A direct, simple, and basic computation of a difference of two dilogarithms

Hervé Gandran-Tomeng.

September 19, 2023

Abstract

The computation of $\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right)$ is performed.

Introduction
Several papers ([1] [2] [3]) contain the proof that,

$$\mathrm{Li}_2\left(\sqrt{2}-1\right) - \mathrm{Li}_2\left(1-\sqrt{2}\right) = \frac{\pi^2}{8} - \frac{1}{2}\ln^2\left(\sqrt{2}-1\right)$$

Following is a direct, simple, and basic proof of this equality.

Proof:
Let,

$$\alpha = \sqrt{2}-1$$

Observe that,

$$\frac{1-\alpha}{1+\alpha} = \alpha$$

Since, for u real,

$$\mathrm{Li}_2\left(u\right) = -\int_0^u \frac{\ln(1-t)}{t}dt$$

Then,

$$
\begin{aligned}
J &= \operatorname{Li}_2(\alpha) - \operatorname{Li}_2(-\alpha) \\
&= -\int_0^{\alpha} \frac{\ln(1-t)}{t} dt + \int_0^{-\alpha} \frac{\ln(1-t)}{t} dt \\
&= -\int_0^{\alpha} \frac{\ln(1-t)}{t} dt - \underbrace{\int_{-\alpha}^{0} \frac{\ln(1-t)}{t} dt}_{w=-t} \\
&= -\int_0^{\alpha} \frac{\ln(1-t)}{t} dt + \int_0^{\alpha} \frac{\ln(1+w)}{w} dw \\
&= -\underbrace{\int_0^{\alpha} \frac{\ln\left(\frac{1-t}{1+t}\right)}{t} dt}_{z=\frac{1-t}{1+t}} = -2\int_{\alpha}^{1} \frac{\ln z}{1-z^2} dz
\end{aligned}
$$

$$
\begin{aligned}
J &= -2\underbrace{\int_0^1 \frac{\ln z}{1-z^2} dz}_{\text{IBP}} + 2\underbrace{\int_0^{\alpha} \frac{\ln z}{1-z^2} dz}_{\text{IBP}} \\
&= \underbrace{\left[\ln\left(\frac{1-z}{1+z}\right)\ln z\right]_0^1}_{=0} - \int_0^1 \frac{\ln\left(\frac{1-z}{1+z}\right)}{z} dz - \left[\ln\left(\frac{1-z}{1+z}\right)\ln z\right]_0^{\alpha} + \\
&\underbrace{\int_0^{\alpha} \frac{\ln\left(\frac{1-z}{1+z}\right)}{z} dz}_{=-J} \\
&= -\frac{1}{2}\int_0^1 \frac{\ln\left(\frac{1-z}{1+z}\right)}{z} dz - \frac{1}{2}\ln^2\alpha \\
&= \frac{\operatorname{Li}_2(1)}{2} + \frac{1}{2}\left(\underbrace{\int_0^1 \frac{\ln\left(1-z^2\right)}{z} dz}_{x=z^2} + \operatorname{Li}_2(1)\right) - \frac{1}{2}\ln^2\alpha \\
&= \frac{3\operatorname{Li}_2(1)}{4} - \frac{1}{2}\ln^2\alpha = \boxed{\frac{\pi^2}{8} - \frac{1}{2}\ln^2\alpha}
\end{aligned}
$$

NB: I assume that,

$$
\operatorname{Li}_2(1) = \frac{\pi^2}{6}
$$

References

[1] F.M.S Lima, New definite integrals and a two-term dilogarithm identity, Indagationes Mathematicae, March 2012.

[2] Jean-Christophe Pain, Relations for the difference of two dilogarithms, `https://arxiv.org/pdf/2304.03349.pdf`

[3] Seán M. Stewart, Some simple proofs of Lima's two-term dilogarithm identity, Irish Math. Soc. Bulletin, 2022.