Sufficiently large number n makes $\sum_{k=n}^{2n-1} \frac{c}{ak+b} = \frac{c}{a}\ln 2$

Tai-choon Yoon[1], Yina Yoon

(Dated: Oct. 23rd, 2023)

According to the Riemann rearrangement theorem, when a sequence converges, the sum can be changed by rearranging the order of the sequence. However, the result cannot be changed simply by rearranging the order of any sequences. In the case of the alternating harmonic series exemplified by Riemann, even if the result was the same by chance, the sum of the series was obtained by ignoring the sum of $\lim_{n\to\infty} \sum_{k=n}^{2n-1} \frac{1}{k+1} = \ln 2$.

Key Words: Harmonic Series, Alternating Harmonic Series, Riemann Series Theorem, Riemann Rearrangement Theorem

Among series in mathematics, the harmonic series is an infinite series written as follows:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty. \tag{1}$$

The sum of this series is divergent.

However, the n fractional sum of this series from k= n+1 to k=2n converges to $\ln 2$,

$$\lim_{n\to\infty} \sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n-2} + \frac{1}{2n-1} + \frac{1}{2n} = \ln 2. \tag{2}$$

if n is a sufficiently large number. We can compute the series that converges

$$\begin{gathered}
\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{2} = 0.5, \quad \text{if } n = 1, \\
\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{19} + \frac{1}{20} = 0.668771, \quad \text{if } n = 10, \\
\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{51} + \frac{1}{52} + \cdots \frac{1}{99} + \frac{1}{100} = 0.688172, \quad \text{if } n = 50, \\
\sum_{k=n+1}^{2n} \frac{1}{k} = \frac{1}{101} + \frac{1}{102} + \cdots + \frac{1}{199} + \frac{1}{200} = 0.690653, \quad \text{if } n = 100, \\
\vdots
\end{gathered} \tag{3}$$

The sum of this series can be derived from an alternating harmonic series. With this result, we may

[1] Electronic address: tcyoon@hanmail.net

have an approximate value of a harmonic series H_n as follows

$$\begin{aligned} H_n &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \cdots \\ &= 1 + \sum_{k=2}^{2}\frac{1}{k} + \sum_{k=3}^{4}\frac{1}{k} + \sum_{k=5}^{8}\frac{1}{k} + \sum_{k=9}^{16}\frac{1}{k} + \cdots + \sum_{k=n+1}^{2n}\frac{1}{k} + \sum_{k=2n+1}^{4n}\frac{1}{k} + \cdots \\ &= 1 + 0.5 + 0.583 + 0.634 + 0.663 + \cdots + \ln 2 + \ln 2 + \cdots. \end{aligned} \tag{4}$$

The alternating harmonic series is defined as follows, and the sum is ln2 as proven by Leonhard Euler[2].

$$\begin{aligned} S &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k} \\ &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots = \ln 2. \end{aligned} \tag{5}$$

Bernhard Riemann proved that rearranging the order of this sequence S changes the result. This is called the Riemann series theorem or the Riemann rearrangement theorem, i.e.,

$$1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{5} - \tfrac{1}{10} - \tfrac{1}{12} + \cdots = \tfrac{1}{2}\ln 2. \tag{6}$$

By using the Σ notation, it is shown as

$$\begin{aligned} &\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2(2k-1)} - \frac{1}{4k}\right) \\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2(2k-1)} - \frac{1}{4k}\right), \quad \because \frac{1}{2k-1} - \frac{1}{2(2k-1)} = \frac{1}{2(2k-1)} \\ &= \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{10} - \tfrac{1}{12} + \cdots + \tfrac{1}{2(2n-1)} - \tfrac{1}{2(2n)} + \cdots \\ &= \tfrac{1}{2}\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots + \tfrac{1}{2n-1} - \tfrac{1}{2n} + \cdots\right) = \tfrac{1}{2}\ln 2\,. \end{aligned} \tag{7}$$

This is called the conditional convergence of the alternating harmonic series.

However, this is only a half correct, because, fundamentally, rearranging the order of a series does not change the sum.

Strictly speaking, the Riemann sequences of the above, $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2k-1}, \frac{1}{2(2k-1)}, \frac{1}{4k}\right)$ and $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2(2k-1)}, \frac{1}{4k}\right)$ are not the same sequence, because the former has three terms in a set in turn, but the latter is just another expression of the alternating harmonic series, i.e.,

---

2 Euler presented the value in his paper "1 – mx + m(m+n)x^2 – etc", 1788, p. 45 [3]

$$\sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k} = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k}\right). \tag{8}$$

Riemann ignored the third term $\left(-\frac{1}{4k}\right)$ of the equation (7). The correct relationship between the Riemann sequence and the alternating harmonic series is consisted with adding $\frac{1}{2}\left(\sum_{k=n}^{2n-1} \frac{1}{k+1}\right)$ from the first term to n^{th} term to eliminate the third term of the equation (7) as follows.

$$\begin{aligned} &\sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2(2k-1)} - \frac{1}{4k}\right) + \frac{1}{2}\sum_{k=n}^{2n-1} \frac{1}{k+1} \\ &\qquad = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k}\right). \end{aligned} \tag{9}$$

Therefore, the Riemann rearrangement theorem is inconsistent, but misleads us to the so-called conditional convergence.

It is clear that we see from the above equation (9), we can compute as follows,

$$\begin{aligned} &\left(1-\frac{1}{2}-\frac{1}{4}\right) + \frac{1}{4} = 1 - \frac{1}{2}, \quad \text{if } n = 1, \\ &\left(1-\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right) + \frac{1}{6} + \frac{1}{8} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}, \quad \text{if } n = 2, \\ &\left(1-\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right) + \left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right) + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} \\ &\qquad = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}, \quad \text{if } n = 3, \\ &\vdots \\ &\left(1-\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right) + \left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2(2n-1)} - \frac{1}{4n}\right) + \cdots \\ &\qquad + \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} + \cdots\right) \\ &\quad = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots + \frac{1}{2n-1} - \frac{1}{2n} + \cdots = \ln 2 \end{aligned} \tag{10}$$

If we look at the additional term $\frac{1}{2}\left(\sum_{k=n}^{2n-1} \frac{1}{k+1}\right)$ to the Riemann sequence, it results the alternating harmonic series. We can identify that the term $\sum_{k=n}^{2n-1}\left(\frac{1}{k+1}\right)$ apparently converges between $\frac{1}{2}$ and 1 if n is bigger enough or tends to infinity,

$$\frac{1}{2} \leq \sum_{k=n}^{2n-1} \frac{1}{k+1} < 1. \tag{11}$$

Riemann ignored this value of (11), but he happened to get the right result of the conditional convergence of $\frac{1}{2}\ln 2$. We can see the below if n tends to infinity from the equation (9),

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2(2k-1)}-\frac{1}{4k}\right)+\frac{1}{2}\lim_{n\to\infty}\sum_{k=n}^{2n-1}\frac{1}{k+1}$$
$$=\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots \tag{12}$$
$$=\ln 2$$

If we subtract $\frac{1}{2}\left(\sum_{k=n}^{2n-1}\frac{1}{k+1}\right)$ from both sides of the above equations (9) or (12), we have from the equation (7)

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2(2k-1)}-\frac{1}{4k}\right)$$
$$=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)-\frac{1}{2}\lim_{n\to\infty}\sum_{k=n}^{2n-1}\frac{1}{k+1} \tag{13}$$
$$=\frac{1}{2}\ln 2.$$

From (13) and the equation (12), we can find that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)=\lim_{n\to\infty}\sum_{k=n}^{2n-1}\frac{1}{k+1}=\ln 2. \tag{14}$$

If we rewrite the above (14) as a general term, we have

$$\lim_{n\to\infty}\sum_{k=n}^{2n-1}\frac{c}{ak+b}=\frac{c}{a}\ln 2, \qquad a\neq 0. \tag{15}$$

where a, b and c are natural numbers.

Another example of a similar rearranged alternating harmonic series is given as follows [5] [6]

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3}{2}\ln 2\ . \tag{16}$$

This can be rewritten by using summation notation Σ,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{4k-3}+\frac{1}{4k-1}-\frac{1}{2k}\right)-\lim_{n\to\infty}\sum_{k=n}^{2n-1}\frac{1}{2k+1}$$
$$=\left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right)+\cdots \tag{17}$$
$$-\left(\frac{1}{2n+1}+\frac{1}{2n+3}+\cdots+\frac{1}{4n-3}+\frac{1}{4n-1}+\cdots\right)$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \cdots - \frac{1}{2n} + \cdots = \ln 2$$

In the above series, compared to n = 1 of the alternating harmonic series, $\frac{1}{3}$ in the first parenthesis is the sequence to be eliminated. And when n = 2, $\frac{1}{5}$ and $\frac{1}{7}$ are to be removed. Thus, the sequence of the above (17) represents the series (16) in general.

And, we have from (15) and (17),

$$\lim_{n\to\infty} \sum_{k=n}^{2n-1} \frac{1}{2k+1} = \lim_{n\to\infty} \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \frac{1}{2n+5} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1}\right) = \frac{1}{2}\ln 2. \qquad (18)$$

As a result, the Riemann rearrangement theorem, which states that when the sum of a certain series converges, the sum of the new series made by rearranging the order of the series is different from the sum of the original series, is incorrect.

By changing the order of the alternating harmonic series, the newly added summation term $\lim_{n\to\infty} \sum_{k=n}^{2n-1} \frac{c}{ak+b}$ for the rearranged series provides with $\frac{c}{a}\ln 2$, if n is a sufficiently large number.

References

[1] Riemann Series theorem, URL: https://en.wikipedia.org/wiki/Riemann_series_theorem, Access Date: Apr. 25, 2021

[2] Harmonic Series (Mathematics), URL: https://en.wikipedia.org/wiki/, Access Date : Apr. 25, 2021

[3] Euler L., « De transformatione seriei divergentis 1 – mx + m(m+n)x2 – m(m+n)(m+2n)x3 + etc. in fractionem continuam », URL : https://scholarlycommons.pacific.edu/euler-works/616/ Access Date : Apr. 25, 2021

[4] « Abel's theorem, URL : https://en.wikipedia.org/wiki/ Access Date: Apr. 25, 2021

[5] "Absolute convergence", URL: https://en.wikipedia.org/wiki/ Access Date : Apr. 25, 2021

[6] «Compute 1+1/3-1/2+1/5+1/7-1/4+1/9+.... », URL : https://math.stackexchange.com/questions/2967089/ Access Date: Apr. 25, 2021