepsilon}$$

We set $C = 2^{m-1}$, and now we have found the constant that always holds the inequality above, namely:

$$C = 2^{m-1}$$

CONCLUSION

In positive integers, there is equation $a + b = c$, and $(a, b) = 1$, when $\forall \varepsilon > 0, \exists C$ can make these triplets (abc) satisfy the following inequality, namely:

$$c < C \cdot (rad\,(a \cdot b \cdot c))^{1+\varepsilon}$$

Example:

$a = 3, b = 5, and\ c = 8, rad(a) = 3, rad(b) = 5, rad(c) = 2,$

$\mathrm{rad(ab)} = 15, \mathrm{rad(abc)} = 30, \mathrm{so}\ X = \{7, 5, 3, 2\}$

So,

$$\mathrm{rad(c)}_{\min} = 2, rad(c)_{\max} = P = 7 \times 5 \times 3 \times 2 = 210$$

So: $m = \dfrac{\log p}{\log 2} \approx 7.71$

So: $C = 2^{m-1} = 2^{7.7143-1} \approx 105.00$

The following inequality holds:

$$c = 8 < C \cdot (rad\,(a \cdot b \cdot c))^{1+\varepsilon} = 105.00 \times 2^{1+\varepsilon}, \forall \varepsilon > 0$$

Conclusion: The ABC conjecture holds.

REFERENCES

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