Fictitious Currents as a Source of Electromagnetic Field

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Oct. 2023†

Abstract

In this paper we introduce the idea of electric fictitious currents for the electromagnetic field. Electric fictitious currents are currents that arise in electrodynamics when we change the topology of space. We show, with a specific example, how fictitious currents may be the source of magnetic moment of a singularity.

Key Words: gravity, electrodynamics, gauge theory.

1 Introduction

The Standard Model of particle physics is a very successful theory describing three out of the four known forces of nature. Its final formulation relay heavily on the use of gauge fields. In gauge theories the Lagrangian of the system (i.e. its dynamics) does not change under local transformations acting in a simply connected region of space-time.

However in the standard model, particle are point objects with no dimension being, as a matter of fact, space-time singularities with fields around them having sometimes infinite values. For example, the classical version of electric field around an electron goes to infinity as $1/r^2$, and in the Standard Models we start from the classical Lagrangian before quantizing.

In this paper we study what happens to a gauge field when we introduce a singularity in space such that the space is not simply connected any more. The major result is that, depending on the topology of the singularity, fictitious currents may arise as a manifestation of the inertia of the system in changing topology. In the example we studied, dealing only with the $U(1)$ symmetry of the Standard Model, these currents may be seen as a sources for magnetic moment of particles.

In section 2 we derive fictitious currents generated by a singularity in space. The reader, that does not want to go through the math, can find a simplified version of the content of this section in [1].

In sections 3 we show how magnetic moment of a particle can arise as a consequence of fictitious currents.

In the sections 4, 5 and 6 we give additional thoughts and conclusions.

2 Fictitious Currents

We start from the Lagrangian density of the electromagnetic field in units where $\mu_0 = 1$:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J_s^{\nu} A_{\nu} \tag{1}$$

where J_s^{ν} is the source four-vector current, $F_{\mu\nu}$ equal to:

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \tag{2}$$

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†Posted at: https://vixra.org/abs/2310.0031 - Current version: v3 - Dec. 2023

is the electromagnetic tensor and A_μ is the four-potential. Moreover, we know that $F_{\mu\nu}$ is a gauge field and that its Lagrangian is invariant with respect to the symmetry:

$$A_\mu \to A_\mu + \partial_\mu \theta \tag{3}$$

where $\theta(x^\mu)$ is any continuous function in a simply connected space-time region Ω.

In the general case, the Lagrangian density $\mathcal{L}(A, \partial A)$, given by Eq. (1), depends on both the four-vector potential and its derivatives. We will consider now the case of the free electromagnetic field (i.e. $J^\nu = 0$) in which the Lagrangian density $\mathcal{L}(\partial A)$ depends only on the derivatives and it can be written as (see Appendix A.1):

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) \tag{4}$$

Let $\xi_\mu = \partial_\mu \theta$ be the gradient of a continuous function in Ω. In this section we will apply the symmetry $A_\mu \to A_\mu + \xi_\mu$ in order to see what happens to the Lagrangian density.

For the first term in (4) we have;

$$\partial_\mu(A_\nu + \xi_\nu)\partial^\mu(A^\nu + \xi^\nu) = \partial_\mu A_\nu \partial^\mu A^\nu + \partial_\mu A_\nu \partial^\mu \xi^\nu + \partial_\mu \xi_\nu \partial^\mu A^\nu + \overbrace{\partial_\mu \xi_\nu \partial^\mu \xi^\nu}^{\text{not needed}} \tag{5}$$

where the last term is not needed and can be omitted because we are interested in the equation of motion and that term does not depend on A_μ and therefore has not effect on the variation of the action with respect to the fields.

For the second term we have:

$$\partial_\nu(A_\mu + \xi_\mu)\partial^\mu(A^\nu + \xi^\nu) = \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\mu \xi^\nu + \partial_\nu \xi_\mu \partial^\mu A^\nu + \overbrace{\partial_\nu \xi_\mu \partial^\mu \xi^\nu}^{\text{not needed}} \tag{6}$$

where the last term once again can be omitted if we are interested in the equation of motion. Putting the two equation above back together, swapping some terms and rearranging the names of dummy indices of the third term in parenthesis below, we have:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A_\nu \partial^\mu \xi^\nu + \partial^\mu A^\nu \partial_\mu \xi_\nu - \partial_\mu A_\nu \partial^\nu \xi^\mu - \partial^\mu A^\nu \partial_\nu \xi_\mu) \tag{7}$$

Applying the Leibniz rule (i.e. $f'g' = (fg')' - fg''$) to the terms in parenthesis above, we have:

$$\partial_\alpha A_\beta \partial^\gamma \xi^\delta = \overbrace{\partial_\alpha(A_\beta \partial^\gamma \xi^\delta)}^{\text{not needed}} - A_\beta \partial_\alpha \partial^\gamma \xi^\delta \tag{8}$$

where in this case the first term is not needed, if we are interested in the equation of motion, because it is a divergence and therefore it depends only on the value of the tensors on the boundary of Ω and has no effect on the variation of the action with respect to fields. Eq. (7) becomes:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(-A_\nu \partial_\mu \partial^\mu \xi^\nu - A^\nu \partial^\mu \partial_\mu \xi_\nu + A_\nu \partial_\mu \partial^\nu \xi^\mu + A^\nu \partial^\mu \partial_\nu \xi_\mu) \tag{9}$$

If we are in nice flat Minkowski space we can raise and lower indices at will, also on the derivative symbols. We have:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(-A_\nu \partial_\mu \partial^\mu \xi^\nu - A_\nu \partial_\mu \partial^\mu \xi^\nu + A_\nu \partial_\mu \partial^\nu \xi^\mu + A_\nu \partial_\mu \partial^\nu \xi^\mu) \tag{10}$$

and eventually:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\nu \partial_\mu(\partial^\nu \xi^\mu - \partial^\mu \xi^\nu) \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\nu A_\nu = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\nu A^\nu
\end{aligned} \tag{11}$$

In the last equation we have put the currents in both covariant and controvariant form where the controvariant form is given by:

$$J^\nu = \partial_\mu(\partial^\nu \xi^\mu - \partial^\mu \xi^\nu) \tag{12}$$

and the covariant form is given by:

$$J_\nu = \partial^\mu(\partial_\nu \xi_\mu - \partial_\mu \xi_\nu) \tag{13}$$

which allow the J to be expressed in terms of exterior derivatives of differential forms.

$$J = \partial d\xi = \partial d^2\theta \tag{14}$$

Statement: Let A be the potential four-vector relevant to the solution of the Maxwell's Equations in a region Ω of flat space-time, with no sources and with given boundary conditions on $\partial\Omega$. Let $\xi^\mu = \partial^\mu\theta$ the gradient of a smooth function θ in Ω. Then, we have:

$$A^\mu \to A^\mu + \xi^\mu \;\Rightarrow\; \mathcal{L} = \mathcal{L} - J^\nu A_\nu \tag{15}$$

with:

$$J^\nu = \partial_\mu(\partial^\nu \xi^\mu - \partial^\mu \xi^\nu) \tag{16}$$

and therefore, changing the four-vector potential by ξ^μ has the same effect on the electromagnetic field of introducing **Fictitious Currents** (Pseudo Currents) J^ν in Ω with the same boundary condition on $\partial\Omega$.

Moreover, form EQ. (14), if Ω is simply connected, clearly $J^\nu = 0$ since d^2 of any differential form in Ω vanishes.

Note that:

$$\mathcal{L}(\partial A) \neq \mathcal{L}(\partial(A + \xi)) \tag{17}$$

only because we have removed terms from the Lagrangian. Otherwise, the two Lagrangians would have been identical since we are applying a symmetry. However, the way we have removed the terms has left the equation of motion unchanged, and in fact, if Ω is simply connected we have $J^\nu = 0$ and this restores the correctness of the equation of motion and leaves the field unchanged.

However, if Ω is not simply connected, then the currents J^ν may be different from zero and act as sources for the fields, and we have called these currents Fictitious Currents (Pseudo Currents). Note that these are electric fictitious currents not to be confused with the magnetic fictitious currents sometimes used in computational electrodynamics as a trick for solving complex problems.

The reason why we call J^ν fictitious, it is due to an analogy with discrete mechanical systems (see Appendix A.2). For discrete mechanical systems, if we act on a symmetry (e.g. shift in space) while the system is evolving, this will result in fictitious forces. For continuous systems, if we act on a symmetry, this will result in fictitious currents.

The analogy is not perfect though. For mechanical systems fictitious forces appear when we have a change of symmetry during the evolution of the system. For the electromagnetic field, fictitious currents appear when we have a change a of the gauge symmetry fields in conjunction with a specific space topology.

With abuse of terminology, we may say that fictitious currents are due to the inertia of the system in changing space topology.

3 Magnetic Moment of One Half Spin Particles

Now that we have defined fictitious currents, we want to find an example where the theory may be used. Let us consider the Lagrangian density of quantum electrodynamics:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{18}$$

and suppose that we have no particles and the electromagnetic field is zero everywhere (at least from a classical field theory point of view). We know that we have the following symmetry:

$$\psi \to \psi e^{i\theta(x^\mu)} \; ; \;\; A \to A + \partial^\mu \theta \;\Rightarrow\; \Delta\mathcal{L} = 0 \tag{19}$$

where A is the four-vector potential.

Suppose now we hit the field with the creation operator and we create a particle. Before the particle is present, the gauge field θ is a continuous function, and since we can choose, we choose it to be zero everywhere. Suppose finally that creating the particle is equivalent to removing from space a cylinder of radius ϵ and height L, with $\epsilon << L$, with the axis of the cylinder $\overline{AB}$ lying on the z axis, with $A = (0, 0, L/2)$ and $B = (0, 0, -L/2)$. The gauge field θ before was continuous everywhere. Now space is not simply connected any more and the gauge field θ has to adapt to the boundary condition we set on the cylinder.

With abuse of terminology, we will call the cylinder a singularity. This is because the results of our analysis on the solution for the fields will be independent from ϵ and therefore we may take the limit for ϵ going to zero, make the cylinder a line segment where the gauge field is step discontinuous and define our particle to be a proper line segment singularity.

We want to evaluate the currents J_μ close to the cylinder and around the middle point of its axis, far as from the two ends A and B. If we are close enough to the cylinder, we can consider it of infinite height and we end up in a solution having cylindrical symmetry where we can assume the solution for θ to be independent from z (i.e. $\xi_3 = \frac{\partial\theta}{\partial_3} = 0$). We can forget the z axis and go back to a 3-dimensional problem. Although the topology has changed, in our case the discontinuity can be represented just by boundary conditions of the field on the cylinder. We are in flat space, and therefore controvariant and covariant forms of the currents are the same. From Eq. (13) we have:

$$J_\nu = \begin{cases} J_0 &= \partial^1(\partial_0\xi_1 - \partial_1\xi_0) + \partial^2(\partial_0\xi_2 - \partial_2\xi_0) \\ J_1 &= \partial^0(\partial_1\xi_0 - \partial_0\xi_1) + \partial^2(\partial_1\xi_2 - \partial_2\xi_1) \\ J_2 &= \partial^0(\partial_2\xi_0 - \partial_0\xi_2) + \partial^1(\partial_2\xi_1 - \partial_1\xi_2) \\ J_3 &= 0 \end{cases} = \begin{cases} J_t &= \partial_y(\nabla\times\hat{\xi})_x - \partial_x(\nabla\times\hat{\xi})_y \\ J_x &= \partial_t(\nabla\times\hat{\xi})_y - \partial_y(\nabla\times\hat{\xi})_t \\ J_y &= \partial_x(\nabla\times\hat{\xi})_t - \partial_t(\nabla\times\hat{\xi})_x \\ J_z &= 0 \end{cases} \tag{20}$$

where $\hat{\xi}$ is the three-vector (t, x, y) and $(\cdot)_i$ means component along the i axis. From the above equation we conclude easily:

$$\begin{cases} \hat{J} &= \nabla\times\nabla\times\hat{\xi} \\ J_z &= 0 \end{cases} \tag{21}$$

where $\hat{J}$ is the three-vector (J_t, J_x, J_y).

For θ we give the following boundary conditions expressed in cylindrical coordinates (r, ϕ, z):

$$\begin{cases} \lim\limits_{r\to\infty} \theta(r,\phi,z) = \theta_\infty \\ \\ \lim\limits_{r\to\epsilon} \theta(r,\phi,z) = \alpha\phi + \theta_0 \quad \text{for} \quad -\frac{L}{2} < z < \frac{L}{2} \end{cases} \tag{22}$$

where α, θ_0 and θ_∞ are constants α can be a positive or negative number. Note that, for $t > 0$ where $\frac{\partial\xi_i}{\partial t} = 0$, and therefore $\xi_t = \xi_z = 0$, we can replace the z axis with the t axis an vice-versa at any time, since the field is independent from both axes. We can find the solution with respect to the (t, x, y) space and this will be the same solution for the (x, y, z) space. The second condition above is the one that defines the discontinuity and, in words, it means that going around the cylinder, by the time we go around the z axis once, the phase of $\Psi e^{i\theta}$ goes around the circle α times.

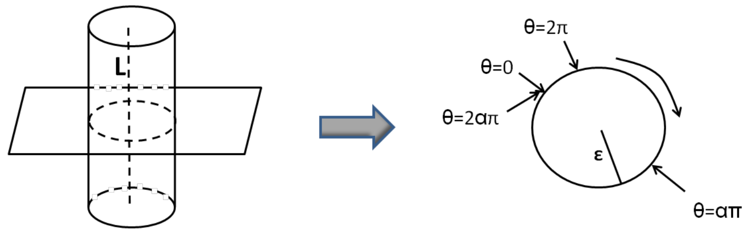

Figure 1: Boundary Conditions in the Singularity

Although we do not need to know θ in space, we want to spend a few words on the way it may look like. We assume that two close points want to have the same phase, which means that there is some energy associated to difference in phase for nearby points. As usual with these problems of finding a minimum for energy in space, we end up with a Laplace equation. We propose the following equation for θ:

$$\nabla^2\theta = 0 \tag{23}$$

that together with the boundary condition allows to evaluate θ in space.

Whatever equation we use for θ, its important to note that, in the (θ, x, y) space, $\theta = \theta(x, y)$ will be a multivalued function shaped like an helicoid and spiralling along the θ axis.

Now we are ready to evaluate the fictitious currents. If we integrate $\hat{\xi}$ on any loop on the (x, y) plane not containing the origin, we get always a vanishing integral because $\nabla \times \nabla\theta = 0$. However, if we integrate $\hat{\xi}$ on any loop on the (x, y) plane containing the cylinder, given Eq. (22) we get $2\pi\alpha$. This is because, given the expression of the gradient in the cylindrical coordinate, we have;

$$\lim_{r\to\epsilon}\hat{\xi} = \lim_{r\to\epsilon}\nabla\theta = \frac{\alpha}{r}\hat{i}_\phi \quad \Rightarrow \quad \hat{\xi} = \frac{\alpha}{r}u(r-\epsilon)\hat{i}_\phi \tag{24}$$

where $u(r)$ is the Heaviside unitary step discontinuous function and we assume vanishing field inside the cylinder. This means we have a discrete curl in the origin. Using Eq. (47) we have:

$$\hat{J} = \nabla\times\nabla\times\hat{\xi} = \nabla\times\nabla\times\left(\frac{\alpha}{r}u(r-\epsilon)\hat{i}_\phi\right) = \frac{\alpha}{\mu_0}\mathring{\delta}_{t,\epsilon}(x, y) \quad [A][m^{-2}] \tag{25}$$

where the definition of $\mathring{\delta}_t$ is given in Appendix A.3. In the above equation we have taken into account that $\hat{\xi}$ has the same units of A^μ ($[kg][m][s^{-2}][A^{-1}]$) and we have put back in the equation μ_0 ($[kg][m][s^{-2}][A^{-2}]$) that was previously set to 1. Moreover, each curl adds an $[m^{-1}]$ units.

The field $\mathring{\delta}_t$ has units of $[m^{-3}]$. The ratio α/μ_0 has units of $[A][m]$. The parameter α is a small integer (e.g. $\alpha = 1$) and therefore the above equation is a new nice interpretation of μ_0 in terms of the currents circulating around our singularity.

The above equation represents a current circulating around z and along the line segment discontinuity. It generates a magnetic moment m per unit length, independent from ϵ and given by Eq. (48). We have:

$$m = \frac{2\pi\alpha}{\mu_0} \quad [A][m] \tag{26}$$

The total magnetic moment M of the line segment discontinuity of length L is therefore:

$$M = mL = \frac{2\pi\alpha L}{\mu_0} \quad [A][m^2] \tag{27}$$

We finish this section by giving an estimation of the length of the discontinuity. For example, if we use the known value of the magnetic moment of the electron[1] μ_e and the value $\alpha = 1$. Given the

[1] $\mu_e = -9.2847647043(28) \times 10^{-24}\ [A][m^2]$

value of the permeability[2] μ_0, we have::

$$L = \frac{\mu_0 |\mu_e|}{2\pi\alpha} = 1.9 \times 10^{-29} \quad [m] \tag{28}$$

4 Charge of One Half Spin Particle

Now we go back to see in more detail what happens during the transition at time $t = 0$, in which we go from a zero gauge to our gauge $\theta(r, \phi, t)$, in cylindrical coordinates, where once again we use the axis t instead of the axis z because the solution is independent from z. The gauge field θ goes from being a continuous field that does not go around the line segment discontinuity, to a field that goes around the discontinuity α times. This is equivalent to have α jumping from zero to its actual value. Given by Eq. (22) we have:

$$\theta(r, \phi, t) = \alpha\phi u(r - d)u(t) \tag{29}$$

where u is the Heaviside unitary step continuous function. From Eq. (21), we evaluate:

$$J_0 = \hat{J}_t = \nabla \times \nabla \times \hat{\xi} = 0 \tag{30}$$

because the field $\hat{\xi}$ lays always on the (x, y) plane. Since J_0 is always vanishing, we conclude that in our geometry there is no generation of charge and therefore it is not possible to use this model to introduce a classical mechanism for the creation of the charge of a particle. Possibly a different geometry should be used.

5 Space Deficiency Model

In [3] we have shown that if we model space as an elastic material, a deficiency in the material (i.e. in space) is equivalent to gravity. This is because, if we remove a ball of material making a hole in it and we identify the boundary of the hole to a point, the material will stretch and the strain field is equivalent to gravitational field. Moreover, two deficiencies in the material will experience an attraction force to each other proportional to $1/r^2$ where r is the distance between the two space deficiencies.

Since space should be conserved, we wonder how a deficiency may be created in space. A possibility is that when a particle is created in a point P, the configuration of space changes from flat Euclidean space to a 3-dimensional manifold attached to the point P by means of a connected sum.

We will illustrate this with a 2-dimensional example.

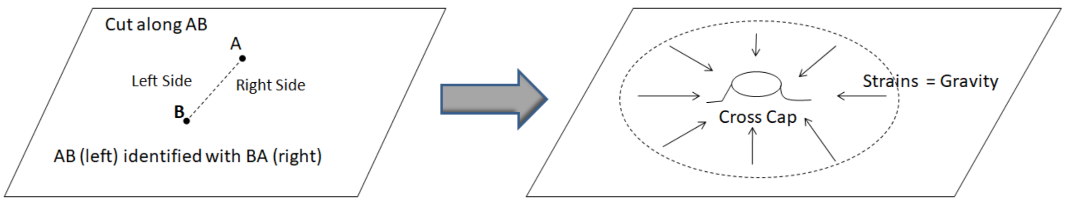

Figure 2: Space Deficiency Model

Given a sheet of elastic material representing space, if we cut along a line segment between two points A and B and we identify the two sides of the cut with opposite orientations, we get a cross cap (i.e. a real projective plane) attached to the sheet by means of a direct sum.

The sheet will pull the cross cap which will shrink till the bending forces inside the cap will balance the pulling forces of the sheet. At the equilibrium, the cross cap will protrude from the sheet and since the quantity of elastic material is conserved in the process, this will be equivalent to a space deficiency.

The sheet will be stretched around the cross cap and the strain field will be equivalent to a gravitation field (see [3]). Note that, if the sheet represents space, fields will change phase when crossing the line segment as in the example we gave in the previous sections.

[2] $\mu_0 = 4\pi \times 10^{-7} \quad [kg][m][s^{-2}][A^{-2}]$

6 Conclusions

In this paper, we have shown that a sufficient condition to have magnetic moment in a point in space (i.e. particle) is to introduce a topological singularity designed to twist the fields in a specific way. Although the paper has been dealing with the $U(1)$ symmetry only, the very same approach may be used to explore what happens with the other two symmetries of the standard Model. We believe that it is worth to further research in order to trying to match particle characteristics to topological characteristics of space singularity (i.e. 3D compact manifolds connected by direct sum to space) with respect to the three symmetries of nature, in an effort to make a one-to-one correspondence between particles and manifolds (or topological singularities).

Although we know that this approach is very unlikely to be a theory that fully describe particles, however, there is also a chance that things may partially match just by mathematical chance, end this would allow to exploit the huge variety of 3D-compact manifold to explain some of the complex characteristics of particles.

We may for example be able to give to some constants of the Standard Model, which now are known by direct measurement, a theoretical derivation. An example that come to mind is the three different families of particles with different mass (e.g. electron, muon and tau). They may just be three different stationary state of the same manifold (i.e. same particle characteristics) like Willmore spheres that are all the same manifold in different stationary state of energy and with different sizes (i.e. mass from a particle point of view).

Another example is the fractional charge of some particles that come in 1/3 and 2/3 the charge of the electron. This may be explained if we find the correct topological singularity to describe them.

A final example may come from dark matter. Maybe dark matter particles are simply manifolds that do not twist the fields (e.g. oriented manifolds) and therefore do not interact to ordinary matter.

Appendix

A.1 Lagrangian Density of Electromagnetic Field

Given the Lagrangian density of the free electromagnetic field:

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{31}$$

we have:

$$\begin{aligned} \mathcal{L} &= \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) && (32) \\ &= \frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + \partial_\mu A_\nu \partial^\mu A^\nu) && (33) \end{aligned}$$

the above terms are equal in pairs, we have:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu) \tag{34}$$

A.2 Fictitious Forces

Given the Lagrangian of a one variable discrete system:

$$L = L(q, \dot{q}) \tag{35}$$

having the following symmetry:

$$q \to q + \phi \;\Rightarrow\; \Delta L = 0 \tag{36}$$

then from the Noether's theorem we know that the quantity $\frac{\partial L}{\partial \dot{q}}$ is conserved.

The above quantity is conserved when we let the system evolve without applying the symmetry to it. However, if we apply the symmetry by changing some symmetry parameter (e.g. for a mechanical system this parameter may be position) while the system is evolving (e.g. for a mechanical system it may correspond to a shift of the whole system in space or to moving the relative position of parts of the system in a way the Lagrangian is not affected), then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$ is not a conserved quantity any more. This physically corresponds to having fictitious forces (pseudo forces) in the system depending from the way we change the symmetry parameter as a function of time.

To address the above case, we need to change the Lagrangian in order to take into account the dependency from symmetry parameters (see [2]). To illustrate that, we will use the same example of [2], where it is shown how to get a new Lagrangian just with a change of coordinates from q_0, the coordinate of the inertial system, to $q = q_0 + \phi$, the coordinate of the non inertial one:

Given the Lagrangian of a particle in an external field:

$$L = \frac{1}{2}m(\dot{x}_0)^2 - U(x_0) \tag{37}$$

where x_0 is the coordinate in the inertial frame and x is the coordinate of the moving frame with velocity $\dot{\phi}$, we consider the change of variable $x_0 = x + \phi(t)$. We have:

$$L'(x+\phi, \dot{x}+\dot{\phi}) = \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{\phi} + \frac{1}{2}m\dot{\phi}^2 - U(x) \tag{38}$$

The term $\frac{1}{2}m\dot{\phi}$ does not depends on x, gives no contribution to $\frac{\partial \mathcal{S}}{\partial q}$ and, if we are interested to the equation of motion, it can be dropped. Regarding the term $m\dot{x}\dot{\phi}$, using the Leibniz rule we have:

$$m\dot{x}\dot{\phi} = m\frac{d}{dt}(x\dot{\phi}) - mx\ddot{\phi} \tag{39}$$

the term $m\frac{d}{dt}(x\dot{\phi})$ is the derivative of a function. Its contribution to the action depends only on its value at the two ends of the time integral. Once again, it gives no contribution to $\frac{\partial \mathcal{S}}{\partial q}$ and it can be dropped. We are left with the following Lagrangian:

$$L' = \frac{1}{2}m\dot{x}^2 - mx\ddot{\phi} - U(x) \tag{40}$$

which, by using the Euler-Lagrange equation, gives the following equation of motion:

$$m\ddot{x} = -\frac{\partial U}{\partial x} - m\ddot{\phi} \tag{41}$$

The term $m\ddot{\phi}$ is what we call a fictitious force (pseudo force) and it is a force experienced by the particle m because it is in a non inertial reference frame.

A.3 Definition of the $\mathring{\delta}$ Function

Let $\tilde{\xi}(r, \phi, z)$ be the following vector field in the (r, ϕ, z) cylindrical coordinates:

$$\tilde{\xi} = \frac{1}{r}u(r-\epsilon)\hat{i}_\phi \tag{42}$$

where $u(r)$ is the Heaviside unitary step discontinuous function, $\hat{i}_\phi$ is the unitary vector (versor) in the ϕ coordinate direction and ϵ is any positive real number.

Using the expression of the curl in cylindrical coordinates we have:

$$\nabla \times \hat{\xi} = \frac{1}{r}\delta(r-\epsilon)\hat{i}_z \tag{43}$$

Given Eq. (43) and using the expression of the curl in cylindrical coordinates we have:

$$\nabla \times (\nabla \times \hat{\xi}) = \left(\frac{1}{r^2}\delta(r-\epsilon) - \frac{1}{r}\delta'(r-\epsilon)\right)\hat{i}_\phi = \frac{2}{r^2}\delta(r-\epsilon)\hat{i}_\phi \tag{44}$$

Where the above is true because, although it is hard to believe, we have that:

$$\frac{1}{r^2}\delta(r-\epsilon) = -\frac{1}{r}\delta'(r-\epsilon) \tag{45}$$

We will prove the equality (45) at the end of the section. For the time being, inspired by Eq. (43), we define the field $\mathring{\delta}_{z,\epsilon}$ as follows:

$$\mathring{\delta}_{z,\epsilon}(x,y) = \frac{2}{\epsilon^2}\delta(r-\epsilon)\hat{i}_\phi \tag{46}$$

which is a field circulating on the (x,y) plane, discrete with respect to the distance from the z axis and present on a circle centred in the origin and having radius ϵ. From Eq. (44) and Eq. (46) we have:

$$\nabla \times \nabla \times \left(\frac{1}{r}\delta(r-\epsilon)\hat{i}_z\right) = \mathring{\delta}_{z,\epsilon}(x,y) \tag{47}$$

We want now to give some physical interpretation of $\mathring{\delta}_{z,\epsilon}$. Suppose $J_0\mathring{\delta}_{z,\epsilon}$ represents a current circulating around the z axis. We want to evaluate the magnetic moment m generated by this current. The magnetic moment of a single coil of a solenoid is given by the intensity of the current multiplied by the area enclosed by the coil. Given Eq. (46), we have that the magnetic moment per unit length along z is given by:

$$m = \pi\epsilon^2 J_0 \frac{2}{\epsilon^2}|\delta(r-\epsilon)|\hat{i}_z = 2\pi J_0 |\mathring{\delta}_{z,\epsilon}(x,y)|\hat{i}_z \tag{48}$$

where with $|\delta(r-\epsilon)|$, we mean the amplitude of the δ. We note that the above results is independent from ϵ and therefore we may say that a $\mathring{\delta}_z(x,y)$ is a distribution of current, as close to the z axis as we like (i.e. $\epsilon \to 0$), and giving a magnetic moment of 2π.

As promised, we finish this paragraph by proving Eq. (45). Let $g(r)$ be a continuous function with $g(r) = 0$ for $r < 0$ and such that:

$$\int_0^\infty g(r)dr = 1 \tag{49}$$

Let f be the generalised function $\frac{1}{r}\delta'(r-\epsilon)$. We have:

$$r^2 f = r^2 \lim_{n\to\infty} \frac{1}{r} n^2 g'(nr-\epsilon) = \lim_{n\to\infty} n[(nr)g'(nr-\epsilon)] = \lim_{n\to\infty} n g_1(nr-\epsilon) \tag{50}$$

where $g_1 = (r+\epsilon)g'(r)$. Clearly $r^2 f$ is a delta function $A\delta(r)$ centred in zero, where the amplitude A is given by:

$$A = \int_0^\infty g_1(r)dr = \int_0^\infty (r+d)g'(r)dr = -\int_0^\infty g(r)dr = 1 \tag{51}$$

where we have used integration by parts. This proves the inequality because we have proven that:

$$r^2 f = -r^2 \frac{1}{r^2}\delta(r-\epsilon) \tag{52}$$

References

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