About an integral in Vălean’s book

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abstract

The evaluation of integrals is an important subject in mathematics, physics and applied sciences. In this note we give some integrals for π^3

1. Introduction

In [1],[2, pp.17, (1.49)],[3, pp. 21, (3.87)] appears the formula

$$\pi^3 = 16 \int_0^1 \frac{\arctan(x)}{x} \ln\left(\frac{1+x^2}{(1-x)^2}\right) dx = 16 \int_0^1 \frac{\arctan(x)\ln\left(1+x^2\right)}{x} dx - 32 \int_0^1 \frac{\arctan(x)\ln(1-x)}{x} dx \tag{1}$$

In this note we give some formulas ralated to (1).

2. Formulas

Entry 1.

$$\pi^3 = -16 \int_0^1 \frac{\ln(1-x)}{x\sqrt{1-x^2}} \arctan\left(\frac{x}{1+\sqrt{1-x^2}}\right) dx \tag{2}$$

$$\pi^3 = -8 \int_0^{\pi/2} \frac{x}{\sin x} \ln(1-\sin x)\, dx \tag{3}$$

$$\pi^3 = -8 \int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) \frac{\ln(1-\cos x)}{\cos x} dx \tag{4}$$

$$\pi^3 = -16 \int_0^{\infty} \arctan\left(\tanh\left(\frac{x}{2}\right)\right) \frac{\ln(1-\tanh x)}{\sinh x} dx \tag{5}$$

$$\pi^3 = -16 \int_0^{\infty} \arctan(e^{-x}) \ln(1-\operatorname{sech} x)\, dx \tag{6}$$

$$\pi^3 = 8 \int_0^{\infty} \sqrt{\frac{e^{-x}}{2-e^{-x}}} \frac{x \arcsin(1-e^{-x})}{1-e^{-x}} dx \tag{7}$$

$$\pi^3 = -32 \int_0^{\pi/4} \frac{x}{\cos x \sin x} \ln(\cos x - \sin x)\, dx \tag{8}$$

$$\pi^3 = -8 \int_0^1 \frac{\arcsin x \ln(1-x)}{x\sqrt{1-x^2}} dx \tag{9}$$

$$\pi^3 = -8 \int_0^{\infty} \arcsin(\tanh x) \frac{\ln(1-\tanh x)}{\sinh x} dx \tag{10}$$

$$\pi^3 = 16\int_0^\infty \arctan(e^{-x})\ln\left(1+e^{-2x}\right)dx - 32\int_0^\infty \arctan(e^{-x})\ln(1-e^{-x})\,dx \tag{11}$$

$$\pi^3 = -8\int_0^1 \arccos x\,\frac{\ln\left(1-\sqrt{1-x^2}\right)}{1-x^2}\,dx \tag{12}$$

$$\pi^3 = 8\int_0^1 \ln\left(\frac{1+\sqrt{1-x^2}}{x^2}\right)\frac{\arccos x}{1-x^2}\,dx \tag{13}$$

$$\pi^3 = -8\int_0^1 \frac{\ln(1-x)}{x}\,\frac{\arccos\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}\,dx \tag{14}$$

$$\pi^3 = -8\int_0^\infty \frac{\arctan(x)}{x\sqrt{1+x^2}}\ln\left(1-\frac{x}{\sqrt{1+x^2}}\right)dx \tag{15}$$

$$\pi^3 = 8\int_0^\infty \frac{\arctan(x)}{x\sqrt{1+x^2}}\ln\left(1+x^2+x\sqrt{1+x^2}\right)dx \tag{16}$$

$$\pi^3 = 8\int_1^\infty \frac{\ln\left(x^2+x\sqrt{x^2-1}\right)}{x^2-1}\arccos\left(\frac{1}{x}\right)dx \tag{17}$$

$$\pi^3 = 32\int_0^\infty \arctan\left(e^{-2x}\right)\ln\left(\frac{\cosh(2x)}{2(\sinh x)^2}\right)dx \tag{18}$$

$$\pi^3 = 8\int_0^\infty \frac{\arctan(x)}{x\sqrt{1+x^2}}\ln\left(\frac{2+2x^2-2\sqrt{1+x^2}}{\left(1+x-\sqrt{1+x^2}\right)^2}\right)dx \tag{19}$$

$$\pi^3 = -8\int_0^\infty \arctan\left(\frac{1}{\sinh x}\right)\ln(1-\operatorname{sech} x)\,dx \tag{20}$$

$$\pi^3 = -8\int_0^\infty \arcsin\left(\frac{1}{\cosh x}\right)\ln(1-\operatorname{sech} x)\,dx \tag{21}$$

$$\pi^3 = 32\int_1^\infty \frac{1}{x^2-1}\arctan\left(\frac{x-1}{x+1}\right)\ln\left(\frac{1+x^2}{2}\right)dx \tag{22}$$

$$\pi^3 = 64\int_0^\infty \frac{x\,e^{-x}}{\left(1-e^{-2x}\right)\sqrt{2-e^{-2x}}}\arctan\left(\frac{1-e^{-2x}}{1+e^{-x}\sqrt{2-e^{-2x}}}\right)dx \tag{23}$$

$$\pi^3 = 64\int_0^{\sqrt{2}-1} \frac{1+x^2}{x\left(1-x^2\right)}\arctan(x)\ln\left(\frac{1+x^2}{1-2x-x^2}\right)dx \tag{24}$$

$$\pi^3 = -32\int_0^{\pi/4} \frac{x}{\cos x\sin x}\ln\left(\sqrt{2}\sin\left(\frac{\pi}{4}-x\right)\right)dx \tag{25}$$

$$\pi^3 = -32\int_0^{\pi/4} \frac{x}{\cos x\sin x}\ln\left(\sqrt{2}\cos\left(\frac{\pi}{4}+x\right)\right)dx \tag{26}$$

$$\pi^3 = -64\int_0^{\pi/4} \frac{\ln\left(\sqrt{2}\sin(x)\right)}{\cos(2x)}\left(\frac{\pi}{4}-x\right)dx \tag{27}$$

$$\pi^3 = 64\int_{\pi/4}^{\pi/2} \frac{\ln\left(\sqrt{2}\cos(x)\right)}{\cos(2x)}\left(x-\frac{\pi}{4}\right)dx \tag{28}$$

$$\pi^3 = 32\int_0^1 \frac{\arctan(x)}{x}\,\operatorname{arctanh}\left(\frac{x}{1-x+x^2}\right)dx \tag{29}$$

$$\pi^3 = 32\int_0^1 \frac{\operatorname{arctanh}(x)}{x\sqrt{1+2x-3x^2}}\,\arctan\left(\frac{2x}{1+x+\sqrt{1+2x-3x^2}}\right)dx \tag{30}$$

$$\pi^3 = 32\int_0^\infty \arctan(e^{-x})\,\operatorname{arctanh}\left(\frac{1}{2\cosh(x)-1}\right)dx \tag{31}$$

$$\pi^3 = 16\int_0^\infty \arctan\left(\frac{1}{\sinh(x)}\right)\operatorname{arctanh}\left(\frac{1}{2\cosh(x)-1}\right)dx \tag{32}$$

$$\pi^3 = 64\int_0^1 \frac{\arctan(x)}{x}\,\operatorname{arctanh}\left(\frac{\sqrt{1+x^2}+x-1}{\sqrt{1+x^2}-x+1}\right)dx \tag{33}$$

$$\pi^3 = 32\int_0^1 \frac{\arctan(x)}{x}\,\operatorname{arctanh}\left(\frac{x^2}{2+x^2}\right)dx + 64\int_0^1 \frac{\arctan(x)}{x}\,\operatorname{arctanh}\left(\frac{x}{2-x}\right)dx \tag{34}$$

$$\pi^3 = 32\int_0^\infty \arctan(e^{-x})\,\operatorname{arctanh}\left(\frac{e^{-2x}}{2+e^{-2x}}\right)dx + 64\int_0^\infty \arctan(e^{-x})\,\operatorname{arctanh}\left(\frac{e^{-x}}{2-e^{-x}}\right)dx \tag{35}$$

$$\pi^3 = 16\int_0^\infty \frac{\ln(1+x)}{x\sqrt{1+2x}}\,\arctan\left(\frac{x}{1+x+\sqrt{1+2x}}\right)dx \tag{36}$$

$$\pi^3 = 32\int_0^\infty \frac{\ln(1+x)}{x\sqrt{1+2x}}\,\arctan\left(\frac{x}{1+x+\sqrt{1+2x}+\sqrt{2(1+x)\left(1+x+\sqrt{1+2x}\right)}}\right)dx \tag{37}$$

Entry 2.

$$\pi^3 = 32\int_1^\infty \frac{\arctan(1-f(x))}{1-f(x)}\,g(x)\ln(x)\,dx \tag{38}$$

where

$$f(x) = \sqrt[3]{\sqrt{\frac{1}{27x^9}+\frac{1}{x^6}}+\frac{1}{x^3}} - \sqrt[3]{\sqrt{\frac{1}{27x^9}+\frac{1}{x^6}}-\frac{1}{x^3}} \tag{39}$$

$$g(x) = \frac{1}{3}\left(\frac{3}{x^4}-\frac{1+18x^3}{2x^{10}\sqrt{\frac{1}{3x^9}+\frac{9}{x^6}}}\right)\left(\sqrt{\frac{1}{27x^9}+\frac{1}{x^6}}-\frac{1}{x^3}\right)^{-2/3} + \frac{1}{3}\left(\frac{3}{x^4}+\frac{1+18x^3}{2x^{10}\sqrt{\frac{1}{3x^9}+\frac{9}{x^6}}}\right)\left(\sqrt{\frac{1}{27x^9}+\frac{1}{x^6}}+\frac{1}{x^3}\right)^{-2/3} \tag{40}$$

3. References

[1] C. I. Vălean, Problem 12054, American Mathematical Monthly, Vol.125, June-July, 2018.

[2] C. I. Vălean, (Almost) Impossible Integrals, Sums, and Series. Problem Books In Mathematics. Springer, 2019.

[3] Abdulhafeez A. Abdulsalam, New closed forms for a class of digamma series and integrals. arXiv:2308.04362v1 [math CA] 8 Aug 2023.