From Coulomb's force to magnetic force and experiments that show magnetic force parallel to current

Kuan Peng 彭宽 titang78@gmail.com

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Abstract: The Lorentz force law is fundamental for electromagnetism. However, it is known long ago that the Lorentz forces between two current elements do not respect the Newton's third law. This seemingly harmless flaw had never been corrected. In physical sciences a discrepancy often hides in it new understanding or unexpected breakthrough. For solving this problem, we give a purely theoretical derivation of magnetic force which respects the Newton's third law in the case of current elements and is identical to the Lorentz force in the case of coils. This new law reveals how electric force is transformed into magnetic force by velocity and is supported by experimental evidences that we will explain and compute with the new law.

1. Introduction

Lorentz force is a fundamental magnetic force which is created by magnetic field on current carrying wire. Let $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$ be current elements and $\boldsymbol{B}_a$ and $\boldsymbol{B}_b$ the magnetic field they create. According to the Lorentz force law the magnetic forces on $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$ are:

$$\boldsymbol{dF}_a = d\boldsymbol{I}_a \times \boldsymbol{B_b}, \quad \boldsymbol{dF_b} = d\boldsymbol{I}_b \times \boldsymbol{B_a} \quad (1)$$

Because the Lorentz forces $d\boldsymbol{F}_a$ and $d\boldsymbol{F}_b$ are action and reaction forces, they should obey the Newton's third law and sum to zero. However, the Figure 1 shows a case where $d\boldsymbol{F}_a$ is perpendicular to $d\boldsymbol{F}_b$, so, $d\boldsymbol{F}_a + d\boldsymbol{F}_b \neq 0$, that is, the Lorentz forces that the two current elements act on each other violate the Newton's third law.

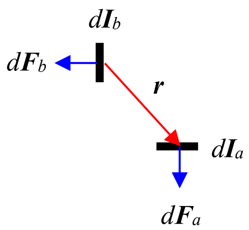

Figure 1

This problem was known for longtime. People justify that the Lorentz forces that two closed loop currents act on each other do satisfy the Newton's third law. Nevertheless, breaking the Newton's third law does not fit scientific standard, even for the Lorentz forces law which is fundamental. The reason for this problem is that being an experimental law Lorentz force law can only describe forces that experiments show. So far, all experimental magnetic forces are perpendicular to current, so the Lorentz force law does not describe magnetic force parallel to the current and consequently cannot respect Newton's third law.

We will try to solve this problem with a new magnetic force law that we have derived with pure theory. This new law respects the Newton's third law in the case of current elements and is identical to the Lorentz force law in the case of coils. This law reveals how electric force is transformed into magnetic force by velocity and is supported by experimental evidences which we will present and compute with the new law at the end.

The new law is derived from the Coulomb's law which defines the Coulomb's force for fixed charges. For moving electrons, the Coulomb's force undergoes relativistic effects and varies with velocity. Although this velocity is very small, the number of free electrons in wires is so huge that relativistic effects show up nevertheless. We have found two relativistic effects in currents: the relativistic dynamic effect and the changing distance effect.

Let us start with the Coulomb's law:

$$\boldsymbol{F} = \frac{q_1 q_2 \boldsymbol{r}}{4\pi\varepsilon_0 |\boldsymbol{r}|^3} \quad (2)$$

with q_1 and q_2 being two electric charges, $\boldsymbol{r}$ the vector radius distance between them, ε_0 the permittivity of free space.

2. Relativistic dynamic effect

a) Couple of charges

Currents flow in neutral wires which contain the same quantity of free electrons and positive charges. For computing the Coulomb's force in neutral material, we use couple of charges. Let ⊖ represent a free electron and ⊕ a fixed positive charge which is a proton in the nucleus of an atom. A couple of charges is represented by (⊕, ⊖) and is neutral. So, the interaction between two couples of charges represents the magnetic force between two neutral materials.

Let ($\oplus_a$, $\ominus_a$) be a couple of charges in the material a and ($\oplus_b$, $\ominus_b$) that in the material b. The vector distance between ($\oplus_a$, $\ominus_a$) and ($\oplus_b$, $\ominus_b$) is $\boldsymbol{r}$, see Figure 2 (a). Between ($\oplus_a$, $\ominus_a$) and ($\oplus_b$, $\ominus_b$) there are 4 interactions : $\oplus_b \rightarrow \oplus_a$, $\ominus_b \rightarrow \oplus_a$, $\oplus_b \rightarrow \ominus_a$, $\ominus_b \rightarrow \ominus_a$. We show them in the line 1 of the Table 1 and in the Figure 2 (b) and (c). We suppose that $\oplus_a$ and $\ominus_a$ are at the same location and $\oplus_b$ and $\ominus_b$ are at the same location. Then, the vector distances between $\oplus_b$ and $\oplus_a$, $\ominus_b$ and $\oplus_a$, $\oplus_b$ and $\ominus_a$, $\ominus_b$ and $\ominus_a$ are all $\boldsymbol{r}$.

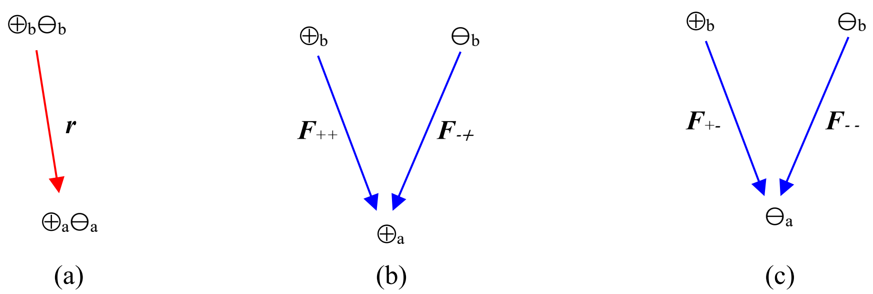

Figure 2

The electric charge of ⊕ is e and that of ⊖ is $-e$, then for the 4 interactions the products of charges $q_1 \cdot q_2$ equal e^2, $-e^2$, $-e^2$ and e^2 respectively, see the line 2 of the Table 1. Applying e^2, $-e^2$, $-e^2$ and e^2 to the Coulomb's law (2) we get the 4 Coulomb's forces which are labeled as $\boldsymbol{F}_{++}$, $\boldsymbol{F}_{-+}$, $\boldsymbol{F}_{+-}$ and $\boldsymbol{F}_{--}$, expressed in the line 3 of the Table 1 and shown in the Figure 2 (b) and (c).

1	$\oplus_b \rightarrow \oplus_a$	$\ominus_b \rightarrow \oplus_a$	$\oplus_b \rightarrow \ominus_a$	$\ominus_b \rightarrow \ominus_a$
2	$q_1 \cdot q_2 = e^2$	$q_1 \cdot q_2 = -e^2$	$q_1 \cdot q_2 = -e^2$	$q_1 \cdot q_2 = e^2$
3	$\boldsymbol{F}_{++} = \frac{e^2}{4\pi\varepsilon_0} \frac{\boldsymbol{r}}{\lvert\boldsymbol{r}\rvert^3}$	$\boldsymbol{F}_{-+} = -\frac{e^2}{4\pi\varepsilon_0} \frac{\boldsymbol{r}}{\lvert\boldsymbol{r}\rvert^3}$	$\boldsymbol{F}_{+-} = -\frac{e^2}{4\pi\varepsilon_0} \frac{\boldsymbol{r}}{\lvert\boldsymbol{r}\rvert^3}$	$\boldsymbol{F}_{--} = \frac{e^2}{4\pi\varepsilon_0} \frac{\boldsymbol{r}}{\lvert\boldsymbol{r}\rvert^3}$

Table 1 Coulomb's forces between the charges

b) Dynamic force across moving frames

The sum of the 4 forces equals zero: $\boldsymbol{F}_{++} + \boldsymbol{F}_{-+} + \boldsymbol{F}_{+-} + \boldsymbol{F}_{--} = 0$. Let us see how the relativistic dynamic effect breaks this equilibrium. In the Figure 3 we have a stationary body $\boldsymbol{b}_1$ and a moving body $\boldsymbol{b}_2$. The frame 1 is attached to $\boldsymbol{b}_1$ and the frame 2 to $\boldsymbol{b}_2$. The velocity of $\boldsymbol{b}_2$ relative to $\boldsymbol{b}_1$ is $\boldsymbol{v}$, so the frame 2 moves at the velocity $\boldsymbol{v}$ in the frame 1.

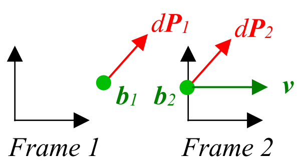

Figure 3

The momentum and time in the frame 1 are labeled as $\boldsymbol{P}_1$ and t_1. The force on $\boldsymbol{b}_1$ equals the time derivation of $\boldsymbol{P}_1$:

$$\boldsymbol{F}_1 = \frac{d\boldsymbol{P}_1}{dt_1} \quad (3)$$

On the other hand, the momentum and time in the frame 2 are labeled as $\boldsymbol{P}_2$ and t_2. Newtonian mechanics applies in the frame 2 and the force on $\boldsymbol{b}_2$ equals the time derivation of $\boldsymbol{P}_2$:

$$\boldsymbol{F}_2 = \frac{d\boldsymbol{P}_2}{dt_2} \quad (4)$$

We transform $\boldsymbol{F}_2$ with the time of the frame 1 t_1:

$$\boldsymbol{F}_2 = \frac{d\boldsymbol{P}_2}{dt_2} = \frac{d\boldsymbol{P}_2}{dt_1}\frac{dt_1}{dt_2} \quad (5)$$

Suppose an impulse hits the body $\boldsymbol{b}_1$ and transfers the differential momentum $d\boldsymbol{P}_1$ to $\boldsymbol{b}_1$. Suppose that the body $\boldsymbol{b}_2$ is in fact the body $\boldsymbol{b}_1$ put in motion, so the impulse hits in fact the body $\boldsymbol{b}_2$ and transfers the differential momentum $d\boldsymbol{P}_2$ to $\boldsymbol{b}_2$. So, $d\boldsymbol{P}_1$ and $d\boldsymbol{P}_2$ are the differential momentum transferred by the same impulse to the same body. Then $d\boldsymbol{P}_1$ equals $d\boldsymbol{P}_2$:

$$d\boldsymbol{P}_1 = d\boldsymbol{P}_2 \quad (6)$$

Note : For a more rigorous demonstration of this equality please see the equation (10) in « Relativistic dynamics: force, mass, kinetic energy, gravitation and dark matter » [1].

Then, we have :

$$\frac{d\boldsymbol{P}_2}{dt_1} = \frac{d\boldsymbol{P}_1}{dt_1} = \boldsymbol{F}_1 \quad (7)$$

The times t_1 and t_2 are in two frames that are in relative motion. According to special relativity, t_1 is converted into t_2 using the Lorentz transformation below :

$$\frac{dt_1}{dt_2} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (8)$$

Introducing (7) and (8) into (5) , the force $\boldsymbol{F}_2$ is expressed with the force $\boldsymbol{F}_1$:

$$\boldsymbol{F}_2 = \frac{\boldsymbol{F}_1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (9)$$

$\boldsymbol{F}_1$ is the Coulomb's force two fixed charges exert on one another; $\boldsymbol{F}_2$ is the Coulomb's force when the two charges move with respect to one another at the velocity v. This relation converts $\boldsymbol{F}_1$ into $\boldsymbol{F}_2$.

c) Relative velocity between charges

The free electrons in the current element $d\boldsymbol{I}_a$ have an average velocity which is denoted as $\boldsymbol{v}_a$; that in $d\boldsymbol{I}_b$ is denoted as $\boldsymbol{v}_b$. For the 4 interactions, we denote the velocity of $\oplus_a$ relative to $\oplus_b$ as $\boldsymbol{v}_{++}$, that of $\oplus_a$ to $\ominus_b$ as $\boldsymbol{v}_{-+}$, that of $\ominus_a$ to $\oplus_b$ as $\boldsymbol{v}_{+-}$, that of $\ominus_a$ to $\ominus_b$ as $\boldsymbol{v}_{--}$. Let $\ominus_1$ and $\ominus_2$ be two charges which move at the velocity $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ respectively. The velocity of $\ominus_1$ relative to $\ominus_2$ is $\boldsymbol{v}_1$ - $\boldsymbol{v}_2$. The relative velocities for the 4 pairs of charges are computed in Table 2.

[1] Kuan Peng, 2021, « Relativistic dynamics: force, mass, kinetic energy, gravitation and dark matter », https://www.academia.edu/49921891/Relativistic_dynamics_force_mass_kinetic_energy_gravitation_and_dark_matter

⊕a	$\boldsymbol{v}_1 = 0$	⊕b	$\boldsymbol{v}_2 = 0$	$\boldsymbol{v}_1 - \boldsymbol{v}_2 = 0$	$\boldsymbol{v}_{++} = 0$
⊕a	$\boldsymbol{v}_1 = 0$	⊖b	$\boldsymbol{v}_2 = \boldsymbol{v}_b$	$\boldsymbol{v}_1 - \boldsymbol{v}_2 = 0 - \boldsymbol{v}_b = -\boldsymbol{v}_b$	$\boldsymbol{v}_{-+} = -\boldsymbol{v}_b$
⊖a	$\boldsymbol{v}_1 = \boldsymbol{v}_a$	⊕b	$\boldsymbol{v}_2 = 0$	$\boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{v}_a - 0 = \boldsymbol{v}_a$	$\boldsymbol{v}_{+-} = \boldsymbol{v}_a$
⊖a	$\boldsymbol{v}_1 = \boldsymbol{v}_a$	⊖b	$\boldsymbol{v}_2 = \boldsymbol{v}_b$	$\boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{v}_a - \boldsymbol{v}_b$	$\boldsymbol{v}_{--} = \boldsymbol{v}_a - \boldsymbol{v}_b$

Table 2 The relative velocities between the charges

d) Relativistic dynamic effect

We apply the relativistic relation (9) to the 4 forces $\boldsymbol{F}_{++}$, $\boldsymbol{F}_{-+}$, $\boldsymbol{F}_{+-}$ and $\boldsymbol{F}_{--}$ given in the line 3 of the Table 1. The forces modified by the relativistic dynamic effect are denoted as : $\boldsymbol{F'}_{++}$, $\boldsymbol{F'}_{-+}$, $\boldsymbol{F'}_{+-}$ and $\boldsymbol{F'}_{--}$. The first modified force is $\boldsymbol{F'}_{++}$ with $\boldsymbol{v}_{++} = 0$:

$$\boldsymbol{F'}_{++} = \frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \quad (10)$$

The second modified force is $\boldsymbol{F'}_{-+}$ with $\boldsymbol{v}_{-+} = -\boldsymbol{v}_b$ **:**

$$\boldsymbol{F'}_{-+} = -\frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\frac{1}{\sqrt{1-\frac{\boldsymbol{v}_b{}^2}{c^2}}} \quad (11)$$

The third modified force is $\boldsymbol{F'}_{+-}$ with $\boldsymbol{v}_{+-} = \boldsymbol{v}_a$ **:**

$$\boldsymbol{F'}_{+-} = -\frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\frac{1}{\sqrt{1-\frac{\boldsymbol{v}_a{}^2}{c^2}}} \quad (12)$$

The fourth modified force is $\boldsymbol{F'}_{--}$ with $\boldsymbol{v}_{--} = \boldsymbol{v}_a - \boldsymbol{v}_b$ **:**

$$\boldsymbol{F'}_{--} = \frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\frac{1}{\sqrt{1-\frac{(\boldsymbol{v}_a-\boldsymbol{v}_b)^2}{c^2}}} \quad (13)$$

As $\boldsymbol{v}_a$ and $\boldsymbol{v}_b$ are very small before the speed of light c, we can do the linear expansion for $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$:

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \quad (14)$$

Then, $\boldsymbol{F'}_{-+}$, $\boldsymbol{F'}_{+-}$ and $\boldsymbol{F'}_{--}$ become:

$$\boldsymbol{F'}_{-+} \approx -\frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\left(1+\frac{1}{2}\frac{\boldsymbol{v}_b{}^2}{c^2}\right) \quad (15)$$
$$\boldsymbol{F'}_{+-} \approx -\frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\left(1+\frac{1}{2}\frac{\boldsymbol{v}_a{}^2}{c^2}\right) \quad (16)$$
$$\boldsymbol{F'}_{--} \approx \frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\left(1+\frac{1}{2}\frac{(\boldsymbol{v}_a-\boldsymbol{v}_b)^2}{c^2}\right) \quad (17)$$

The velocity squared $\boldsymbol{v}^2$ is the vector $\boldsymbol{v}$ dotted by itself :

$$\boldsymbol{v}^2 = \boldsymbol{v}\cdot\boldsymbol{v} \quad (18)$$

We develop $(\boldsymbol{v}_a - \boldsymbol{v}_b)^2$:

$$(\boldsymbol{v}_a-\boldsymbol{v}_b)^2 = (\boldsymbol{v}_a-\boldsymbol{v}_b)\cdot(\boldsymbol{v}_a-\boldsymbol{v}_b) = \boldsymbol{v}_a{}^2 - 2\boldsymbol{v}_a\cdot\boldsymbol{v}_b + \boldsymbol{v}_b{}^2 \quad (19)$$

Introducing (19) into (17) gives :

$$F'_{--} \approx \frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\left(1+\frac{\boldsymbol{v_a}^2 - 2\boldsymbol{v}_a\cdot\boldsymbol{v}_b + \boldsymbol{v}_b{}^2}{2c^2}\right) \quad (20)$$

The sum $\boldsymbol{F'}_{++} + \boldsymbol{F'}_{-+} + \boldsymbol{F'}_{+-} + \boldsymbol{F'}_{--}$ is :

$$\boldsymbol{F'}_{ba} \approx \frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\left(1-\left(1+\frac{\boldsymbol{v}_b{}^2}{2c^2}\right)-\left(1+\frac{\boldsymbol{v}_a{}^2}{2c^2}\right)+\left(1+\frac{\boldsymbol{v}_a{}^2 - 2\boldsymbol{v}_a\cdot\boldsymbol{v}_b + \boldsymbol{v}_b{}^2}{2c^2}\right)\right) \quad (21)$$

The terms $\left(1+\frac{v_b{}^2}{2c^2}\right)$ and $\left(1+\frac{v_a{}^2}{2c^2}\right)$ cancel out and the force $\boldsymbol{F'}_{ba}$ becomes :

$$\boldsymbol{F'}_{ba} \approx -\frac{e^2}{4\pi\varepsilon_0 c^2}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}(\boldsymbol{v}_a\cdot\boldsymbol{v}_b) \quad (22)$$

$\boldsymbol{F'}_{ba}$ is the resultant force that the couple of charges ($\oplus_b$, $\ominus_b$) exerts on ($\oplus_a$, $\ominus_a$). I have obtained this expression in «Length-contraction-magnetic-force between arbitrary currents»[2] using the length-contraction effect of relativity. But deriving from couples of charges is a more fundamental approach because the basic force is on individual electric charges.

3. Changing distance effect

a) Distance change

The second relativistic effect is caused by the position change of electrons. For example, in the Figure 4 the electron moves from ⊖ to ⊖' . Because $\boldsymbol{r}$ is different from $\boldsymbol{r}_0$ the Coulomb's force it acts on the positive charge ⊕ varies from $\boldsymbol{F}_0$ to $\boldsymbol{F}$. In the couples of charges ($\oplus_a$, $\ominus_a$) and ($\oplus_b$, $\ominus_b$), the electrons $\ominus_a$ and $\ominus_b$ are moving and change constantly positions, so the resultant Coulomb's force that ($\oplus_b$, $\ominus_b$) acts on ($\oplus_a$, $\ominus_a$) changes too. We call this change of the resultant Coulomb's force the "Changing distance effect".

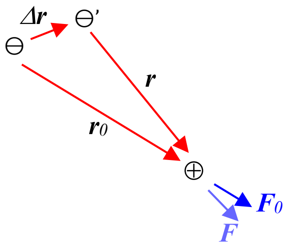

Figure 4

For the distance $\boldsymbol{r}$ the Coulomb's force on ⊕ is:

$$\boldsymbol{F} = -\frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \quad (23)$$

For computing the varying value of $\boldsymbol{F}$ we express $\boldsymbol{r}$ as $\boldsymbol{r} = \boldsymbol{r}_0 + \Delta\boldsymbol{r}$. Then $\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$ is expressed with $\Delta\boldsymbol{r}$:

$$\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} = (\boldsymbol{r_0} + \Delta\boldsymbol{r})|\boldsymbol{r_0} + \Delta\boldsymbol{r}|^{-3} \quad (24)$$

The expression for $|\boldsymbol{r_0} + \Delta\boldsymbol{r}|^{-3}$ is:

[2] Kuan Peng, 2017, «Length-contraction-magnetic-force between arbitrary currents», https://www.academia.edu/32815401/Length-contraction-magnetic-force_between_arbitrary_currents

$$|\boldsymbol{r_0} + \Delta\boldsymbol{r}|^{-3} = (\boldsymbol{r}^2)^{-\frac{3}{2}} = (\boldsymbol{r_0}^2 + 2\boldsymbol{r_0}\cdot\Delta\boldsymbol{r} + \Delta\boldsymbol{r}^2)^{-\frac{3}{2}} \quad (25)$$

We consider the case where $\Delta\boldsymbol{r}$ is very small before $\boldsymbol{r_0}$. So, we do the linear expansion :

$$(\boldsymbol{r_0}^2 + 2\boldsymbol{r_0}\cdot\Delta\boldsymbol{r} + \Delta\boldsymbol{r}^2)^{-\frac{3}{2}} \approx |\boldsymbol{r_0}|^{-3}\left(1 - 3\frac{\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{|\boldsymbol{r_0}|^2}\right) \qquad (26)$$

and the $\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$ of (24) becomes :

$$\begin{aligned}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} &\approx (\boldsymbol{r_0} + \Delta\boldsymbol{r})|\boldsymbol{r_0}|^{-3}\left(1 - 3\frac{\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{|\boldsymbol{r_0}|^2}\right) \\ &= |\boldsymbol{r_0}|^{-3}\left(\boldsymbol{r_0} + \Delta\boldsymbol{r} - 3\boldsymbol{r_0}\frac{\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{|\boldsymbol{r_0}|^2} - 3\Delta\boldsymbol{r}\frac{\boldsymbol{r_0}\cdot\Delta\boldsymbol{r}}{|\boldsymbol{r_0}|^2}\right)\end{aligned} \qquad (27)$$

Let $\boldsymbol{v}$ be the velocity of the electron and $\Delta\boldsymbol{r}$ be the distance traveled by the electron in time t :

$$\Delta\boldsymbol{r} = \boldsymbol{v}t \quad (28)$$

We replace $\Delta\boldsymbol{r}$ with $\boldsymbol{v}t$ in (27) and $\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$ becomes :

$$\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3} \approx |\boldsymbol{r_0}|^{-3}\left(\boldsymbol{r_0} + \boldsymbol{v}t - 3\boldsymbol{r_0}\frac{\boldsymbol{r_0}\cdot\boldsymbol{v}t}{|\boldsymbol{r_0}|^2} - 3\boldsymbol{v}t\frac{\boldsymbol{r_0}\cdot\boldsymbol{v}t}{|\boldsymbol{r_0}|^2}\right) \quad (29)$$

b) Changing distance effect

The 4 Coulomb's forces modified by the changing distance effect are labeled as $\boldsymbol{F}''_{++}$, $\boldsymbol{F}''_{-+}$, $\boldsymbol{F}''_{+-}$ and, $\boldsymbol{F}''_{--}$ and are computed by introducing the $\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}$ of (29) into the expressions of $\boldsymbol{F}_{++}$, $\boldsymbol{F}_{-+}$, $\boldsymbol{F}_{+-}$ and $\boldsymbol{F}_{--}$ given in Table 1. The relative velocities are given in Table 2. For the interaction $\oplus_b\rightarrow\oplus_a$, the relative velocity is zero, then $\boldsymbol{F}''_{++}$ equals $\boldsymbol{F}_{++}$:

$$\boldsymbol{F}''_{++} = \frac{e^2}{4\pi\varepsilon_0}\frac{\boldsymbol{r}_0}{|\boldsymbol{r_0}|^3} \quad (30)$$

For the interaction $\ominus_b\rightarrow\oplus_a$, the relative velocity is $-\boldsymbol{v}_b$ and $\boldsymbol{F}''_{-+}$ is

$$\boldsymbol{F}''_{-+} \approx -\frac{e^2}{4\pi\varepsilon_0}|\boldsymbol{r_0}|^{-3}\left(\boldsymbol{r_0} + (-\boldsymbol{v}_b)t - 3\boldsymbol{r_0}\frac{\boldsymbol{r_0}\cdot(-\boldsymbol{v}_b)t}{|\boldsymbol{r_0}|^2} - 3(-\boldsymbol{v}_b)t\frac{\boldsymbol{r_0}\cdot(-\boldsymbol{v}_b)t}{|\boldsymbol{r_0}|^2}\right) \quad (31)$$

For the interaction $\oplus_b\rightarrow\ominus_a$, the relative velocity is $\boldsymbol{v}_a$ and $\boldsymbol{F}''_{+-}$ is

$$\boldsymbol{F}''_{+-} \approx -\frac{e^2}{4\pi\varepsilon_0}|\boldsymbol{r_0}|^{-3}\left(\boldsymbol{r_0} + \boldsymbol{v}_a t - 3\boldsymbol{r_0}\frac{\boldsymbol{r_0}\cdot\boldsymbol{v}_a t}{|\boldsymbol{r_0}|^2} - 3\boldsymbol{v}_a\Delta t\frac{\boldsymbol{r_0}\cdot\boldsymbol{v}_a t}{|\boldsymbol{r_0}|^2}\right) \quad (32)$$

For the interaction $\ominus_b\rightarrow\ominus_a$ the relative velocity is $\boldsymbol{v}_a$-$\boldsymbol{v}_b$ and $\boldsymbol{F}''_{--}$ is

$$\boldsymbol{F}''_{--} \approx \frac{e^2}{4\pi\varepsilon_0}|\boldsymbol{r_0}|^{-3}\left(\boldsymbol{r_0} + (\boldsymbol{v}_a - \boldsymbol{v}_b)t - 3\boldsymbol{r_0}\frac{\boldsymbol{r_0}\cdot(\boldsymbol{v}_a - \boldsymbol{v}_b)t}{|\boldsymbol{r_0}|^2} - 3(\boldsymbol{v}_a - \boldsymbol{v}_b)t\frac{\boldsymbol{r_0}\cdot(\boldsymbol{v}_a - \boldsymbol{v}_b)t}{|\boldsymbol{r_0}|^2}\right) \quad (33)$$

The sum $\boldsymbol{F}''_{++} + \boldsymbol{F}''_{-+} + \boldsymbol{F}''_{-+} + \boldsymbol{F}''_{-+}$ is

$$\boldsymbol{F}''_t = \frac{e^2}{4\pi\varepsilon_0}|\boldsymbol{r_0}|^{-3}\begin{pmatrix} \boldsymbol{r_0} \\ -\left(\boldsymbol{r_0} + (-\boldsymbol{v}_b)t - 3\boldsymbol{r_0}\frac{\boldsymbol{r_0}\cdot(-\boldsymbol{v}_b)t}{|\boldsymbol{r_0}|^2} - 3(-\boldsymbol{v}_b)t\frac{\boldsymbol{r_0}\cdot(-\boldsymbol{v}_b)t}{|\boldsymbol{r_0}|^2}\right) \\ -\left(\boldsymbol{r_0} + \boldsymbol{v}_a t - 3\boldsymbol{r_0}\frac{\boldsymbol{r_0}\cdot\boldsymbol{v}_a t}{|\boldsymbol{r_0}|^2} - 3\boldsymbol{v}_a\Delta t\frac{\boldsymbol{r_0}\cdot\boldsymbol{v}_a t}{|\boldsymbol{r_0}|^2}\right) \\ +\left(\boldsymbol{r_0} + (\boldsymbol{v}_a - \boldsymbol{v}_b)t - 3\boldsymbol{r_0}\frac{\boldsymbol{r_0}\cdot(\boldsymbol{v}_a - \boldsymbol{v}_b)t}{|\boldsymbol{r_0}|^2} - 3(\boldsymbol{v}_a - \boldsymbol{v}_b)t\frac{\boldsymbol{r_0}\cdot(\boldsymbol{v}_a - \boldsymbol{v}_b)t}{|\boldsymbol{r_0}|^2}\right) \end{pmatrix} \quad (34)$$

The term $\boldsymbol{r_0} + (\boldsymbol{v}_a - \boldsymbol{v}_b)t - 3\boldsymbol{r_0}\frac{\boldsymbol{r_0}\cdot(\boldsymbol{v}_a-\boldsymbol{v}_b)t}{|\boldsymbol{r_0}|^2}$ cancels out and $\boldsymbol{F''}_t$ becomes:

$$\boldsymbol{F}''_t = 3t^2\frac{e^2}{4\pi\varepsilon_0}|\boldsymbol{r_0}|^{-5}\Big(\boldsymbol{v}_b(\boldsymbol{r_0}\cdot\boldsymbol{v}_b) + \boldsymbol{v}_a(\boldsymbol{r_0}\cdot\boldsymbol{v}_a) - (\boldsymbol{v}_a - \boldsymbol{v}_b)\big(\boldsymbol{r_0}\cdot(\boldsymbol{v}_a - \boldsymbol{v}_b)\big)\Big) \quad (35)$$

We develop the term $(\boldsymbol{v}_a - \boldsymbol{v}_b)\big(\boldsymbol{r_0}\cdot(\boldsymbol{v}_a - \boldsymbol{v}_b)\big)$:

$$(\boldsymbol{v}_a - \boldsymbol{v}_b)\big(\boldsymbol{r_0}\cdot(\boldsymbol{v}_a - \boldsymbol{v}_b)\big) = \boldsymbol{v}_a(\boldsymbol{r_0}\cdot\boldsymbol{v}_a) - \boldsymbol{v}_a(\boldsymbol{r_0}\cdot\boldsymbol{v}_b) - \boldsymbol{v}_b(\boldsymbol{r_0}\cdot\boldsymbol{v}_a) + \boldsymbol{v}_b(\boldsymbol{r_0}\cdot\boldsymbol{v}_b) \quad (36)$$

and introduce it into (35). Then the term $\boldsymbol{v}_b(\boldsymbol{r_0}\cdot\boldsymbol{v}_b) + \boldsymbol{v}_a(\boldsymbol{r_0}\cdot\boldsymbol{v}_a)$ cancels out and $\boldsymbol{F''}_t$ becomes :

$$\boldsymbol{F}''_t = 3t^2\frac{e^2}{4\pi\varepsilon_0}|\boldsymbol{r_0}|^{-5}\big(\boldsymbol{v}_a(\boldsymbol{r_0}\cdot\boldsymbol{v}_b) + \boldsymbol{v}_b(\boldsymbol{r_0}\cdot\boldsymbol{v}_a)\big) \quad (37)$$

Because the force is changing, we take its average over time. Let us compute the average value of $\boldsymbol{F''}_t$ over the time period from $t = 0$ to $t = t_e$, which equal the time integral of $\boldsymbol{F''}_t$ divided by t_e :

$$\boldsymbol{F}''_{ba} = \frac{1}{t_e}\int_0^{te}\boldsymbol{F}''_t dt \quad (38)$$

Using (37), (38) becomes :

$$\begin{aligned}\boldsymbol{F}''_{ba} &= \frac{1}{t_e}\int_0^{t_e} 3t^2\frac{e^2|\boldsymbol{r_0}|^{-5}}{4\pi\varepsilon_0}\big(\boldsymbol{v}_a(\boldsymbol{r_0}\cdot\boldsymbol{v}_b) + \boldsymbol{v}_b(\boldsymbol{r_0}\cdot\boldsymbol{v}_a)\big)dt \\ &= {t_e}^2\frac{e^2|\boldsymbol{r_0}|^{-5}}{4\pi\varepsilon_0}\big(\boldsymbol{v}_a(\boldsymbol{r_0}\cdot\boldsymbol{v}_b) + \boldsymbol{v}_b(\boldsymbol{r_0}\cdot\boldsymbol{v}_a)\big)\end{aligned} \quad (39)$$

c) What is t_e?

We notice that $\boldsymbol{F''}_{ba}$ depends on the time t_e. But what is the value of t_e ? The answer is in the trajectory of electrons. Let us see the Figure 5 in which the trajectory of an electron is drawn. The electron passes through the nodes ⊖0, ⊖1, ⊖2, ⊖3 and ⊖4 … . At a distance is a couple of charges ⊕⊖ which receives the Coulomb's force from the electron. As currents flow in closed loop, the trajectory of the electron is also a closed loop. So, with a finite number of nodes the electron will go over the entire trajectory. The set of all the nodes ⊖$_i$, for $i=0$ to n, constitutes the complete trajectory of the electron.

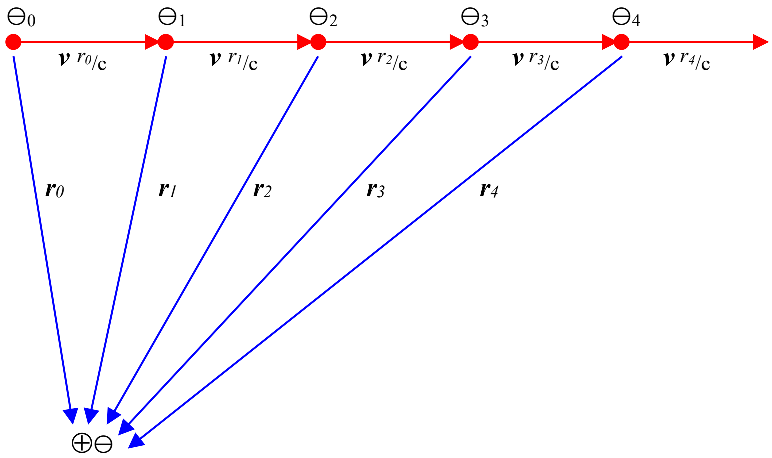

Figure 5

The Coulomb's force on ⊕⊖ is the time derivative of the momentum $\boldsymbol{P}$ emitted by the electron and received by ⊕⊖. Suppose that at time $t = 0$, the electron is at the node ⊖0 and emits the momentum $d\boldsymbol{P}_0$. The radial distance between ⊖$_0$ and ⊕⊖ is $\boldsymbol{r}_0$. According to relativity all electromagnetic signals travel at the speed of light c, so the momentum $d\boldsymbol{P}_0$ travels at the speed of light c and arrives at ⊕⊖ at the time $t = |\boldsymbol{r}_0|/c$. At the same time the electron arrives at the node ⊖$_1$ and emits from ⊖$_1$ the momentum $d\boldsymbol{P}_1$.

The radial distance between ⊖$_1$ and ⊕⊖ is $\boldsymbol{r}_1$. Traveling at the speed of light $d\boldsymbol{P}_1$ arrives at ⊕⊖ at the time $t = |\boldsymbol{r}_0|/c + |\boldsymbol{r}_1|/c$. From the time $t = |\boldsymbol{r}_0|/c$ to $t = |\boldsymbol{r}_0|/c + |\boldsymbol{r}_1|/c$ the couple of charges ⊕⊖ receives the momentum emitted from the time $t = 0$ to $t = |\boldsymbol{r}_0|/c$. So, the average Coulomb's force on ⊕⊖ equals the integral of the momentum from $t = 0$ to $t = |\boldsymbol{r}_0|/c$ divided by $|\boldsymbol{r}_1|/c$:

$$\boldsymbol{F}''_{ba} = \frac{1}{|\boldsymbol{r}_1|/c} \int_0^{|\mathbf{r}_0|/c} d\boldsymbol{P} \quad (40)$$

As the velocity of the electron is extremely small with respect to c, the radial distances $\boldsymbol{r}_0$ and $\boldsymbol{r}_1$ are almost equal and we can write $\boldsymbol{r}_0 \approx \boldsymbol{r}_1$:

$$\boldsymbol{F}''_{ba} \approx \frac{1}{|\boldsymbol{r}_0|/c} \int_0^{|\mathbf{r}_0|/c} d\boldsymbol{P} \quad (41)$$

By comparing (41) with (38) we determine that $t_e = |\boldsymbol{r}_0|/c$. For generalization we call the journey of the electron from ⊖$_1$ to ⊖$_2$ the step 1, the journey from ⊖$_2$ to ⊖$_3$ the step 2, the journey from ⊖$_i$ to ⊖$_{i+1}$ the step i. So, for the step i the time period t_e equals $|\boldsymbol{r}_i|/c$.

Let $\boldsymbol{r}$ be the radial distance from an arbitrary electron to the couple of charges ⊕⊖, the time period for computing the average Coulomb's force is

$$t_e = \frac{|\boldsymbol{r}|}{\boldsymbol{c}} \quad (42)$$

One may question : Since $\boldsymbol{F}''_{ba}$ is an averaged force, why cannot t_e be bigger or smaller than $|\boldsymbol{r}|/c$? Let us take the step where the radial distance is $\boldsymbol{r}_0$. If t_e is big the distance $\boldsymbol{v}t_e$ is big, then the position $\boldsymbol{r}_e = \boldsymbol{r}_0 + \boldsymbol{v}t_e$ will be out of the step defined by $\boldsymbol{r}_0$, the average force $\boldsymbol{F}''_{ba}$ which is proportional to t_e^2 will become too large and irrelevant for the distance $\boldsymbol{r}_0$ used in (39).

Let us see Figure 6 where $t_e > |\boldsymbol{r}|/c$. In this case, the integral can start before the time $t = 0$ or end after $t = |\boldsymbol{r}|/c$, then the momenta emitted before $t = 0$ and after $t = |\boldsymbol{r}|/c$ are included in the integral. Because these momenta overlap that emitted before ⊖$_0$ and after ⊖$_1$, the integral would include momenta that are not from the step 1. If $t_e > |\boldsymbol{r}|/c$ for all the steps, the average force from the complete trajectory of the electron will contain momenta that are counted twice and the average Coulomb's force would be wrong. So, t_e should not be bigger than $|\boldsymbol{r}|/c$.

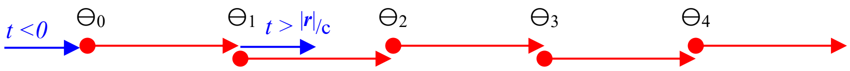

Figure 6 $t_e > |\boldsymbol{r}|/c$

The Figure 7 shows the case where $t_e < |\boldsymbol{r}|/c$. We see in this figure that the distance traveled by the electron is shorter than the distance of the step. Then the sum of all the distances included in the integrals would be shorter than the full trajectory of the electron and the value of $\boldsymbol{F}''_{ba}$ would be wrong. So, t_e should not be smaller than $|\boldsymbol{r}|/c$.

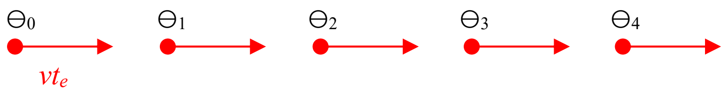

Figure 7 $t_e < |\boldsymbol{r}|/c$

In consequence, $t_e = {}^{|r|}/c$ and $\boldsymbol{F''}_{ba}$ in (39) is :

$$\boldsymbol{F}"_{ba} = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{c^2|\boldsymbol{r}|^3}\big(\boldsymbol{v}_a(\boldsymbol{r}\cdot\boldsymbol{v}_b) + \boldsymbol{v}_b(\boldsymbol{r}\cdot\boldsymbol{v}_a)\big) \quad (43)$$

$\boldsymbol{F''}_{ba}$ is the magnetic force due to the changing distance effect. This formula was already derived in «Changing distance effect»[3]. But the derivation of the changing distance effect and the explanation for "What is t_e?" were different.

4. Complete magnetic force

a) Magnetic force on couples of charges

The complete magnetic force that the couple of charge ($\oplus_b$, $\ominus_b$) exerts on ($\oplus_a$, $\ominus_a$) equals the sum of the magnetic force due to the relativistic dynamic effect given in (22) and that due to the changing distance effect given in (43). So, the complete magnetic force is

$$\boldsymbol{F}_{ba} = \boldsymbol{F}'_{ba} + \boldsymbol{F}"_{ba} \approx -\frac{e^2}{4\pi\varepsilon_0 c^2}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}(\boldsymbol{v}_a\cdot\boldsymbol{v}_b) + \frac{e^2}{4\pi\varepsilon_0 c^2|\boldsymbol{r}|^3}\big(\boldsymbol{v}_a(\boldsymbol{r}\cdot\boldsymbol{v}_b) + \boldsymbol{v}_b(\boldsymbol{r}\cdot\boldsymbol{v}_a)\big) \quad (44)$$

which we rearrange to express the complete magnetic force the couple of charge ($\oplus_b$, $\ominus_b$) exerts on ($\oplus_a$, $\ominus_a$):

$$\boldsymbol{F}_{ba} \approx \frac{e^2}{4\pi\varepsilon_0 c^2|\boldsymbol{r}|^3}\big(-\boldsymbol{r}(\boldsymbol{v}_a\cdot\boldsymbol{v}_b) + \boldsymbol{v}_a(\boldsymbol{r}\cdot\boldsymbol{v}_b) + \boldsymbol{v}_b(\boldsymbol{r}\cdot\boldsymbol{v}_a)\big) \quad (45)$$

b) Magnetic force on current elements

Let ($\oplus_a$, $\ominus_a$) be one couple of charge in $d\boldsymbol{I}_a$ and ($\oplus_b$, $\ominus_b$) one couple of charge in $d\boldsymbol{I}_b$. One ($\oplus_b$, $\ominus_b$) exerts the magnetic force $\boldsymbol{F}_{ba}$ on one ($\oplus_a$, $\ominus_a$). A current element contains a huge number of couples of charges. Let m be the number of couples in $d\boldsymbol{I}_a$, then one ($\oplus_b$, $\ominus_b$) exerts on $d\boldsymbol{I}_a$ the magnetic force $m{\cdot}\boldsymbol{F}_{ba}$. Let n be the number of couples in $d\boldsymbol{I}_b$, then n couples exert on $d\boldsymbol{I}_a$ the magnetic force $n{\cdot}m{\cdot}\boldsymbol{F}_{ba}$. So, the total magnetic force that $d\boldsymbol{I}_b$ exerts on $d\boldsymbol{I}_a$ is, see (45) :

$$\begin{aligned}\boldsymbol{F}_{Iba} &= m\cdot n\cdot\boldsymbol{F}_{ba}\\ &= \frac{1}{4\pi\varepsilon_0 c^2|\boldsymbol{r}|^3}\big(-\boldsymbol{r}(em\boldsymbol{v}_a\cdot en\boldsymbol{v}_b) + em\boldsymbol{v}_a(\boldsymbol{r}\cdot en\boldsymbol{v}_b) + en\boldsymbol{v}_b(\boldsymbol{r}\cdot em\boldsymbol{v}_a)\big)\end{aligned} \quad (46)$$

The charge of one electron is $-e$, then the charge of m electrons is $-m{\cdot}e$. The velocity of the free electrons is $\boldsymbol{v}_a$ in $d\boldsymbol{I}_a$, then the current element $d\boldsymbol{I}_a$ equals - (- $m{\cdot}e$) times $\boldsymbol{v}_a$:

$$d\boldsymbol{I}_a = m\cdot e\boldsymbol{v}_a \quad (47)$$

In the same way, the current element $d\boldsymbol{I}_b$ has n free electrons and the velocity of the free electrons is $\boldsymbol{v}_b$. Then the current element $d\boldsymbol{I}_b$ equals :

$$d\boldsymbol{I}_b = n\cdot e\boldsymbol{v}_b \quad (48)$$

We replace $m{\cdot}e{\cdot}\boldsymbol{v}_a$ with $d\boldsymbol{I}_a$ and $n{\cdot}e{\cdot}\boldsymbol{v}_b$ with $d\boldsymbol{I}_b$ in (46) . Then, $\boldsymbol{F}_{Iba}$ is expressed with $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$:

$$\boldsymbol{F}_{Iba} = \frac{1}{4\pi\varepsilon_0 c^2|\boldsymbol{r}|^3}\big(-\boldsymbol{r}(d\boldsymbol{I}_a\cdot d\boldsymbol{I}_b) + d\boldsymbol{I}_a(\boldsymbol{r}\cdot d\boldsymbol{I}_b) + d\boldsymbol{I}_b(\boldsymbol{r}\cdot d\boldsymbol{I}_a)\big) \quad (49)$$

This is the magnetic force that $d\boldsymbol{I}_b$ exerts on $d\boldsymbol{I}_a$ which is derived from the Coulomb's law and the two relativistic effects. So, we call it "Coulomb magnetic force". I have already derived $\boldsymbol{F}_{Iba}$ in this from in «Coulomb magnetic force»[4].

[3] Kuan Peng, 2018, «Changing distance effect», https://www.academia.edu/36272940/Changing_distance_effect
[4] Kuan Peng, 2018, «Coulomb magnetic force», https://www.academia.edu/36278169/Coulomb_magnetic_force

c) Force a coil exerts on a current element

From (49) we derive the magnetic force that a coil exerts on one current element by using the double vector product identity:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = -\boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}) + \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) \quad (50)$$

By putting :

$$d\boldsymbol{I}_a = \boldsymbol{A}, \quad d\boldsymbol{I}_b = \boldsymbol{B}, \quad \boldsymbol{r} = \boldsymbol{C} \quad (51)$$

(49) becomes :

$$\boldsymbol{F}_{Iba} = \frac{1}{4\pi\varepsilon_0 c^2 |\boldsymbol{r}|^3}\left(d\boldsymbol{I}_a \times (d\boldsymbol{I}_b \times \boldsymbol{r}) + d\boldsymbol{I}_a(\boldsymbol{r} \cdot d\boldsymbol{I}_b)\right) \quad (52)$$

The magnetic force that a coil exerts on the current element $d\boldsymbol{I}_a$ equals the closed line integral of $\boldsymbol{F}_{Iba}$ over the coil, with $d\boldsymbol{I}_a$ constant in the integral and $d\boldsymbol{I}_b$ a current element of the coil, see Figure 8. The integrated magnetic force is

$$\boldsymbol{F}_{\text{Coil}} = \oint \frac{1}{4\pi\varepsilon_0 c^2}\left(\frac{d\boldsymbol{I}_a \times (d\boldsymbol{I}_b \times \boldsymbol{r})}{|\boldsymbol{r}|^3} + \frac{d\boldsymbol{I}_a(\boldsymbol{r} \cdot d\boldsymbol{I}_b)}{|\boldsymbol{r}|^3}\right) \quad (53)$$

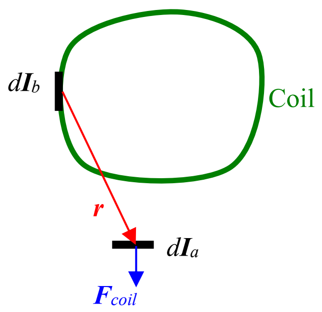

Figure 8

The closed line integral of $\frac{d\boldsymbol{I}_a(\boldsymbol{r} \cdot d\boldsymbol{I}_b)}{|\boldsymbol{r}|^3}$, the last term in (53), equals zero :

$$\oint \frac{d\boldsymbol{I}_a(\boldsymbol{r} \cdot d\boldsymbol{I}_b)}{|\boldsymbol{r}|^3} = d\boldsymbol{I}_a \oint \frac{\boldsymbol{r} \cdot d\boldsymbol{I}_b}{|\boldsymbol{r}|^3} = 0 \quad (54)$$

Then, the magnetic force the coil exerts on the current element $d\boldsymbol{I}_a$ is

$$\boldsymbol{F}_{\text{Coil}} = \oint \frac{1}{4\pi\varepsilon_0 c^2}\frac{d\boldsymbol{I}_a \times (d\boldsymbol{I}_b \times \boldsymbol{r})}{|\boldsymbol{r}|^3} \quad (55)$$

The integrand of (55) is

$$d\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{d\boldsymbol{I}_a \times (d\boldsymbol{I}_b \times \boldsymbol{r})}{|\boldsymbol{r}|^3} \quad (56)$$

We recognize that (56) is the Lorentz force that one $d\boldsymbol{I}_b$ of the coil exerts on $d\boldsymbol{I}_a$.

5. Consequences

a) The relation $\mu_0\varepsilon_0 c^2 = 1$

The Lorentz force that the magnetic field $\boldsymbol{B}$ exerts on $d\boldsymbol{I}_a$ is

$$\boldsymbol{F}_{\text{Lorentz}} = d\boldsymbol{I}_a \times \boldsymbol{B} \quad (57)$$

By comparing (57) with (56) we find that the magnetic field of one $d\boldsymbol{I}_b$ of the coil is :

$$\boldsymbol{B_b} = \frac{1}{\varepsilon_0 c^2}\frac{1}{4\pi}\frac{d\boldsymbol{I}_b \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \quad (58)$$

By comparing (58) with the Biot–Savart law below :

$$\boldsymbol{B_b} = \mu_0 \frac{1}{4\pi}\frac{d\boldsymbol{I}_b \times \boldsymbol{r}}{|\boldsymbol{r}|^3} \quad (59)$$

We find the relation $\mu_0\varepsilon_0 c^2 = 1$:

$$\mu_0 = \frac{1}{\varepsilon_0 c^2} \quad (60)$$

Historically, the values of μ_0 , ε_0 and the speed of light c were measured experimentally. It was James Clerk Maxwell who noticed that c equals $\frac{1}{\sqrt{\mu_0\varepsilon_0}}$. So, until now the relation $\mu_0\varepsilon_0 c^2 = 1$ was an empirical law.

In our derivation this relation emerged naturally from both relativistic dynamic effect and changing distance effect. So, we have theoretically proven this relation and in consequence, the relation $\mu_0\varepsilon_0 c^2 = 1$ is now a theoretical law.

b) Biot–Savart law

The equation (58) is identical to the Biot–Savart law (59) but is derived with pure theory. So, the Biot–Savart law becomes a theoretical law too.

c) Lorentz force law

Because $\frac{1}{\varepsilon_0 c^2} = \mu_0$, (56) can be written as :

$$\boldsymbol{F}_{\text{Lorentz}} = \frac{\mu_0}{4\pi}\frac{d\boldsymbol{I}_a \times (d\boldsymbol{I}_b \times \boldsymbol{r})}{|\boldsymbol{r}|^3} \quad (61)$$

(61) is the Lorentz force that one $d\boldsymbol{I}_b$ exerts on $d\boldsymbol{I}_a$. So, we have derived the Lorentz force law from the Coulomb's law.

d) Magnetic force vs. Newton's third law

We have explained in the introduction that the elementary Lorentz force law violates Newton's third law. Let us compute the sum of the Lorentz force that $d\boldsymbol{I}_b$ exerts on $d\boldsymbol{I}_a$ and the back Lorentz force that $d\boldsymbol{I}_a$ exerts on $d\boldsymbol{I}_b$. The first force is given in (61) :

$$d\boldsymbol{F}_a = \frac{\mu_0}{4\pi|\boldsymbol{r}|^3} d\boldsymbol{I}_a \times (d\boldsymbol{I}_b \times \boldsymbol{r}) = \frac{\mu_0}{4\pi|\boldsymbol{r}|^3}\big(-\boldsymbol{r}(d\boldsymbol{I}_a \cdot d\boldsymbol{I}_b) + d\boldsymbol{I}_b(\boldsymbol{r} \cdot d\boldsymbol{I}_a)\big) \quad (62)$$

The back Lorentz force is obtained from (62) by replacing $\boldsymbol{r}_0$ with $-\boldsymbol{r}_0$, $d\boldsymbol{I}_a$ with $d\boldsymbol{I}_b$ and $d\boldsymbol{I}_b$ with $d\boldsymbol{I}_a$:

$$d\boldsymbol{F}_b = \frac{\mu_0}{4\pi|\boldsymbol{r}|^3} d\boldsymbol{I}_b \times \big(d\boldsymbol{I}_a \times (-\boldsymbol{r})\big) = \frac{\mu_0}{4\pi|\boldsymbol{r}|^3}\Big(-(-\boldsymbol{r})(d\boldsymbol{I}_b \cdot d\boldsymbol{I}_a) + d\boldsymbol{I}_a\big((-\boldsymbol{r}) \cdot d\boldsymbol{I}_b\big)\Big) \quad (63)$$

Adding (62) with (63) we obtain the sum $d\boldsymbol{F}_a + d\boldsymbol{F}_b$ which is not zero :

$$\begin{aligned} d\boldsymbol{F}_a + d\boldsymbol{F}_b &= \frac{\mu_0}{4\pi|\boldsymbol{r}|^3}\Big(-\boldsymbol{r}(d\boldsymbol{I}_a \cdot d\boldsymbol{I}_b) + d\boldsymbol{I}_b(\boldsymbol{r} \cdot d\boldsymbol{I}_a) - (-\boldsymbol{r})(d\boldsymbol{I}_b \cdot d\boldsymbol{I}_a) + d\boldsymbol{I}_a\big((-\boldsymbol{r}) \cdot d\boldsymbol{I}_b\big)\Big) \\ &= \frac{\mu_0}{4\pi|\boldsymbol{r}|^3}\big(d\boldsymbol{I}_b(\boldsymbol{r} \cdot d\boldsymbol{I}_a) - d\boldsymbol{I}_a(\boldsymbol{r} \cdot d\boldsymbol{I}_b)\big) \\ &\neq 0 \end{aligned} \tag{64}$$

In the contrary, the sum of the magnetic force (49) and its back force is zero. The back magnetic force for (49) is obtained from (49) by replacing $\boldsymbol{r}_0$ with $-\boldsymbol{r}_0$, $d\boldsymbol{I}_b$ with $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_a$ with $d\boldsymbol{I}_b$:

$$\boldsymbol{F}_{Iab} = \frac{1}{4\pi\varepsilon_0 c^2|\boldsymbol{r}|^3}\Big(-(-\boldsymbol{r})(d\boldsymbol{I}_b \cdot d\boldsymbol{I}_a) + d\boldsymbol{I}_b\big((-\boldsymbol{r}) \cdot d\boldsymbol{I}_a\big) + d\boldsymbol{I}_a\big((-\boldsymbol{r}) \cdot d\boldsymbol{I}_b\big)\Big) \tag{65}$$

Adding $\boldsymbol{F}_{Iba}$ given in (49) with $\boldsymbol{F}_{Iab}$ yields $\boldsymbol{F}_{Iba} + \boldsymbol{F}_{Iab} = 0$ because all the terms cancel out :

$$\boldsymbol{F}_{Iba} + \boldsymbol{F}_{Iab} = \frac{1}{4\pi\varepsilon_0 c^2|\boldsymbol{r}|^3}\begin{pmatrix} -\boldsymbol{r}(d\boldsymbol{I}_a \cdot d\boldsymbol{I}_b) + d\boldsymbol{I}_a(\boldsymbol{r} \cdot d\boldsymbol{I}_b) + d\boldsymbol{I}_b(\boldsymbol{r} \cdot d\boldsymbol{I}_a) \\ -(-\boldsymbol{r})(d\boldsymbol{I}_b \cdot d\boldsymbol{I}_a) + d\boldsymbol{I}_b\big((-\boldsymbol{r}) \cdot d\boldsymbol{I}_a\big) + d\boldsymbol{I}_a\big((-\boldsymbol{r}) \cdot d\boldsymbol{I}_b\big) \end{pmatrix} = 0 \tag{66}$$

So, the magnetic force law (49) satisfies the Newton's third law for current elements .

Why the Lorentz force law violates the Newton's third law for current elements? By comparing (62) with (49) we find that (62) lacks the term $\frac{d\boldsymbol{I}_a(\boldsymbol{r}\cdot d\boldsymbol{I}_b)}{|\boldsymbol{r}|^3}$. Because all experiments are done with closed loop coil over which the integral of this term equals zero, see (54), the magnetic force corresponding to this term never appear and does not exist in experiments. Being an experimental law, the Lorentz force law does not describe a force that does not exist and thus, lacks this term. So, it cannot satisfy Newton's third law.

Thanks to the fully theoretical derivation, the magnetic force law (49) contains the missing term $\frac{d\boldsymbol{I}_a(\boldsymbol{r}\cdot d\boldsymbol{I}_b)}{|\boldsymbol{r}|^3}$ and consequently, satisfies Newton's third law.

6. Experimental evidences

a) My experiments

What is the magnetic force corresponding to the term $\frac{d\boldsymbol{I}_a(\boldsymbol{r}\cdot d\boldsymbol{I}_b)}{|\boldsymbol{r}|^3}$? This force is parallel to the current element $d\boldsymbol{I}_a$; such force has never been detected so far. Since this force did not appear in experiments because the test coils were all closed loop, we have specially designed experiments with non-closed wire and successfully shown magnetic force parallel to the current.

The first experiment is «Continuous rotation of a circular coil experiment»[5]. The video of this experiment is: https://www.youtube.com/watch?v=9162Qw-wNow[6]. In this video we see a round coil that rotates in its plane. Because the coil is round the driving force must be parallel to the wire, that is, the driving force is parallel to the current. This force cannot be Lorentz force which is perpendicular to the current. A detailed technical explanation is in the paper «Showing tangential magnetic force by experiment»[7] .

I have also made a « Circular motor driven by tangential magnetic force »[8] . The video of this experiment is: https://www.youtube.com/watch?v=JkGUaJqa6nU&list=UUuJXMstqPh8VY4UYqDgwcvQ[9]. The technical details of this experiment is: « Detail of my circular motor using tangential force and the equivalence with homopolar motor » [10].

[5] Kuan Peng, 2017, «Continuous rotation of a circular coil experiment», https://www.academia.edu/33604205/Continuous_rotation_of_a_circular_coil_experiment
[6] Kuan Peng, 2017, Video https://www.youtube.com/watch?v=9162Qw-wNow
[7] Kuan Peng, 2018, «Showing tangential magnetic force by experiment», https://www.academia.edu/36652163/Showing_tangential_magnetic_force_by_experiment
[8] Kuan Peng, 2014, « Circular motor driven by tangential magnetic force », https://www.academia.edu/6227926/Circular_motor_driven_by_tangential_magnetic_force
[9] Kuan Peng, 2014, Video https://www.youtube.com/watch?v=JkGUaJqa6nU&list=UUuJXMstqPh8VY4UYqDgwcvQ
[10] Kuan Peng, 2014, « Detail of my circular motor using tangential force and the equivalence with homopolar motor » , https://www.academia.edu/7879755/Detail_of_my_circular_motor_using_tangential_force_and_the_equivalence_with_homopolar_motor

b) Experiment of wire fragmentation

In 1961, Jan Nasilowski in Poland has carried out an experiment which consisted of passing a huge current in a thin wire. The wire exploded into small pieces. The interesting thing is that the wires were not melted but teared apart by mechanical force. The Figure 9 is a photograph of the exploded wires which shows that all the small pieces have approximately the same length. Jan Nasiowski has published his result in two papers [11, 12] which are cited by Lars Johansson in his Thesis[13] .

Figure 9

The magnetic force shown in this experiment is parallel to the current and is strong enough to tear the wire apart. Let us explain this experiment with the magnetic force law (49). Take two current elements from the wire, $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$, the vector distance from $d\boldsymbol{I}_b$ to $d\boldsymbol{I}_a$ is $\boldsymbol{r}$. Let us compute the magnetic force that $d\boldsymbol{I}_b$ exerts on $d\boldsymbol{I}_a$ using (52). Because $d\boldsymbol{I}_b$ is parallel to $\boldsymbol{r}$, then $d\boldsymbol{I}_b \times \boldsymbol{r} = 0$ and the magnetic force is

$$\boldsymbol{F}_{Iba} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{d\boldsymbol{I}_a(\boldsymbol{r}\cdot d\boldsymbol{I}_b)}{|\boldsymbol{r}|^3} \quad (67)$$

The back magnetic force that $d\boldsymbol{I}_a$ exerts on $d\boldsymbol{I}_b$ is

$$\boldsymbol{F}_{Iab} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{d\boldsymbol{I}_b(-\boldsymbol{r}\cdot d\boldsymbol{I}_a)}{|\boldsymbol{r}|^3} \quad (68)$$

$d\boldsymbol{I}_b$, $d\boldsymbol{I}_a$ and $\boldsymbol{r}$ are parallel, we write them in (69) with $\boldsymbol{e}_r$ being their common unit vector :

$$d\boldsymbol{I}_a = I_a dl_a \boldsymbol{e}_r, \quad d\boldsymbol{I}_b = I_b dl_b \boldsymbol{e}_r, \quad \boldsymbol{r} = r\boldsymbol{e}_r \quad (69)$$

We introduce the expressions for $d\boldsymbol{I}_b$, $d\boldsymbol{I}_a$ and $\boldsymbol{r}$ into (67) and (68) :

$$\boldsymbol{F}_{Iba} = \frac{I_a dl_a \boldsymbol{e}_r(r\boldsymbol{e}_r \cdot I_b dl_b \boldsymbol{e}_r)}{4\pi\varepsilon_0 c^2 r^3} = \frac{r I_a dl_a I_b dl_b}{4\pi\varepsilon_0 c^2 r^3}\boldsymbol{e}_r \quad (70)$$

$$\boldsymbol{F}_{Iab} = \frac{I_b dl_b \boldsymbol{e}_r\big((-r\boldsymbol{e}_r)\cdot I_a dl_a \boldsymbol{e}_r\big)}{4\pi\varepsilon_0 c^2 r^3} = -\frac{r I_a dl_a I_b dl_b}{4\pi\varepsilon_0 c^2 r^3}\boldsymbol{e}_r \quad (71)$$

We have drawn in Figure 10 the forces $\boldsymbol{F}_{Iba}$ and $\boldsymbol{F}_{Iab}$ and the current elements $d\boldsymbol{I}_a$ and $d\boldsymbol{I}_b$. Let S be a point on the wire. $d\boldsymbol{I}_a$ is on the right of the point S and $\boldsymbol{F}_{Iba}$ pulls it to the right; $d\boldsymbol{I}_b$ is on the left of the point S and $\boldsymbol{F}_{Iab}$ pulls it to the left. So, the point S is under a tension that tears it. If the tension is strong enough, the wire breaks, which was the result of Jan Nasilowski's experiment.

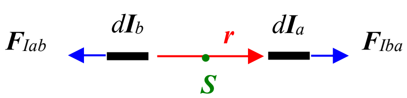

Figure 10

[11] Jan Nasiowski Phenomena Connected with the Disintegration of Conductors Overloaded by Short-Circuit Current (in Polish) Przeglad Elektrotechniczny, 1961, p.397-403 52

[12] Jan Nasiowski Unduloids and striated Disintegration of Wires Exploding Wires, W.G. Chase, H.K. Moore Eds., Vol.3, Plenum, N.Y., 1964

[13] Lars Johansson, 1996, "Longitudinal electrodynamic forces | and their possible technological applications", https://deanostoybox.com/hot-streamer/temp/LongitudinalMSc.pdf

A wire is broken at a point by the tension created by the segments of the wire on either side of the point . The magnetic force per unit length is the same everywhere in the wire because the current is constant. The segments of the same length create the same tension. This is why all the pieces of an exploded wire have approximately the same length.

In consequence, the magnetic force law (49) explains well the experiment of Jan Nasiowski.

7. Conclusion

1. We have derived a new magnetic force law from Coulomb's law and relativity without experimental data.
2. We have proven theoretically the relation $\mu_0\varepsilon_0c^2 = 1$ which was an experimental law. Now the three fundamental constants μ_0, ε_0 and c are reduced into two primary constants : ε_0 and c.
3. The Biot–Savart law and the Lorentz force law are derived with pure theory.
4. This new law has a component of magnetic force parallel to the current. Because the Lorentz force law lacks this force, it violates the Newton's third law.
5. We have found two relativistic effects: the relativistic dynamic effect and the changing distance effect. These effects are the deep mechanism that creates magnetic force from electric force.
6. We have presented two of my experiments that show the existence of magnetic force parallel to the current.
7. Our new magnetic force law explains well the experiment of Jan Nasiowski : for the breaking of the wires as well as for the regularity of the lengths of the small pieces of an exploded wire.
8. These experiments give strong evidences for the existence of magnetic force parallel to the current.

So, the new magnetic force law (49) correctly describes magnetic force. Because the new law gives the same prediction as the Lorentz force law for closed loop currents, it works for electromagnetism as the Lorentz force law. However, the component of magnetic force parallel to the current is new and shown to be rather significant. So, it could be used as the driving force for new devices.

Since the Biot–Savart law, the Lorentz force law and the relation $\mu_0\varepsilon_0c^2 = 1$ are derived with pure theory, the deep mechanism that transforms electric force into magnetic force is revealed to be the two relativistic effects. Consequently, electromagnetism is much better understood, which will surely unblock the way to unsuspected physical discoveries.