Frenet's trihedron of the second order

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Abstract

Based on the remarkable property of the Darboux vector to be perpendicular to the normal, I define a new trihedron associated with curves in space and prove that this trihedron also satisfies Frenet's formulas. Unlike the previous paper ([2]), where I used the trigonometric form of Frenet's formulas for simplicity, in this paper I construct a proof based only on curvature and torsion, respectively, darbuzian and lancretian.

Key words: Frenet formulas, Frenet matrix, Frenet trihedron, curvature, torsion, Darboux vector, recursiveness

MSC: 53A04, 15B10, 85-10, 70F10

1 Introduction

Thoroughly studying elements of the theory of curves in space from vast and deep works ([6], [8]), I synthesized the formulated ideas and found that several trihedrals associated with a curve can be constructed, which trihedrals also respect Frenet's formulas. In the following, in order to be able to describe my findings as fluently as possible, I will use the notations recently established in the field of the differential geometry of curves in space ([10], [11]). Thus, for reasons of clarity, I will simply note it without an arrow above T, N and B tangent, normal and binormal versors, versors that can be associated with any smooth curve in space and any vector field line. I will also note with a dot above the derivative of the quantities with respect to the canonical parameter. I also denote by κ the curvature of the curve at a certain point of the curve and by τ the torsion of the curve at the same point.

Under these conditions, the famous formulas of Frenet can be written in matrix form ([11]), as follows:

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

Noting $\mathbf{T} = \begin{pmatrix} T \\ N \\ B \end{pmatrix}$ and $\mathbf{F} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$,

Frenet's formulas can be written more condensed

$$\dot{\mathbf{T}} = \mathbf{F} \cdot \mathbf{T}.$$

I also remind you that Darboux's vector ([9], [10]), also noted here with Ω, is the vector with the property that

$$\dot{T} = \Omega \times T,$$

$$\dot{N} = \Omega \times N,$$

$$\dot{B} = \Omega \times B$$

and can be interpreted as the "speed of rotation" of Frenet's trihedron.

Then it can be written

$$\Omega = \tau T + \kappa B,$$

matters already well known to informed readers.

Next I will note with $d = \sqrt{\kappa^2 + \tau^2}$ and with $l = \frac{\kappa}{\tau}$, calling these parameters "darbuzian" and "lancretian" respectively, in honor of the French mathematicians Darboux and Lancret whose contribution to the theory of curves was, as is well known ([3], [5]), overwhelming.

2 Definition of the second-order trihedron

Due to the fact that the Darboux vector has no component on the normal, being thus perpendicular to the normal, I asked myself whether an interesting trihedron can be constructed starting from this remarkable property. The answer is positive.

For this, similar to the proposal of the Croatian mathematician Stanko Bilinski ([1]), who defined the second-order tangent as the first-order normal, I will define as a tangent versor of the second order exactly the versor of the Darboux vector, as I did in the previous work ([2]) which I drafted at a time when I was unaware of Mr. Bilinski's results. I mean, I'm going to write

$$T_2 = \frac{1}{d}(\tau T + \kappa B) = \frac{\tau}{d}T + \frac{\kappa}{d}B.$$

Then I define the binormal versor of the second order as the opposite of the normal versor, that is

$$B_2 = -N.$$

Finally, the second-order normal versor will be given by the vector product between the second-order binormal and the second-order tangent, defined above. Namely

$$N_2 = B_2 \times T_2 = -N \times (\frac{\tau}{d}T + \frac{\kappa}{d}B) =$$

$$= -\frac{\tau}{d}N \times T - \frac{\kappa}{d}N \times B =$$

$$= \frac{\tau}{d}B - \frac{\kappa}{d}T = \frac{1}{d}(-\kappa T + \tau B) = \frac{1}{d}\dot{N}.$$

Synthesizing in matrix form, we have Frenet's trihedral of the second order:

$$\begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix} = \frac{1}{d}\begin{pmatrix} \tau & 0 & \kappa \\ -\kappa & 0 & \tau \\ 0 & -d & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

relationship that can be written even more condensed

$$\mathbf{T_2} = \frac{1}{d}\mathbf{A} \cdot \mathbf{T},$$

where I noted

$$\mathbf{A} = \begin{pmatrix} \tau & 0 & \kappa \\ -\kappa & 0 & \tau \\ 0 & -d & 0 \end{pmatrix}.$$

3 Derivatives of second-order versors written in terms of first-order versors

I will now calculate the derivatives of the second-order versors in turn with respect to the canonical parameter and write the results in terms of the first-order versors.

3.1 Derivative of the tangent of the second order

$$\dot{T}_2 = \frac{d}{ds}\left(\frac{\tau}{d}T + \frac{\kappa}{d}B\right) =$$

$$= \frac{\dot{\tau}d - \dot{d}\tau}{d^2}T + \frac{\tau}{d}\dot{T} + \frac{\dot{\kappa}d - \dot{d}\kappa}{d^2}B + \frac{\kappa}{d}\dot{B} =$$

$$= \frac{\dot{\tau}d - \dot{d}\tau}{d^2}T + \frac{\tau\kappa}{d}N + \frac{\dot{\kappa}d - \dot{d}\kappa}{d^2}B - \frac{\tau\kappa}{d}N =$$

$$= \frac{\dot{\tau}d - \dot{d}\tau}{d^2}T + \frac{\dot{\kappa}d - \dot{d}\kappa}{d^2}B =$$

$$= \frac{1}{d^2}[(\dot{\tau}d - \dot{d}\tau)T + (\dot{\kappa}d - \dot{d}\kappa)B] =$$

$$= \left(\frac{\dot{\tau}}{d}\right)T + \left(\frac{\dot{\kappa}}{d}\right)B.$$

3.2 The derivative of the second order normal

$$\dot{N}_2 = \frac{d}{ds}\left(\frac{\tau}{d}B - \frac{\kappa}{d}T\right) =$$

$$= \left(\frac{\dot{\tau}}{d}\right)B + \frac{\tau}{d}\dot{B} - \left(\frac{\dot{\kappa}}{d}\right)T - \frac{\kappa}{d}\dot{T} =$$

$$= \left(\frac{\dot{\tau}}{d}\right)B - \frac{\tau^2}{d}N - \left(\frac{\dot{\kappa}}{d}\right)T - \frac{\kappa^2}{d}N =$$

$$= \left(\frac{\dot{\tau}}{d}\right)B - dN - \left(\frac{\dot{\kappa}}{d}\right)T =$$

$$= -\left(\frac{\dot{\kappa}}{d}\right)T - dN + \left(\frac{\dot{\tau}}{d}\right)B.$$

3.3 The derivative of the second order binormal

$$\dot{B}_2 = -\dot{N} = -(-\kappa T + \tau B) = \kappa T - \tau B.$$

3.4 Matrix synthesis of the results

The previous results can be summarized matrixally as follows:

$$\begin{pmatrix} \dot{T}_2 \\ \dot{N}_2 \\ \dot{B}_2 \end{pmatrix} = \begin{pmatrix} \dot{\left(\frac{\tau}{d}\right)} & 0 & \dot{\left(\frac{\kappa}{d}\right)} \\ -\left(\frac{\kappa}{d}\right) & -d & \left(\frac{\tau}{d}\right) \\ \kappa & 0 & -\tau \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

4 Derivatives of second order versors written in terms of second order versors

Now I will write the derivatives of the second-order versors in terms of the second-order versors, not the first-order ones. So, how can we write $\dot{T}_2$, $\dot{B}_2$, and $\dot{B}_2$ in terms of T_2 , N_2 , and B_2 ?

We will have to invert the matrix

$$\mathbf{A} = \begin{pmatrix} \tau & 0 & \kappa \\ -\kappa & 0 & \tau \\ 0 & -d & 0 \end{pmatrix}$$

of passing from the relation that connects the second-order versors to the first-order versors, i.e. from the relation

$$\begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} \tau & 0 & \kappa \\ -\kappa & 0 & \tau \\ 0 & -d & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix}.$$

After we invert it, we'll multiply it to the left in this relation, and we'll get

an inverse relation where we can write the first-order versors in terms of the second-order versors, a necessary relation for our goal.

4.1 Calculation of the inverse of the matrix A

So we calculate $\mathbf{A}^{-1}$.

We have first

$$\det \mathbf{A} = \kappa^2 d + \tau^2 d = d^3.$$

Then

$$\mathbf{A}^t = \begin{pmatrix} \tau & -\kappa & 0 \\ 0 & 0 & -d \\ \kappa & \tau & 0 \end{pmatrix}.$$

Because of the fact that

$a_{11} = \tau d$, $a_{12} = -\kappa d$, $a_{13} = 0$

$a_{21} = 0$, $a_{22} = 0$, $a_{23} = -d^2$

$a_{31} = \kappa d$, $a_{32} = \tau d$, $a_{33} = 0$,

we will have

$$\mathbf{A}^* = \begin{pmatrix} \tau d & -\kappa d & 0 \\ 0 & 0 & -d^2 \\ \kappa d & \tau d & 0 \end{pmatrix},$$

which allows us to obtain the inverse matrix we are looking for:

$$\mathbf{A}^{-1} = \frac{1}{d^2} \begin{pmatrix} \tau & -\kappa & 0 \\ 0 & 0 & -d \\ \kappa & \tau & 0 \end{pmatrix} = \frac{1}{d^2} \cdot \mathbf{A}^t.$$

4.2 Writing the inverse relationship that gives the first-order versors as a function of the second-order versors

Now multiplying to the left by the matrix $\mathbf{A}^{-1}$ in relation

$$\begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} \tau & 0 & \kappa \\ -\kappa & 0 & \tau \\ 0 & -d & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

as we set out, we achieve

$$\frac{1}{d^2} \begin{pmatrix} \tau & -\kappa & 0 \\ 0 & 0 & -d \\ \kappa & \tau & 0 \end{pmatrix} \cdot \begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix} = \frac{1}{d^2} \begin{pmatrix} \tau & -\kappa & 0 \\ 0 & 0 & -d \\ \kappa & \tau & 0 \end{pmatrix} \cdot \frac{1}{d} \begin{pmatrix} \tau & 0 & \kappa \\ -\kappa & 0 & \tau \\ 0 & -d & 0 \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

so

$$\frac{1}{d^2} \begin{pmatrix} \tau & -\kappa & 0 \\ 0 & 0 & -d \\ \kappa & \tau & 0 \end{pmatrix} \cdot \begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix} = \frac{1}{d^3} \begin{pmatrix} d^2 & 0 & 0 \\ 0 & d^2 & 0 \\ 0 & 0 & d^2 \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix} = \frac{1}{d} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

namely

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \frac{1}{d} \begin{pmatrix} \tau & -\kappa & 0 \\ 0 & 0 & -d \\ \kappa & \tau & 0 \end{pmatrix} \cdot \begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix}.$$

Now we can, finally, rewrite the relation that gave us the derivatives with respect to the first-order versors

$$\begin{pmatrix} \dot{T}_2 \\ \dot{N}_2 \\ \dot{B}_2 \end{pmatrix} = \begin{pmatrix} \dot{\left(\frac{\tau}{d}\right)} & 0 & \dot{\left(\frac{\kappa}{d}\right)} \\ -\dot{\left(\frac{\kappa}{d}\right)} & -d & \dot{\left(\frac{\tau}{d}\right)} \\ \kappa & 0 & -\tau \end{pmatrix} \cdot \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

thus

$$\begin{pmatrix} \dot{T}_2 \\ \dot{N}_2 \\ \dot{B}_2 \end{pmatrix} = \begin{pmatrix} \dot{\left(\frac{\tau}{d}\right)} & 0 & \dot{\left(\frac{\kappa}{d}\right)} \\ -\left(\frac{\kappa}{d}\right) & -d & \left(\frac{\tau}{d}\right) \\ \kappa & 0 & -\tau \end{pmatrix} \cdot \frac{1}{d} \begin{pmatrix} \tau & -\kappa & 0 \\ 0 & 0 & -d \\ \kappa & \tau & 0 \end{pmatrix} \cdot \begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix},$$

namely

$$\begin{pmatrix} \dot{T}_2 \\ \dot{N}_2 \\ \dot{B}_2 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} \dot{\left(\frac{\tau}{d}\right)} & 0 & \dot{\left(\frac{\kappa}{d}\right)} \\ -\left(\frac{\kappa}{d}\right) & -d & \left(\frac{\tau}{d}\right) \\ \kappa & 0 & -\tau \end{pmatrix} \cdot \begin{pmatrix} \tau & -\kappa & 0 \\ 0 & 0 & -d \\ \kappa & \tau & 0 \end{pmatrix} \cdot \begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix}.$$

Multiplying the matrices that appear in the previous relation, we get

$$\begin{pmatrix} \dot{T}_2 \\ \dot{N}_2 \\ \dot{B}_2 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} \tau\dot{\left(\frac{\tau}{d}\right)} + \kappa\dot{\left(\frac{\kappa}{d}\right)} & -\kappa\dot{\left(\frac{\tau}{d}\right)} + \tau\dot{\left(\frac{\kappa}{d}\right)} & 0 \\ -\tau\left(\frac{\kappa}{d}\right) + \kappa\left(\frac{\tau}{d}\right) & \kappa\left(\frac{\kappa}{d}\right) + \tau\left(\frac{\tau}{d}\right) & d^2 \\ 0 & -d^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix}.$$

But considering the wonderful relationship

$$\begin{aligned} \tau\dot{\left(\frac{\tau}{d}\right)} + \kappa\dot{\left(\frac{\kappa}{d}\right)} &= \frac{\tau d\dot{\tau} - \tau^2\dot{d} + \kappa d\dot{\kappa} - \kappa^2\dot{d}}{d^2} = \\ &= \frac{\tau\dot{\tau} + \kappa\dot{\kappa}}{d} - \frac{(\kappa^2+\tau^2)\dot{d}}{d^2} = \\ &= \frac{\tau\dot{\tau} + \kappa\dot{\kappa}}{d} - \dot{d} = \\ &= \frac{\tau\dot{\tau} + \kappa\dot{\kappa}}{d} - \frac{d}{ds}\sqrt{\kappa^2+\tau^2} = \\ &= \frac{\tau\dot{\tau} + \kappa\dot{\kappa}}{d} - \frac{2(\kappa\dot{\kappa} + \tau\dot{\tau})}{2\sqrt{\kappa^2+\tau^2}} = 0, \end{aligned}$$

we get that the last matrix relation becomes

$$\begin{pmatrix} \dot{T}_2 \\ \dot{N}_2 \\ \dot{B}_2 \end{pmatrix} = \frac{1}{d} \begin{pmatrix} 0 & -\kappa\dot{\left(\frac{\tau}{d}\right)} + \tau\dot{\left(\frac{\kappa}{d}\right)} & 0 \\ -\tau\dot{\left(\frac{\kappa}{d}\right)} + \kappa\dot{\left(\frac{\tau}{d}\right)} & 0 & d^2 \\ 0 & -d^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix}.$$

Finally, if we write $\kappa_2 = \frac{1}{d} \cdot \left(-\kappa\dot{\left(\frac{\tau}{d}\right)} + \tau\dot{\left(\frac{\kappa}{d}\right)}\right)$ and $\tau_2 = d$, then the last matrix relation can be written suggestively

$$\begin{pmatrix} \dot{T}_2 \\ \dot{N}_2 \\ \dot{B}_2 \end{pmatrix} = \begin{pmatrix} 0 & \kappa_2 & 0 \\ -\kappa_2 & 0 & \tau_2 \\ 0 & -\tau_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} T_2 \\ N_2 \\ B_2 \end{pmatrix},$$

relationship in which you can easily recognize *Frenet's second-order formulas* which relates the derivatives of the second-order versors to the second-order versors. Thus, we have achieved the goal of proving that not only the first-order Frenet trihedron obeys Frenet's formulas, but also *second-order trihedron*!

5 About second-order curvature

I would like to add the fact that the second-order curvature, denoted by κ_2, it can also be written still

$$\kappa_2 = \frac{1}{d} \cdot \left(-\kappa \left(\frac{\dot{\tau}}{d} \right) + \tau \left(\frac{\dot{\kappa}}{d} \right) \right) =$$

$$= \frac{1}{d} \cdot \frac{-\kappa d\dot{\tau} + \kappa\tau\dot{d} + \tau d\dot{\kappa} - \kappa\tau\dot{d}}{d^2} =$$

$$= \frac{-\kappa d\dot{\tau} + \tau d\dot{\kappa}}{d^3} = \frac{-\kappa\dot{\tau} + \tau\dot{\kappa}}{d^2} = \frac{\dot{\left(\frac{\kappa}{\tau}\right)}\tau^2}{d^2} = \frac{\tau^2}{d^2}\dot{l} =$$

$$= \frac{1}{1+l^2}\dot{l},$$

where, I remind you, l is the lancretian, and $\dot{l}$ is its derivative according to the canonical parameter.

This last result, which can be found (with difficulty) in a similar form in other valuable works ([4], [7]), highlights the fact that the second-order curvature depends on the derivative of the lancretian. In other words, if the lancretian (the ratio of curvature to torsion) is constant, as in the case of curves called "helices" that maintain a fixed direction in space, then the second curvature is zero. This means that the second-order tangent becomes constant, being exactly the versor of that fixed direction in space around which the first-order tangent of the helix precesses.

Of course, if the derivative of the lancretian is not zero, then the second-order tangent is no longer fixed, but it also precesses around another „tangent of the third order", the phenomenon being recursive ([1], [2]) and having an important physical significance in the study of turbulent motions.

6 Final word

Of course, all the reasoning presented here can be easily generalized by mathematical induction to higher-order trihedrons, opening up new horizons for solving open problems in the curve theory. But especially Physics can bear fruit on the path we opened, redefining its notions more rigorously, then incorporating these reasonings into the study of motion of any kind, including gravitational, turbulent or quantum.

There are a lot of people I should thank for making this material exist, people without whom I would have been hungry, bored, or out of ideas. I give everyone a big hug with my intense thought.

May these notes produce a click in the mind of someone who is dissatisfied with the world in which we live and thus help him to improve it.

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