A new formulation of Mertens function

Juan Moreno Borrallo

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email: juan.morenoborrallo@gmail.com

Abstract

In this brief note there are showed original formulations for $\sum_{k=1}^{n}\frac{\mu(k)}{k}$, where $\mu(k)$ is the Möbius function, and Mertens function $M(n)=\sum_{k=1}^{n}\mu(k)$.

For any positive integer n, we define de Möbius function $\mu(n)$ as having the following values depending on the factorization of n into prime factors:

- $\mu(n)=1$ if n is a square-free positive integer with an even number of prime factors.
- $\mu(n)=-1$ if n is a square-free positive integer with an odd number of prime factors.
- $\mu(n)=0$ if n has a squared prime factor.

Therefore, we have that

$$\sum_{k=1}\frac{\mu(k)}{k}=1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}-\frac{1}{7}+\frac{1}{10}-\frac{1}{11}-\frac{1}{13}+... \tag{1}$$

Where k runs over the square-free integers.

It is straightforward from the definition of $\mu(n)$ to note that

$$\sum_{k\leq n}\frac{\mu(k)}{k}=1-\sum_{p_i\leq n}\left(\frac{1}{p_i}\right)+\sum_{p_i<p_j\leq\frac{n}{p_i}}\left(\frac{1}{p_ip_j}\right)-\sum_{p_i<p_j<p_k\leq\frac{n}{p_ip_j}}\left(\frac{1}{p_ip_jp_k}\right)+\dots \tag{2}$$

Other hand, Merten's function $M(n)$is defined for all positive integers as

$$M(n)=\sum_{k=1}^{n}\mu(k) \tag{3}$$

Starting from (2), applying the inclusion-exclusion principle term by term, it is pretty straightforward to obtain that

$$M(n) = 1 - \pi(n) + \sum_{p_i \leq \frac{n}{p_i}} \left(\pi\left(\frac{n}{p_i}\right) - i \right) - \sum_{p_i < p_j \leq \frac{n}{p_i p_j}} \left(\pi\left(\frac{n}{p_i p_j}\right) - j \right) +$$

$$\sum_{p_i < p_j < p_k \leq \frac{n}{p_i p_j p_k}} \left(\pi\left(\frac{n}{p_i p_j p_k}\right) - k \right) - \ldots \qquad (4)$$