The Prescribed Measurement Problem: Toward a Contention-Free Formulation of Quantum Physics

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Abstract

Quantum mechanics, though empirically validated, confronts numerous interpretative challenges, predominantly centered around the quantum measurement problem. Addressing these challenges, we introduce the "Prescribed Measurement Problem," which serves as an inversion to the traditional wavefunction collapse problem. Rather than axiomatizing the entire framework, our approach emphasizes the axiomatization of a sequence of prescribed measurements, highlighting their complex-phase attributes and inherent linearity. Leveraging entropy maximization techniques specific to these measurements, we recover the core elements of quantum mechanics: the Schrödinger equation, Born rule, complex Hilbert spaces, unitary evolution, and self-adjoint operators. Collectively, this approach offers a comprehensive and equivalent formulation of quantum mechanics that integrates measurement outcomes while sidestepping the traditional measurement problem.

1 Introduction

The quest for a unified and coherent understanding of the natural world has been a driving force in the evolution of physics. Through rigorous empirical observations and mathematical formalism, the field has created theoretical frameworks that have profoundly advanced our grasp of the universe. These successes, however, have not been without their challenges[1]. In constructing these frameworks, the discipline has sometimes faced foundational debates, inconsistencies, and paradoxes that pose questions about the conceptual integrity of the established theories.

In striking contrast to the persistent debates surrounding the foundations of quantum mechanics, the bedrock principles of statistical mechanics have gained almost unanimous acceptance. One could argue that this consensus stems from the theory's direct empirical basis—a straightforward yet profound axiom[2]. This axiom is rooted in what we directly observe in the laboratory: a sequence of energy measurements $E_1, E_2, \dots$ that converge to an average value $\overline{E}$. This average, mirroring the unmistakable and direct empirical data we obtain from repeated measurements, serves as an unwavering constraint within the theoretical framework:

$$\overline{E} = \sum_{q \in \mathbb{Q}} \rho(q) E(q) \tag{1}$$

To derive a probability distribution, $\rho(q)$, that maximizes entropy while adhering to this constraint, the theory employs a Lagrange multiplier equation[3].

$$\mathcal{L}(\rho, \lambda, \beta) = -k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) + \lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right) + \beta \left(\overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q) \right) \tag{2}$$

Solving this yields the well-established Gibbs measure.

$$\rho(q) = \frac{\exp(-\beta E(q))}{\sum_{q \in \mathbb{Q}} \exp(-\beta E(q))} \tag{3}$$

This axiomatic structure reflects our empirical observations and interactions with the physical world. By beginning with a sequence of prescribed energy measurements, the theory leverages rigorous mathematical techniques to determine the least biased probability distribution. This mathematical structure emerges naturally from a singular, empirically rooted constraint. This lends statistical mechanics a degree of coherence and conceptual clarity, differentiating it from quantum mechanics, which, despite its empirical validity, offers multiple contentious interpretations.

In stark contrast, quantum mechanics operates on a tapestry of abstract mathematical axioms[4, 5]. These axioms encompass a variety of concepts that form the theoretical skeleton of the discipline:

1. State spaces are described as complex Hilbert spaces.
2. Observables correspond to Hermitian operators within these Hilbert spaces.
3. Unit vectors in the Hilbert spaces represent physical states.
4. The evolution of these states follows the Schrödinger Equation.

To stitch these intricate mathematical axioms to empirical reality, the theory resorts to ad hoc elements:

5. Probability outcomes use the Born Rule.
6. Wave function collapses via the Measurement Postulate.

While the foundation of statistical mechanics moves from the empirical (the measurement constraint) to the theoretical (the probability measure), quantum mechanics moves from a preconceived theoretical landscape (the wavefunction/complex Hilbert spaces) to an empirical endpoint (Born rule/measurements); that is, in the opposite direction. This fundamental difference in direction may account for the general consensus around the axiomatic structure of statistical mechanics and the ongoing controversies in the foundations of quantum mechanics.

The central aim of this paper is to introduce and explore the "Prescribed Measurement Problem." This new approach seeks to build quantum mechanics based on a method similar to statistical mechanics: starting from prescribed measurements and using them as the primary constraint. By then maximizing the entropy, we aim to recover the full foundation of quantum mechanics not as axioms, but as a central theorem.

The challenge of the wavefunction collapse problem lies in elucidating how a deterministically evolving wavefunction can become non-deterministic upon observation. In contrast, the prescribed measurement problem seeks to interpret a series of non-deterministic measurements as a constraint to devise an optimal theoretical framework. In this sense, the prescribed measurement problem is an inversion of the wavefunction collapse problem.

Our endeavor is far more than a revision of existing theories; it calls for a profound shift in the very philosophy underlying the foundations of physics. In stark contrast to the prevalent view that the empirical validity of a set of axioms is sufficient for their adoption, we argue that this is not nearly enough to be free of contention. We assert that the only axioms capable of such must themselves directly represent measurements made in nature—our unequivocal, verifiable touchpoints with reality. It is not just that our axioms should align with empirical data; rather, they should be the empirical data. Both our mathematical and empirical starting points should therefore focus on a prescribed sequence of measurements. This not only forges a direct pathway from empirical observation to theoretical formulation, but also aims to correct what we identify as a departure from essential empiricism. This has led the field into persistent ontological debates and interpretive challenges.

Should our framework achieve its objectives, the impact could extend beyond a reevaluation of quantum mechanics, potentially establishing a new standard for theoretical development in multiple scientific disciplines. At its core, our aim is to restore the axiomatic foundation of physics to a state of empirical immediacy, free from abstract mathematical detours, thereby enriching the discourse on the empirical coherence and ontological validity of our scientific models.

2 Results

The primary aim of this section is to demonstrate how the "Prescribed Measurement Problem" can be adapted to the realm of quantum mechanics, offering a new perspective on its foundational concepts.

In classical statistical mechanics, a sequence of prescribed measurements typically converges to an average value, often represented by scalar energy values in the domain of real numbers. This approach is adequate for generating the Gibbs measure. However, it falls short when applied to quantum mechanics, which requires two unique and fundamental attributes for energy measurements: linearity and complex-phase invariance. These attributes are encapsulated in the subsequent constraint equation (linearity from matrices, and complex-phase invariance from the trace):

$$\operatorname{tr}\begin{bmatrix} 0 & -\overline{E} \\ \overline{E} & 0 \end{bmatrix} = \sum_{q\in\mathbb{Q}} \rho(q) \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{4}$$

Upon establishing this constraint, we will demonstrate that it singularly serves as the complete foundational basis for quantum mechanics, rendering supplementary axioms superfluous. This formulation represents one of the most parsimonious and efficient formulations of quantum mechanics to date.

Our next procedural step entails solving the corresponding Lagrange multiplier equation, a process that mirrors the methodology employed in statistical mechanics. Ensuring the probability sums to unity, and utilizing the relative Shannon entropy[6, 7][1] instead of the Boltzmann entropy, we deploy the following Lagrange multiplier equation:

$$\mathcal{L} = -\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)} + \lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right) + \tau \left(\operatorname{tr}\begin{bmatrix} 0 & -\overline{E} \\ \overline{E} & 0 \end{bmatrix} - \sum_{q\in\mathbb{Q}} \rho(q) \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{5}$$

We solve for $\partial\mathcal{L}(\rho, \lambda, t)/\partial\rho = 0$ as follows:

$$\begin{aligned}
\frac{\partial\mathcal{L}(\rho,\lambda,\tau)}{\partial\rho(q)} &= -\ln\frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} && (6)\\
0 &= \ln\frac{\rho(q)}{p(q)} + 1 + \lambda + \tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} && (7)\\
\implies \ln\frac{\rho(q)}{p(q)} &= -1 - \lambda - \tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} && (8)\\
\implies \rho(q) &= p(q)\exp(-1-\lambda)\exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) && (9)\\
&= \frac{1}{Z(\tau)} p(q) \exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) && (10)
\end{aligned}$$

where $Z(\tau)$ is obtained as

[1] Also known as the Kullback–Leibler divergence.

$$1 = \sum_{q \in \mathbb{Q}} p(q) \exp(-1-\lambda) \exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{11}$$

$$\implies (\exp(-1-\lambda))^{-1} = \sum_{q \in \mathbb{Q}} p(q) \exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{12}$$

$$Z(\tau) := \sum_{q \in \mathbb{Q}} p(q) \exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{13}$$

The final result is:

$$\rho(q) = \frac{\exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right)}{\sum_{q \in \mathbb{Q}} \exp\left(-\tau \operatorname{tr}\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right)} p(q) \tag{14}$$

Although the result may not initially appear in its traditional guise, what unfolds serves as an implementation of the Born rule within the context of wavefunctions. This is the natural outcome of employing our unique constraint and solving the corresponding Lagrange multiplier equation to maximize the relative Shannon entropy.

By utilizing fundamental equivalences and substituting $\tau = t/\hbar$ in a manner analogous to $\beta = 1/(k_B T)$, by noting that the trace drops down from the exponential into the determinant, and that the determinant of such a matrix is equivalent to a complex norm, we can rearticulate this into its more commonly recognized form:

$$\rho(q) = \frac{\|\exp(-itE(q)/\hbar)\|}{\sum_{q \in \mathbb{Q}} \|\exp(-itE(q)/\hbar)\|} p(q) \tag{15}$$

Here, the role of time, t, emerges in resemblance to how temperature surfaces in conventional statistical mechanics.

The wavefunction can be envisioned as a vector in a complex Hilbert space, with the partition function acting as its inner product. Expressing this relation:

$$\sum_{q \in \mathbb{Q}} \|\exp(-itE(q)/\hbar)\| = Z = \langle \psi | \psi \rangle \tag{16}$$

and where $\psi = \exp(-itE(q)/\hbar)$.

With solutions already normalized by the entropy-maximization process, physical states correlate with unit vectors. The probability for a particular state becomes:

$$\rho(q,t) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q,t))^\dagger\psi(q,t). \tag{17}$$

Upon moving the solution out of its eigenbasis through unitary transformations, we find that energy, $E(q)$, generally transforms as an Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar)\,|\psi(0)\rangle \tag{18}$$

Any self-adjoint operator abides by the condition $\langle\mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$. Measured in its eigenbasis, it aligns with a real-valued observable in statistical mechanics for the given constraint.

The dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\begin{aligned}
\frac{\partial}{\partial t}|\psi(t)\rangle &= \frac{\partial}{\partial t}(\exp(-it\mathbf{H}/\hbar)\,|\psi(0)\rangle) && (19)\\
&= -i\mathbf{H}/\hbar\exp(-it\mathbf{H}/\hbar)\,|\psi(0)\rangle && (20)\\
&= -i\mathbf{H}/\hbar\,|\psi(t)\rangle && (21)\\
\implies \mathbf{H}\,|\psi(t)\rangle &= i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle && (22)
\end{aligned}$$

Which is the Schrödinger equation.

The measurement postulate is captured in $\rho(q,\tau)$ being a probability measure. It gives the likelihood of finding the state at parameter q upon observation. The act of observation, then, associates to sampling from this distribution, and the post-collapse state is merely the value of this sampling.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative. Specifically, linear energy measurements, when endowed with complex-phase invariance, are sufficient to entail the full foundations of quantum mechanics through the principle of entropy maximization.

3 Discussion

The work presented herein introduces a new conceptual paradigm: the "Prescribed Measurement Problem," aiming to align the foundation of quantum mechanics with the same empirical immediacy enjoyed by statistical mechanics. This approach has far-reaching implications that are not limited to resolving controversies in the foundations of quantum mechanics, but potentially sets a new standard for theoretical development across various scientific disciplines.

The fundamental shift we advocate for centers on the reconceptualization of quantum mechanics' foundational axioms. While abstract mathematical constructs hold intrinsic value in pure mathematics, in physics, the distance between these constructs and direct empirical measurements can induce contention. We contend that axioms should be more closely aligned with these empirical measurements to reduce such gaps and the debates they spawn. By considering the prescribed sequence of linear energy measurements endowed with complex-phase invariance and axiomatizing it as a constraint, we establish a cornerstone that not only offers an alternative to the ad hoc postulates that have traditionally characterized quantum mechanics, but also reduces the mathematical gap between the theoretical framework and empirical measurements to zero.

Our formulation demonstrates that this constraint, coupled with mathematical techniques of entropy maximization, is sufficient to yield the complete set of principles governing quantum mechanics, including the Schrödinger equation and the Born rule. Consequently, our framework argues for the elimination of these traditional axioms, now rebranded as theorems, simplifying the conceptual landscape of quantum mechanics.

In general, this new structure is mathematically encapsulated in the following Lagrange equation formulation:

$$\underbrace{\mathcal{L}(\rho,\lambda,\boldsymbol{\tau})}_{\substack{\text{a}\\ \text{maximization}\\ \text{problem}}} = \underbrace{-\sum_{q\in\mathbb{Q}}\rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\text{on the quantity of information}\\ \text{associated with sampling}\\ \text{the distribution}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\text{over all predictive theories}} \underbrace{+\tau\left(\operatorname{tr}\begin{bmatrix}0 & -\overline{E}\\ \overline{E} & 0\end{bmatrix}-\sum_{q\in\mathbb{Q}}\rho(q)\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right)}_{\text{of nature}} \tag{23}$$

This Lagrange equation formulation encapsulates the core principles of the Prescribed Measurement Problem. Importantly, any physical theory that emerges as a solution to such an equation is inherently consistent with both the principles of the Prescribed Measurement Problem and the philosophy of essential empiricism. Conversely, any theory that does not satisfy this criterion is not in alignment with these foundational perspectives. By changing the natural constraint, different frameworks can be obtained.

With this foundational shift in perspective, let's explore the key insights and findings our approach yields:

Key Findings:

1. <u>Core Axioms Derived as a Central Theorem:</u> Our work transcends the axiomatic approach to quantum mechanics by successfully deriving all axioms as theorems from a single entropy-maximization principle. This development offers a new perspective that seeks to disentangle quantum mechanics from a posited formulation, recontextualizing these ”axioms” as corollaries of a more primary theory. This paradigm shift not only demystifies their origin but also opens the door for further investigations into the foundational principles underlying quantum mechanics.

2. <u>Harmonization of Classical and Quantum Probability:</u> The boundary between classical and quantum probability has long been a subject of debate and exploration. While our approach doesn't entirely dissolve this bound-

ary, it does offer a compelling insight: both classical and quantum probabilities can be derived from the same entropy-maximization procedure. The key difference lies in the constraints applied to energy measurements. In classical systems, scalar measurements are considered, while quantum systems involve linear measurements endowed with phases. This distinction serves as the basis for the differing probabilities observed in classical and quantum mechanics. Thus, the "quantum character" of probabilities isn't an inherent feature; rather, it emerges naturally from the structure of the measurements serving as constraints in entropy maximization problems.

3. Rationalization of the Born Rule: The Born Rule has long been an unsettling addition to quantum mechanics, introduced without a rigorous theoretical basis. Our study provides that grounding by demonstrating that the Born Rule is not an arbitrary insertion but rather an automatic by-product of the entropy-maximization problem. This settles many theoretical apprehensions and provides a unified framework that incorporates the Born Rule as an inevitable outcome, rather than an imposed feature.

4. Ontological Status of the Wavefunction: Traditionally, the wavefunction's axiomatic foundation led to various interpretations that attribute to it an ontological status. However, our research radically transforms this perspective. We show that the wavefunction is no more ontological than the Gibbs measure in statistical mechanics; it is merely a least-biased probability measure for predicting outcomes of energy measurements.

5. Unparalleled Conceptual and Mathematical Efficiency: One of the most striking advantages of our formulation lies in its conceptual and mathematical efficiency. By rooting the axioms in empirical measurements and utilizing a prescribed sequence of measurements, our framework dramatically simplifies the philosophical and theoretical underpinnings of quantum mechanics. This leads to a more parsimonious set of principles that govern quantum behavior, significantly reducing the mathematical complexity typically associated with traditional quantum mechanical frameworks. This efficiency not only makes the theory more accessible but also lays a solid foundation for future theoretical developments, encouraging deeper inquiries into the fundamental nature of quantum systems.

6. Addressing the Collapse Paradox and the Inversion Concept: Quantum mechanics has long grappled with the concept of the 'collapse', a central paradox wherein the deterministic evolution of a wavefunction appears to become non-deterministic upon measurement. Before examining how our framework provides a resolution, it's essential to highlight the inversion our approach represents, differentiating it from traditional perspectives:

   (a) **Wavefunction Collapse Problem:** Traditionally, quantum mechanics has been plagued by the question: "How and why does the wavefunction, which evolves deterministically, appear to collapse non-deterministically upon measurement?"

(b) **Prescribed Measurement Problem:** Diverging from traditional paradigms, our approach is rooted in empirical observations. Instead of attempting to reconcile pre-established theoretical constructs with observations, we derive the entire quantum mechanical framework directly from these observations.

This paradigm shift is more than just a change in perspective. While the wavefunction collapse problem grapples with reconciling theory with measurement outcomes, the prescribed measurement problem constructs the theory based on these outcomes.

Building on this inversion, we now examine the collapse issue in the light of our framework...

Reassessing the Collapse Paradox:

Our approach revealed that quantum mechanics, like statistical mechanics, is constrained by a foundational relation on average energy, and is recovered from the same entropy maximization procedure. Consequently, they share a conceptual basis, and as such, the problems of one ought to be the problems of the other. Thus, if the wavefunction collapse is seen as an issue in quantum mechanics, then, logically, the 'collapse' of the Gibbs measure in statistical mechanics should raise the same concerns. Yet, since the latter isn't a source of contention, it underscores that the wavefunction collapse might not be the foundational issue it's often perceived to be. .

In our formulation of quantum mechanics, the foundation is built not on specific measurement outcomes, but on their collective average effect. This means that the foundational framework lacks the information to specify individual outcomes. To address this, we treat the sequence of prescribed measurements as foundational axioms—unexplainable, but fundamental truths. This approach seamlessly integrates these measurements as the central tenet of the theory.

The Role of Information in Bridging Theory and Reality

Shannon entropy serves to quantify the information associated with the random selection of an outcome from a set of potential results. Given the probabilistic nature of quantum measurement outcomes, the theory of quantum mechanics inherently lacks the information to specify any individual outcomes.

The entropy in our approach measures the informational gap between quantum theory and the measurement outcomes. This difference quantifies the additional information an observer possesses beyond what is provided by the quantum theory, as the result of measurements. While the theory offers a broad probabilistic perspective on potential outcomes, the observer's reality is distinct, specific, and enriched with this supplementary information.

Thus, the entropy can be viewed as representing the information necessary to "pin" the observer to the present universe, as selected from the set of all possible measurement outcomes, providing specificity to their experience.

Crucially, an observed physical system isn't defined merely by its wavefunction. A comprehensive specification demands that the wavefunction be paired with the information associated with the prescribed measurement, as quantified by the entropy. Our framework offers a unique perspective by quantifying the information necessary to complete this description.

7. <u>Measurement Outcomes: The True Bedrock of Physical Reality</u>

   Positing the laws of physics as axioms has given rise to the view that these laws are the foundational truths about the universe. However, it's typically appreciated within the physics community that the laws are derived from and constrained by measurement outcomes and observations. A cloud of points, each representing a distinct measurement outcome, sets the boundaries and dictates the formulation of the laws of physics. Our methodology, particularly the entropy maximization techniques, provides a unique mathematical formulation consistent with this perspective, emphasizing that reality is fundamentally anchored in measurement outcomes. Thus, the traditional laws of physics are not intrinsic truths but frameworks derived to encapsulate and articulate these foundational measurement outcomes.

These foundational insights collectively define a new interpretation of quantum mechanics, termed the 'Prescribed Measurement Interpretation'.

4 Conclusion

In this investigation, we demonstrate the transformative potential of the Prescribed Measurement Problem. Unlike traditional frameworks, this approach insists that the axioms anchoring any physical theory should be direct measurements made in nature—our irrefutable, empirically-validated touchpoints with reality. The conceptual structure of quantum mechanics, under this new paradigm, emerges not as a collection of theoretical constructs, but as a theorem deduced from a singular, empirically-grounded constraint.

The constraint enforces linearity and complex-phase invariance in energy measurements, attributes that have been empirically validated through a century of quantum mechanics research. As such, the constraint—and by extension, the entire theory—stands devoid of significant contention.

This paradigm challenges the existing methodologies in theoretical physics by advocating for a return to essential empiricism. It aims to circumvent ontological debates and interpretive challenges, ultimately serving as a standard-bearer for theoretical advancement across scientific disciplines. By centering solely the empirical as the axiomatic, it promises to reduce the gap between observation and theory, facilitating a more coherent and unified comprehension of the natural world.

Statements and Declarations

Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

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