The non-uniqueness of the set of natural numbers

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Abstract: Some counterexamples to the uniqueness of the set of natural numbers were given and the non-uniqueness of the set of natural numbers was proved on the basis of strict definition of the number of elements of any set. It was proved that the number of elements in any infinite set is more than that in its proper subset and any one-to-one correspondence cannot be established between two sets with different number of elements according to the definition of bijection. As a result, an infinite set cannot correspond one-to-one to its proper subset. So, there is no infinite hotel paradox and Galilean paradox. The set of natural numbers that corresponds one-to-one to the rational numbers set Q is not the proper set of Q, but another set of natural numbers. The part does not equal the whole. The number of digits after the decimal point for infinite decimals may also be different for infinite decimals. When the number is the same, there is no one-to-one correspondence between different dimensional spaces and there is no one-to-one correspondence between different length segments. The real numbers are countable. There is no uncountable set. Almost all of Cantor's counterintuitive theories are wrong in their logic, so the conflict between intuition and logic is not normal and intuitionism and logicism must be united. It is possible to reach limit but the infinite process can never be completed. There is no Zeno paradox. The fact that there are so many errors in the most rigorous mathematics and that they remain uncorrected for so long shows that human logical thinking ability needs to be greatly improved and no one can be sure that he is correct. Therefore, it is foolish to impose one's own views, including ideology and values, on others, even by force, to push mankind towards self-destruction in nuclear and biological wars.

Keywords: set of natural numbers; the number of elements of infinite set; one-to-one correspondence; Galilean paradox; infinite hotel paradox; Zeno paradox; uncountable set; intuitionism and logicism

0 introduction

The existence of a unique set of natural numbers is the basic belief of set theory, and also the root of almost all counterintuitive errors and paradoxes. Once the non-uniqueness of the set of natural numbers is recognized, all these errors and paradoxes are eliminated.

When I say that the set of natural numbers is not unique, most people will be confused and even opposed except my friends who are familiar with my ideas. Indeed, from a static point of view, if the set of natural numbers is regarded as an infinite set that has been completed and will no longer change, it is easy to prove that the set of natural numbers is unique: if there are two sets of natural numbers that will no longer change, then the elements can only be natural numbers and will no longer change, so their extensions must be equal, and of course they are the same set.

In mathematics, the set of natural numbers is usually defined as a set that already contains all natural numbers. This definition seems also clear enough to draw the conclusion that the set of natural numbers is unique: since all natural numbers are included, of course, these natural numbers can't be different. How can there be different sets of natural numbers?

Everything seems clear.

However, is it really that simple?

1 Is the set of natural numbers unique？

Mathematics is rigorous and cannot introduce any assumptions that may be unreliable.

Unfortunately, human thinking is far from rigorous enough to ensure that unreliable assumptions are not introduced. On the contrary, in many deductions, people often unconsciously introduce some specious assumptions, which leads to the unreliability of the conclusions.

Take the above deduction as example. There is actually an unproven assumption that there is a complete set of unchanging natural numbers. For the purpose of discussion, this hypothesis is called natural number set's completion hypothesis, which is called completion hypothesis for short.

It is not entirely correct to say that this hypothesis has not been proved. According to the viewpoint of real infinity, the infinite process can be completed, so as an infinite set, the formation of the set of natural numbers can also be completed. So the completion hypothesis can be proved by the viewpoint of real infinity.

The problem is that the idea of real infinity itself is a hypothesis that no one can strictly prove, and there are many counterexamples.

For example, no matter according to human mathematical practice since ancient times, or according to Peano's axiom or infinite axiom, natural numbers can be continuously increased by adding 1, that is, any natural number has successor.

Some people may think that the reason why we can't list all natural numbers is because time and space are limited. It is implied that as long as time and space are infinite, we can list all natural numbers. This argument is illogical: even if people are given infinite time and space, the process of listing natural numbers will never be completed. This is because once this process is completed, it means that there are no successors.

There should be no counterexample for any universally applicable viewpoint. Therefore, the view of real infinity is not universally correct, and the completion hypothesis can not be proved.

On the other hand, any mathematical definition must first ensure the existence of the object it defines.

For example, there is no regular decahedron in Euclidean space, so it is meaningless to define regular decahedron in Euclidean space. If we have to define it, it may form another non-Euclidean geometry that may have no practical significance.

Likewise, the definition of the set of natural numbers must be based on the existence of the set.

For example, if N is defined as a set that already contains all natural numbers, then the existence of the concept of "all natural numbers" must first be guaranteed.

In mathematics, "all natural numbers" usually have two meanings. First, we can't exclude any kind of natural numbers. For example, there are no odd numbers in an even set, so although any even number is a natural number, we can't call an even set a natural number set because it excludes the odd numbers. From this perspective alone, the concept of "all natural numbers" is of course valid. However, another meaning is that we can't find any natural numbers outside of "all natural numbers". This concept is problematic. As mentioned above, natural numbers can be increased by adding successors. Unless this process can be terminated, that is, there are no successors, it is possible to form "all natural numbers". However, this is impossible, so it will never be possible to form "all natural numbers".

Using the axiom of infinity or Peano's axiom only proves that any natural number can be formed from 0 by means of +1 continually, but it does not prove that the +1 process can be ended to form all natural numbers. It should be noted here that “any" and "all" are not necessarily the same: the fact that “any" natural number can be formed by +1 continuously does not mean that the process of formation can be ended to form "all" natural numbers: as long as the process is still in progress, new natural numbers are constantly produced and "all" natural numbers can never be formed.

Mathematical logic's confusion of the obviously different "any" and "all" is the root cause of a large number of errors in set theory. "any" can be used to describe any mathematical object in the process of growing, for example, to describe any natural number in the sequence of natural numbers formed by adding 1, but "all" can only describe the mathematical objects after the process of growth has ended, such as the natural number in so-called “all natural numbers”. If the process of adding 1 can never be finished, there can never be the “all natural numbers”.

Of course, it is normal to produce a lot of paradoxes from this rough logic's rule, such as the vase paradox.

2 Some counter examples to the uniqueness of the set of natural numbers

Any theory, if it is correct, will absolutely not produce any contradiction in the concrete application, otherwise it directly proves that the theory is wrong. For example, if the uniqueness of the set of natural numbers leads to any error or absurdity in its application, it proves that the set of natural numbers is not unique.

For example, when we use the natural number set, the numbers often have specific units. Under this circumstance, the set of natural numbers corresponding to different units is usually different. For example, if there are infinite dollars, expressed by a set of natural numbers, it can be expressed as {1 cent, 2 cents, 3 cents ...} or {$1, $2, $3 ... } . If the set of natural numbers is unique, it will lead to the absurd conclusion that 1 cent equals 1 dollar.

Another non unique example of natural number set is an infinite school composed of two infinite classes: suppose that for every student admitted to Class A, Class B must admit two students, and the enrollment of the two classes will never stop.

The result of arranging students in a class is called to be the class number, so if they are also arranged in schools, there are also school numbers. The class numbers of Class *A* and Class *B* and school numbers can be expressed as three sets of natural numbers. Obviously, Class *B* has twice as many people as class *A*, and the school has three times as many people as class A, therefore， the three sets of natural numbers cannot be the same set, otherwise, we won't be able to distinguish between class A, class B and the school.

Not only is it enough to overthrow the proposition that the set of natural numbers is unique with any counter example, but it can also be strictly proved that the set of natural numbers is not unique by deduction.

3 The formation of the set of natural numbers and the number of its elements

3.1 The formation of the set of natural numbers

Science is nothing more than a system of concepts to describe facts.

For example, when we want to define and study the set of natural numbers, we must first study how the set of natural numbers is formed. If we skip this crucial step and go straight to the set of natural numbers, it is possible that the set being studied may never be formed, and therefore certainly does not exist. It is scientifically meaningless to study something that does not exist.

It is an obvious fact that any natural number is obtained by starting from 1 and adding some ones. As an element of the set of natural numbers is certainly no exception. It is not difficult to see that as soon as this process to obtain the elements stops, it can only stop at a certain natural number n^*, and then can only form a finite set of elements with the number n^*. Therefore, in order to form an infinite set containing infinite elements, the above process can never stop, and the number of elements of the set must also be greater than any given natural number n^*.

That is to say, only when the process of forming natural numbers by the +1 method may never be stopped, can an infinite set of natural numbers whose number of elements is greater than any pre-specified n^* be formed.

From this we can see that the above facts can only be described by establishing the following theory:

(1) The infinite set of natural numbers is the set of natural numbers whose number of elements is greater than any given n^*;

(2) The process of forming the infinite set of natural numbers can never be stopped, that is, it can never be completed, so there is only the infinite set of natural numbers in the process of being formed, and no set of natural numbers that has been completed;

(3) The set of infinite natural numbers in the process of formation can be different when it is in different stages of formation, for example, when the number of elements of the set of natural numbers A is just greater than some n^*, the number of elements of the set of natural numbers B may already be greater than $2n^*$. Obviously, A and B cannot be the same set. Because no set of natural numbers that has been completed, therefore, there cannot be a unique set that already contains all the natural numbers.

In order to ensure the rigor of derivation, the concept of the number of elements should be defined first.

3.2 **The number of elements of the set**

Definition 1 A variable whose value starts from 1 and keep on increasing by +1 is called an increasing natural number variable. If there is no upper bound for this variable, the variable is called an unbounded increasing natural number variable or unbounded increasing variable for short.

Property of Definition 1:

Property 1: The value of unbounded increasing variable can be greater than any given natural number n^*.

Proof: If the values of unbounded increasing variable always$\leqslant n^*$, then there is an upper bound n^*+1, contradicting its definition.

Definition 2 An unbounded increasing variable with a value greater than any given natural number n^* is called an infinite variable and is represented by n_∞ (or simply by∞).

3.3 Differences and similarities of symbol ∞ between mathematical analysis and this paper

It can be seen by Definition 2 that Infinite variables are not different from the definition of the abnormal limit Lim $_{n\to\infty}n=\infty$ for sequence $\{n\}$: 1,2,3… in mathematical analysis because the latter actuarily also only states that Lim $_{n\to\infty}n=\infty$ is also greater than any given natural number n^*. This is why the variable is referred to in this paper as an infinite variable and is represented by ∞.

This concept clearly guarantees that any infinite phenomena related to natural numbers can be described in terms of infinite variable.

But in this paper, the precise mathematical meaning of ∞ is given: an arbitrarily large value of an unbounded increment variable, so that it can be studied more scientifically.

For example， although the value of ∞ can be greater than any given natural number n^*, this follows from the definition of the variable (see the proof of Property 1) and does not change the fact that ∞ is a variable only: variables whose values without an upper bound are also variables. Since ∞ is simply a variable, according to Aristotle's deductive syllogism, formulas that can be applied to any variable x, such as $x+1>x$, $x+x=2x>x$, and $x/2<x$, $2^x>x$ and so on can also be applied to ∞. That is, $\infty+1>\infty$, $\infty+\infty=2\infty>\infty$, $\infty>\infty/2$, $2^\infty>\infty$ and so on are also true. Then， we have

Properties of Definition 2:

Property 1 Infinite variable ∞ can participate in any kind of mathematical operation just like any other mathematical variable.

Since infinite variable ∞ is nothing more than a variable taking on natural numbers, therefore

Property 2: Infinite variable ∞ can be studied by mathematical induction.

Properties 1 and 2 of Definition 2 build an ingenious bridge between finite natural numbers and ∞, and the infinite problems that have puzzled mankind for thousands of years will become very simpler: The so-called infinite is only a finite value that is always increasing without end.

Obviously, these properties are very important: before this paper, no one knows these properties of symbol ∞, its application is also very confusing.

However, it should be noted here that in mathematical analysis, any abnormal limit of any divergent sequence is represented by ∞. This representation is too rough to find the distinguish between different divergent sequences. But in this paper, ∞ refers to the abnormal limit of the sequence $\{n\}$ only, and infinity in other cases is often represented by $f(\infty)$ according to Properties 1,2, so as to clarify the relationship between the various infinities.

Definition 3: For the sequence $\{n\}=\{1,2,3....n-1,n,n+1.....\}$ and $\{m\}=\{1,2,3....m-1, m,m+1.....\}$, when there is a function, called relational function, $m=f(n)$ so that for any $n > n^*$, $m=f(n)>f(n^*)$, $\{m\}$ is called to be an associative infinite sequence of $\{n\}$.

Theorem 1: The abnormal limit of sequence $\{m\}$ in Definition 3 is $f(\infty)$.

Proof：Apply Definition 2 to infinity variables n and m respectively, we have

$\operatorname{Lim}_{n\to\infty} n=\infty$ and $\operatorname{Lim}_{m\to\infty} m=\infty_m$,

Since $m=f(n)$ and for any $n > n^*$, $m=f(n)>f(n^*)$, i.e., when $m\to\infty$, $n\to\infty$ also, so $\infty_m =\operatorname{Lim}_{m\to\infty} m= \operatorname{Lim}_{n\to\infty}f(n)= f(\infty)$. The last "=" holds true according to Property 1 of Definition 2. □

Definition 3 and Theorem 1 studies various infinite relations quantitatively and strictly, which is also of great significance for the study of infinite problems in mathematics.

For example, if the student number of class *A* in the previous examples represented by $\{n\}=\{1,2,3....n-1,n,n+1.....\}$ and the student number of class *B* is represented by $\{m\}=\{1,2,3....2n-1,2n,2n+1.....\}$ according to the admission rules. Since there is a function $m=2n$ so that for any $n > n^*$, $m>2n^*$, it can be known from **Definition 3** that $\{m\}$ is an associative infinite sequence of $\{n\}$ with an abnormal limit of $\infty_m=2\infty_n$ or obtain it directly from

$$\operatorname{Lim}_{n\to\infty} 2n=2\operatorname{Lim}_{n\to\infty} n =2\infty$$

For another example, if $\{n\}$ is used to represent the number of decimals after decimal point and $\{m\}$ is the number of decimals, then for binary system, there is a function $m=2^n$ so that for any $n > n^*$, $m>2^{n^*}$, it can be known from **Definition 3** that $\{m\}$ is an associative infinite sequence of $\{n\}$ with an abnormal limit of $\infty_m=2^\infty$ or obtain it directly from

$$\operatorname{Lim}_{n\to\infty}(m)= \operatorname{Lim}_{n\to\infty}(2^n)=2^\infty$$

Note also here that $m=f(n)$ does not necessarily determine the value of every element in $\{m\}$, but it does determine the relationship between ∞ and ∞_m.

Obviously, since the method can reveal the relationship between different infinities, it is possible to study the relationship between different infinities in more detail than mathematical analysis.

The infinite variable can be used to represent the number of elements in an infinite set very clearly.

Definition 4 The result of counting elements is called the number of elements.

Because the counting process itself is an addition process of "adding 1 to the counting result for each counted element":

counting result =1+1+1……

When we count the number of elements, all we get is an increasing variable. In this case, there are only two possibilities:

1) If the counting process can end, then the increasing variable can end at a finite natural number n^*,

counting result =1+1+1……= n^*

and obviously the set being counted is a finite set;

Properties of Definition 4:

Property 1: The number of elements of a finite set can be represented by a natural number n^*.

2) If the set being counted is not a finite set, then the counting process can never end, and only infinite variable whose value is larger than any given n^* can be obtained. Therefore, we have:

counting result =1+1+1……=∞

Property 2: The number of elements of a set can also be an infinite variable.

Definition 5: A set whose number of elements is an infinite variable is called infinity set.

Since there is no third possibility, for infinite sets, we can never get a fixed infinite value, but only an increasing variable, once the process of counting has ended, it can only end at a finite number of natural numbers, so there cannot be a fixed infinite value.

Theorem 2: There is no fixed infinite number to describe the number of elements of the set of natural numbers.

Proof: According to definition 5 and properties 1 and 2 of definition 4, the number of elements of an infinite set of natural numbers is ∞ and the number of elements of a finite set of natural numbers is n^* and there is no third kind of the set of natural numbers, so, there is no fixed infinite numbers to describe the number of elements of any set of natural numbers. □

The mathematical significance of Theorem 2 is obvious: Since here is no fixed infinite number to describe the number of elements of the set of natural numbers, there is also no infinite set with constant extension. All theories based on the fixed infinite numbers, such as the super finite number theory, are wrong.

The misconception that there are fixed infinite numbers fundamentally confuses the distinction between finite and infinite sets because only for finite set, its number of elements can be a fixed number.

Science is only used to study and describe facts, and any imagination that conflicts with facts is wrong and must be abandoned. On the contrary, to replace facts with subjective wishes and imagination, and to use non-existent things as the starting point of theories is completely cutting the foot to fit the shoe and putting the cart before the horse.

In fact, in mathematics, the recognition of variables and various treatments of variables are already a very simple and mature matter, and the extension of set theory rejects variables and remains in the primitive stage of constant mathematics, which is really ironic.

According to Property 1 of Definition 2 and Definition 5, we have the property of Definition 5:

Property: The number of elements of infinite set is additive.

For example, for sets infinite sets *A* and *B*, if their intersection set is empty, $\infty_{A\cup B}=\infty_A+\infty_B$

Although the concept of the number of elements of an infinite set is in fact widely but imprecisely applied, this paper defines the concept strictly for the first time and gives its properties.

Note that the cardinality concept of Cantor, which are usually not additive like ∞. For example, for the number of elements, $\infty+\infty=2\infty$, but for cardinals $a+a=a$.

In addition, although we can't give a fixed value of the number of elements in any infinite set, we may give a fixed relative value according to the Properties. For example, if $N_0 = \{0\} \cup N$, according to the Property of Definition 5, when there are no 0 elements in *N*, N_0 has just one more element than *N*. That is, $\infty_{N0}=1+\infty_N$

The result can be understood as follows: *N* is a proper subset of N_0, so every element that belongs to *N* also belongs to N_0, so although the number of elements of *N* is increasing, the number of elements of N_0 is increasing simultaneously. But the element 0 will always belong to N_0 , not to *N*, that is, N_0 will always have one more element 0 than *N*.

Note that this fact, although very simple and clear, is very important, which can be expressed as:

Theorem 3 The number of elements in any infinite set is more than its any proper subset.

Proof: For any infinite set *A*, if it contains a proper subset *B*, then $A=B \cup (A-B)$. According to Property 4 of definite 3, $\infty_A=\infty_B+\infty_{A-B}$ because the intersection of *B* and (*A*-*B*) is Φ. Since $A-B\neq\Phi$, that is, $\infty_{A-B}>0$, thus $\infty_A=\infty_B+\infty_{A-B}>\infty_B$.□

Theorem 4 Any one-to-one correspondence cannot be established between two sets with different numbers of elements.

Proof: Let the number of elements of the set *A* and *B* be different and without loss of generality, let $\infty_A>\infty_B$, then there are always elements $\infty_A-\infty_A$ in *A*, there is no preimage in *B*, that is, the injective between *A* and *B* is not surjective.□

For example, there are a set of rooms and a set of passengers with the same number of elements. if there's a new passenger, according to Property of Definition 5, the set of passengers has one more element than that of rooms, according to Theorem 4, there is no one-to-one correspondence between the two set. As a result,

Corollary 1 There is no infinite hotel paradox.

The comparison of Theorem3 and Theorem 4 clearly leads to Theorem 5:

Theorem 5 An infinite set cannot correspond one-to-one to its proper subset.

In the same way,

Corollary 2 The set of natural numbers cannot correspond one-to-one to its even proper subset.

Corollary 3 Rational numbers do not correspond one-to-one to its proper set of natural numbers.

However, although the above derivation is very rigorous and clear and does not have any question, it is clearly in conflict with existing theories. For example, Cantor did establish a one-to-one correspondence between the set of rational numbers and the set of natural numbers, and he seems also to have established a one-to-one correspondence between above N_0 and N.

Where is the problem?

4 The theoretical basis of the non-uniqueness of the set of natural numbers

With the exact concept of the number of elements, it is easy to prove the non-uniqueness of the set of natural numbers strictly.

Theorem 6 The set of natural numbers is not unique.

Proof 1: Assuming that the set N = {1, 2, 3,...} is unique, that is, there is no other set of natural numbers, add 1 to each element of N to get the set T = {2, 3, 4 ...}. Its elements can strictly one-to-one correspond to N, that is, $\infty_N=\infty_T$, then N^*= {1} U {2,3,4 ...} = {1, 2, 3... } is also a set of natural numbers, but according to the Property of Definition 5, $\infty_{N*}=\infty_N+1$, that is, the extension of $N^*$is different from that of N, so set of natural numbers $N^*\neq N$, which contradicts the assumption that N is the unique set of natural numbers. □

Proof 2: Assume set N={1,2,3 ... } is unique. Let $A_1=\{y|y=2x, x\in N\}$={2,4,6,....}, $A_2=\{y|y=2x-1|x\in N\}$={1,3,5,...}. Since both A_1 and A_2 one-to-one correspond strictly to N, i.e. according to Theorem 4, $\infty_{A1}=\infty_{A2}=\infty N$,. Therefore, according to Property of Definition 5, the set of natural numbers $A_3 = A_1 \text{U} A_2$ has twice the number of elements of N, is not the set N, and contradicts the assumption that N is the unique set of natural numbers. □

Many mathematical problems can become very clear after strictly proving the non-uniqueness of the set of natural numbers.

For example, as what mentioned before, for $N_0=\{0\}\cup N=\{0\}\cup\{1,2,3,...\}=\{0,1,2,3,...\}$, $\infty_{N0}=1+\infty_N$, according to Theorem 5, N_0 cannot correspond one-to-one to its proper subset N, but Cantor established the following so-called one-to-one correspondence of $N\rightarrow N_0$

N_0 :   0, 1, 2, 3,...
          | | | |
N :   1, 2, 3, 4,...

Of course, the so-called one-to-one correspondence above is wrong according to Theorem 4 and 5.

According to Theorem 4, the real one-to-one correspondence should be

N_0 :   0, 1, 2, 3,...
          | | | |
N_1 :   1, 2, 3, 4,...

Here， N_1 is another set of natural numbers with $\infty_{N1}=\infty_{N0}=1+\infty_N$ according to Theorem 4.

Clearly, Cantor's mistake was to mistake N_1 for N, so the fact that N_1 corresponds to N_0 is mistaken for the idea that N_0 corresponds to its proper subset N. The root cause of his error was his failure to recognize the non-uniqueness of the set of natural numbers: although both N and N_1 can be written as {1,2,3... }, are sets of natural numbers, but they can be different sets of natural numbers with different numbers of elements.

Here are some of the more important applications.

5 The Applications of the non-uniqueness of the set of natural numbers

5.1 The complete digestion of Galileo's paradox

Galileo's paradox, although it has a history of nearly 400 years, has had a great influence on the history of mathematics, which refers to the number of even numbers in the natural numbers, of course, only half of the natural numbers, but even numbers can establish a one-to-one correspondence with the natural numbers (see set A_1), which shows that the number of even numbers and natural numbers N are exactly the same, and then the paradox appeared: $\infty/2=\infty$?

In order to "eliminate" this paradox, Cantor had to avoid the intuitive and accurate concept of the number of elements and establish the so-called cardinal number theory, openly confusing half of the elements with all of the elements，that is, they have the same cardinal number！

It seems that when I really can't tell the difference between ∞ and $\infty/2$, I say $\infty=\infty/2$, and then I don't have to say the difference between ∞ and $\infty/2$.

Is this scientific research? Or just a joke?

Galileo paradox can very easily be explained by the non-uniqueness of the set of natural numbers: since the set of natural numbers is not unique, the set of even numbers can certainly not be unique. Therefore, if let E denote the even proper subset of N, $E\neq A_1$ is entirely possible. On the other hand, $\infty_E\neq\infty_{A1}$ is perfectly normal when $E\neq A_1$. so:

There is no Galilean paradox.

Mistaking A_1 for the proper subset of N is the root of errors.

A_1 is not the proper subset of N: A_1 and N have the same number of elements, and a proper subset of N must have fewer elements than that of N according to Theorem 3, so how can A_1 be a proper subset of N? This is the crux of the problem, and neither Galileo nor Hilbert nor Cantor got it correctly.

In addition, there is also no one-to-one correspondence between set N and E according to Theorem 5: set E can only form an injection with half the elements in set N, so this injection is not surjective because the other half of elements in set N cannot find preimage in set E. Therefore, the conclusion that "any infinite set can correspond to its proper subset" just is a misinterpretation of Galileo's paradox and is completely wrong, as shown in Theorem 4 and Theorem 5.

5.2 Dispel the fog over *N* and *Q*

Cantor established a one-to-one correspondence between the set of rational numbers Q and the proper set N. This fact also seems puzzling and obviously counterintuitive: every natural number is rational number, but not every rational number is a natural number, and so the number of natural numbers must be much smaller than that of rational numbers. But according to Theorem 4, two sets in strict one-to-one correspondence must have exactly the same number of elements, which is obviously contradictory. Moreover, according to Theorem 5, N is no more than a proper subset of Q, and it is impossible to establish a one-to-one correspondence between them.

Before author, no one had been able to explain exactly what was going on. However, according to the non-uniqueness of the set of natural numbers, this problem is also very easily solved: the set of natural numbers corresponding to Q cannot be its proper subset according to Theorem 5, but can be another set (denoted by N_2) of natural numbers according to the non-uniqueness of the set of natural numbers.

The set of natural numbers that one-to-one corresponds to the rational numbers set Q is not the proper set of Q, but another set of natural numbers N_2.

Cantor's mistake was to mistake N_2 for N, so the fact that N_2 corresponds to Q is mistaken for the idea that Q corresponds to its proper subset N. The root cause of his error was also his failure to recognize the non-uniqueness of the set of natural numbers.

If, on the contrary, we insist that Q can correspond to N one-to-one, we will fall into the counterintuitive paradox that part can be equal to the whole. It's wrong, thus

The part does not equal the whole.

5.3 One-to-one correspondence between different length line segments and between different dimensional spaces

Because the set of natural numbers is not unique, the set formed by the number of decimal places (the number of digits after the decimal point) for infinite decimals that can correspond to the set of natural numbers one-to-one is also not unique.

The number of decimal places for infinite decimals is not unique.

Therefore, it is impossible to discuss the relevant problem precisely without making clear the number of decimal places.

For example, for two concentric circles to correspond one to one, the number of points on the circles must be the same.

The number of points on the unit length is called the linear density. Clearly, the smaller the radius is, the greater the linear density is for the concentric circles. When the ratio of the radius of the two concentric circles is X, the ratio of their linear density is $1/X$ in order to ensure that the number of points on the two circles is the same. Let the number of decimal places of the points on the two concentric circles of A and B are ∞_A, ∞_B respectively, then, for binary decimals, the linear density is equal to 2^{∞_A} and 2^{∞_B} respectively, ratio of the linear density is $X=2^{\infty_A-\infty_B}$. When $X=2$, $\infty_A-\infty_B=1$, that is, the number of decimal places differs by 1.

The problem of concentric circles has been solved in this paper for the first time.

Without talking about the number of decimal places, it is meaningless to discuss one-to-one correspondence.

Cantor simply did not pay attention to these details, so his theory was too loose and careless.

The problem of concentric circles has puzzled mankind since the Middle Ages. Galileo therefore saw no point in discussing infinite problems. Instead of solving the problem, Cantor amplified and generalized the errors, such as arguing that not only could points of different lengths correspond one-to-one with each other, but also points of finite lengths could correspond one-to-one to infinite lengths, and thus, if there were more than one universe, there are as many points on a single nanometer as there are on a line spanning several universes!

According to the analysis in this paper, since there is no mention of decimal places, these are not strict and unmeaning.

Cantor's discussion of the one-to-one correspondence between different dimensional spaces was more loose and more absurd.

Let the number of decimal places of binary system is ∞_A, then the number of decimals on the unit length is 2^{∞_A}, the number of decimals on the unit square is $2^{2^{\infty_A}}$, it can be seen that if the decimal places are same, the number of decimal points in one-dimensional space and two-dimensional space is different, according to Theorem 4, there is no one-to-one correspondence between one dimensional space and two dimensional space. But Cantor did establish a one-to-one correspondence between one dimensional space and two-dimensional space, so what's going on here?

In fact, it is very simple. We just look at the line segment with the number of decimal places ∞_B and let $2^{\infty_B}=2^{2^{\infty_A}}$,i.e., let $\infty_B=2\infty_A$, we can establish a one-to-one correspondence between the points on line segment with ∞_B decimal places and the points on the plane with $\infty_A=0.5\infty_B$ decimal places. In other words, when the number of decimal places in a line segment is twice the number of decimal places in a plane, a one-to-one correspondence between oen-dimensional line segment and two-dimensional plane can be established. This is exactly what Cantor did: he took the odd or even decimal places of the points on the line as that on the plane, and obviously the number of decimal places on the plane is exactly half of that on the line.

It's just that he didn't know anything about the non-uniqueness of number of decimal places, so no one including himself could explain his theory before author. And without mentioning the number of decimal places, these theories make no universal sense.

The above discussion can be summarized as follows.

When the number of decimal places for infinite decimals is the same, there is no one-to-one correspondence between different dimensional spaces and there is no one-to-one correspondence between different length segments.

5.4 Is there are noncountable sets

The introduction of the non-uniqueness of the set of natural numbers not only exposes the logical errors of the above counterintuitive statements, but also exposes the logical errors of other counterintuitive statements.

For example, if we don't know that countable and uncountable sets exist at all, there is no reason to think that the things that exist cannot be listed one-to-one. Therefore, if we have an infinite number of infinite decimals, of course we can list them one-to-one.

a_1:0.$a_{11}a_{12}a_{13}$....
a_2:0.$a_{21}a_{22}a_{23}$.... (1)
a_3:0.$a_{31}a_{32}a_{33}$....
....

...

Where, the subscript j (j = 1,2,3 ... .)of a_{ij} denotes the number of decimal places , subscript i (i = 1,2,3 ... .) denotes the number of decimals themselves.

Since subscripts can only be represented by natural numbers, any set of subscripts can only be a set of natural numbers. But in both finite and infinite cases, the number of decimals is always much greater than the number of decimal places, taking binary system as an example, in the finite case, the number of decimals m is equal to 2^n, here, n is the number of decimal places, in infinite case, both the number of decimal places and that of decimals can be represented as sets of natural numbers, represented by N and N_3 respectively.

According to Theorem 1, we can obtain $\infty_{N3}=2^{\infty_N}$, this means

$$N \neq N_3 \qquad (2)$$

exactly in line with Theorem 6.

However, In Diagonal Argument, Cantor let.

b=0.$b_1b_2b_3$..., (3)

here,

$b_k \neq a_{kk}$, (k=1,2,3,...) (4)

From (3)(4), b seems different from any of the decimals in (1), and thus seems to contradict that (1) has already listed all the decimals.

According to Cantor, contradictions arise from the countability hypothesis. If his deduction had been just the only countability hypothesis all along, it would have been fine.

However, what Cantor did not see was that both the subscripts of a_k and the first subscript in a_{kk} belong to the set N_3, while the second subscript in a_{kk} belongs to the set N, **since all three subscripts are always represented by the same *k*, this means that the eq(4) holds only under the false assumption that $N_3 = N$, that is, $\infty_{N_3}=\infty_N$**

Clearly the introduction another false assumption in the deduction makes it impossible to determine which hypothesis is responsible for the contradiction. So, the Diagonal Argument does not prove that the real numbers are uncountable.

Strict proof by contradiction should have only one assumption, and no other assumption should be introduced.

However, once we admit (2), that is, remove the false assumption of $N=N_3$, we find that the contradiction no longer exists.

In fact, the contradiction only occurs when $N_3 = N$. At this assumption, the row subscripts set and the column subscripts set have exactly the same number of elements, forming an infinite $\infty_N*\infty_N$ square matrix, b is obviously not in this square matrix. However, since the existence of real number b proves that $\infty_{N_3} \geqslant 1+\infty_N$ only, clearly, there actually is no contradiction with (2) because $\infty_{N3}=2^{\infty_N} \geqslant 1+\infty_N$.

Therefore, the so-called contradiction arises only from the artificial false assumption that $N_3 = N$, that is, the Diagonal Argument only proves (2), and has nothing to do with whether N_3 is countable. In fact, N_3 itself is a set of natural numbers, so how can it be uncountable?

Since (1) does not actually cause any contradiction, and N_3 is itself a set of natural numbers, (1) directly proves that

The real numbers are countable.

In fact, many people at home and abroad have doubts about the Diagonal Argument. Imagine, a clear and rigorous theorem, like the Pythagorean theorem, would anyone question it?

If we apply Cantor's theorem to the set of natural numbers, as is well known, the power set $P(N)$ of N can correspond to binary decimals, and therefore $P(N)$ corresponds to the set of natural numbers N_3, so long as it is recognized that there are different sets of natural numbers, neither the Diagonal Argument nor Cantor's theorem proves that the real numbers are uncountable.

Cantor's initial method of proving that the real numbers are uncountable is the closed interval method. The idea is also to assume that the real numbers are countable first and list them one-to-one [see (1)], if one can find a real number

$$\theta \neq a_i (i=1,2,3...) \qquad (5)$$

He thinks he has proved that real numbers are uncountable. To do this, Cantor divided the interval into

$$I_1 \supset I_2 \supset I_3 \supset ... \qquad (6)$$

and try to make (5) holds true.

There are also two different sets of natural numbers: in addition to the N_3 set, which corresponds to the decimal one-to-one[see (1)], there is also a set of natural numbers, which is formed by the subscripts in (6) and noted by N_4. It is clear that (5) holds true only when

$$\infty_{N3} <= \infty_{N4} \qquad (7)$$

This is because the method can only guarantee that the subscript of the common point is not in N_4 only, thus, if $\infty_{N3} > \infty_{N4}$, it cannot guarantee the point not in N_3.

However, Cantor did not prove (7), so the method does not prove that the real numbers are uncountable.

In fact, when we divide the intervals into three parts, the end points of the intervals are rational numbers only, so $\infty_{N4} < \infty_{N3}$.

On the basis of proving that the set of natural numbers is not unique, it is not difficult to prove that

Theorem 7 There is no uncountable set.

Proof: Every element in any infinite set is distinct from each other, therefore, each of these elements can certainly have an increasing numeric subscript, and the subscripts can certainly form a set of natural numbers. The proof of Cantor's theorem shows that there are infinite sets that do not correspond one-to-one with each other, and therefore there must be relevant sets of natural numbers that do not correspond one-to-one with each other, which, according to Theorem 6, does not lead to any contradiction (The so-called contradiction, as mentioned earlier, encountered by Cantor in his Diagonal Argument is only due to his carelessness in confusing the row and column subscripts, the root cause is the confusion of different sets of natural numbers. The contradiction in fact do not exist), that is, there is no uncountable set. □

It should be noted that the proof citers only to the proof of Cantor's theorem, but not to Cantor's theorem itself. This is because the expression of Cantor's theorem itself uses the concept of cardinal numbers, but we do not use the concept of cardinal numbers, which is easy to conflict with the precise and clear concept of the number of elements proposed in this paper. Only the conclusion that there are infinite sets that cannot correspond one by one with each other is needed in this proof.

For example, as mentioned earlier, the set of rational numbers Q corresponds one-to-one to a set of natural numbers (but not the set of natural numbers contained in set Q), and the set of real numbers R corresponds one-to-one to another set of natural numbers N_3[See eq.(1)]. Even if the power set of the power set of N, i.e. $P(P(N))$, can also corresponds one-to-one to a set of natural numbers......

Since there is no uncountable set, therefore, the so-called Continuum Hypothesis and the theory of large cardinal numbers have no meaning.

The history of mathematics has taken a big detour.

6 Discussion

This paper has discovered the non-uniqueness of the natural numbers and therefore explained a lot of problems.

This paper at least discusses five different sets of natural numbers: N, N_1~N_4, all of which can be written as {1,2,3... }, but the number of elements varies. Cantor confused these different sets of natural numbers and produced a large number of paradoxes and errors. For example, he failed to see that N_1 has one more element than N, so he mistakenly assumed that N_0 could correspond to N one by one, and mistakenly concluded that the infinite hotel paradox and the infinite set could correspond to its proper subset one by one, leading to the partial equals all paradox; For example, he did not see the difference between N_2 and N, and thus mistakenly believed that N could correspond to $Q(Q \supseteq N)$, a set of rational numbers much larger than N, and he did not see the difference between N_3 and N, and thus mistakenly believed that the real numbers were uncountable...

But people who like to explore the truth may still wonder, since the elements of the set of natural numbers are natural numbers only, what causes them to be different?

The answer is that the process of forming the sets of natural numbers never ends, so there is no common end point. Since the sets of natural numbers are always in the process of formation, it cannot be guaranteed that the sets of natural numbers in all forms must be the same. That's why the set of natural numbers may be different.

Taking the infinite school as an example again, for every additional student in class A, there are two additional students in class B, and the resulting sets cannot be the same at any time including infinite time.

Therefore, the crux of the problem is the infinite view in this paper that infinity cannot have an end, i.e., cannot be completed.

Whether infinity can be completed is not a problem that can be solved by guessing puzzles or taking sides, but a scientific problem of what the objective facts are.

Pay particular attention to the relationship between the completability of the problem relating to infinity and the in-completability of infinity itself. A typical example is the limit of a convergent sequences: the limit value of the sequences itself is deterministic, but the terms of the sequences are infinite.

As is generally known, the limits of convergent sequences are invariant, for example, when $n\to\infty$, $\mathrm{Lim}(1/n)=0$, where limit zero is a invariant value. Since the sequence limit is a concept related to infinity, the invariance property of the limit is easily misunderstood as the completability of infinity itself. Cantor, for example, who was seriously lacking in rigor because his thinking was too arbitrary and free, understood it in this way, and thus emphasized the idea of the real infinity view that believe that infinity can be completable. In fact, he got it all wrong: the certainty of limits is based on the in-completability of Infinity. Still take the above sequences limit as an example, if the infinite process $n\to\infty$ can be terminated or completed, according to the definition of sequences limit, it can only be terminated or completed at a finite value n^*, then the value obtained is a finite value $1/n^*$ greater than 0, that is, only when infinite process $n\to\infty$ has no end, that is, it cannot be completed, the value of the sequences item can tend to an invariable 0. Or the in-completability of Infinity is a necessary condition for the limit value to be fixed.

Note that the above discussion has nothing to do with time. For example, when $n\to\infty$, it can be specified that n increases by 1 every second, or it can be specified that n increases by 1 every zero second, but regardless of the stipulation, the conclusion that infinity cannot be completed remains unchanged.

Obviously in the former case, it takes an infinite amount of time to reach the limit, and in the latter case, limit zero is reached when time is greater than zero. It follows that the limit is not necessarily unreachable, but the infinity can never be completed.

Another example is 0.3+0.03+0.003+... The limit is 1/3, and the necessary condition for reaching the limit is that the number of terms of addition is infinite, which cannot be terminated or completed: if it can be terminated or completed, then the sum <1/3.

In fact, the endless debate on infinity stems from a failure to correctly understand the in-completability of infinity. For example, if the in-completability of infinity is regarded as the unreachability of limits, various paradoxes such as Zeno's paradox will arise, and denying in-completability of Infinity will result in various paradoxes of set theory, such as those based on the uniqueness of the set of natural numbers discussed in this paper.

In Zeno's paradox, If the relative distance between the fast runner and the turtle is 1/3 meter, the relative speed is 0.3 meters/second and the limit of 1/3 cannot be reached, the fast runner will never catch up with the turtle, which is inconsistent with the fact. The fact is that both fast runners and turtles walk on their feet only and has nothing to do with how the distance is divided in the mind and whether this division requires time, so we can let zero second to divide 1/3 to infinite distances, and of course the limit can be reached. So, the essence of Zeno's paradox is simply to confuse what's actually happening with what's going on in people's mind.

Since the infinite part of the traditional set theory rests on the basis that one can obtain the fixed extension set that already contains all of the natural numbers, the basis is wrong and the result must be wrong. Before the foundation is not solid, it is suggested that the education department temporarily cancel the teaching related to infinite sets, so as not to mislead students.

This paper also finds that almost all of Cantor's counterintuitive theories are wrong in their logic.

In the history of mathematics, there was a debate between intuitionism and logicism. Once, logicism prevailed, and intuition was considered unreliable. Cantor's theory is widely accepted on this philosophical basis. The mathematics teaching material of our country still stays at this level when introducing this period of history.

But according to Zou Xiaohui of Tsinghua University, recently， intuitionism has gradually developed more rapidly.

In fact, with the increasing expansion and refinement of people's observation range， wrong intuition is usually easy to find and correct. On the contrary, some hidden logical errors are often not easy to find and correct. The evidence is the existence of a large number of paradoxes. Therefore, if there is a contradiction difficult to solve between intuition and logic for a long term, most likely the logic is wrong.

In the process of scientific development, the theory that only emphasizes one side can only exist in a certain stage of development, and in the relatively mature natural sciences， we should not have a theory that only emphasizes one side for a long-term.

To my way of thinking, this is the reasons why intuitionism and logicism must be united.

Since people don't have very high logical thinking skills, when logic and intuition conflict, people must first carefully check the errors in logic, and should not fantasize that "logic is reliable and intuition is unreliable" and then let the possible errors continue.

For example, intuition tells us that anything that does exist can be listed one by one, and anything that can be listed one by one can be numbered by natural numbers, that is, any set, no matter how big or small it is， always can correspond one-to-one to one of sets of natural numbers. Since the set of natural numbers is not unique, the above operation should not lead to any contradiction. However, Cantor's careless confusion of different subscripts artificially leads to contradictions that do not exist in the first place, and thus "proves" the existence of uncountable sets.

This paper found that Cantor's thought, although imaginative and creative, seriously lacks rigor and meticulousness: many details and many obvious contradictions were not seen by him. Lack of strict logical thinking ability, but also superstitious logic, is the reason why his theory is full of errors.

The fact that there are so many errors in the most rigorous mathematics and that they remain uncorrected for so long shows that human logical thinking ability needs to be greatly improved and no one can be sure that he is correct. Therefore, it is foolish to impose one's own views, including ideology and values, on others, even by force, to push mankind towards self-destruction in nuclear and biological wars.