On the Frequency of Light in Vacuum Space

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Abstract: Based on the fact that the frequency of light is reduced at the same rate (redshift) across a far distance, and the fact that the speed of light is constant, we can yield a mathematical law of light's frequency loss. This law states for light traveling through vacuum space, its frequency must reduce an amount equal to H_o= H/MPC = $2.26*10^{-18}$ Hertz for every cycle (or every 1 wavelength of travel) of the light wave, where H is Hubble Constant and MPC = Megaparsecs = 30,856,775,814,913,673,000 km or $\Delta f/f = -H_o t$. This frequency is reducing exponentially against time ($f(t) = f_o e^{-H_o t}$). In Mathematics, some properties of Digamma function $\psi(x)$ is used to deliver the law. By carefully comparing space's expansion and this law of frequency loss, we conclude the expansion of space cannot be the reason of light's frequency reduction, since they contradict one another. Instead, light traveling in space will lose a very small and constant amount of energy for every of its wavelengths that it travels. Finally an experiment is proposed to prove the theory and to find the Hubble Constant without needing to look at galaxies millions of light years away.

1. Frequency of light decrease exponentially against time.

Let us denote f_t to be the frequency of light traveling at time t and f_o to be the original frequency of the light. In this section, we will show that

$$f_t = f_o e^{-H_0 t} \qquad (1)$$

Where $H_0 = \frac{H}{MPC}$, H is the Hubble constant, MPC = Mparsec = 30,856,775,814,913,673,000 km, and e is Euler's number or the base of the natural logarithm.

When we observe light from a far distance, we notice that its frequency spectrum has been shifted to the right. The shift distance is in direct variance to the frequency variance of the light. So we have the following postulate:

Postulate 1: For any light traveling in a vacuum during time t, the frequency decreases at the same rate as any other light traveling with a different initial frequency in a different space. This rate depends only on time t (i.e. the rate at which frequency decreases is the same for any f_o whether it is a microwave or a gamma ray).

$$f_t = f_o a(t) \qquad (2)$$

Where a(t) is the decreasing function of time t and f_o is the original frequency of the light.

Now let us use Postulate 1 to derive the formula (1). Assume $t_1 < t$, and C to be the point the light reached at time t_1, as illustrate in the Figure (a).

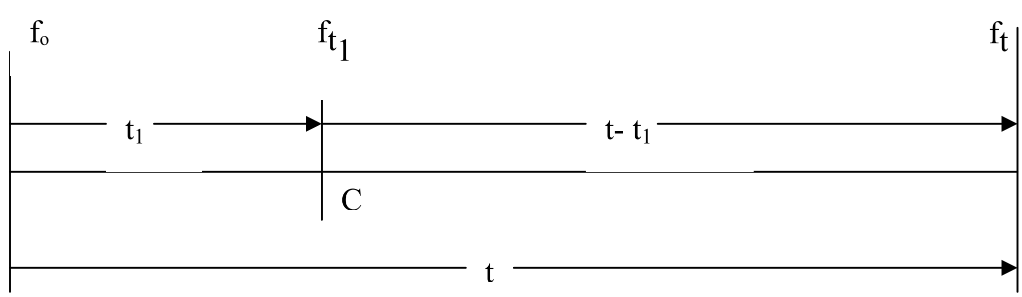


Figure (a)

We have

$$f_{t_1} = f_o a(t_1)$$

$$f_t = f_{t_1} a(t - t_1)$$

$$f_t = f_o a(t)$$

Combining the last 3 equations, we have

$$a(t) = a(t_1)\, a(t - t_1)$$

This equation holds for any t_1 and t with $t_1 < t$. By Theorem 1 in appendix, we have

$$a(t) = e^{-ht}$$

Where $h = -\ln(a(1))$ and t is time in seconds.

By the Taylor series of e^{ht} we have

$$\frac{f_o}{f_t} = e^{ht} = 1 + ht + \frac{1}{2!}(ht)^2 + \frac{1}{3!}(ht)^3 + \ldots$$

The redshift, Z, is defined by

$$Z = \frac{f_o}{f_t} - 1 = e^{ht} - 1 = ht + \frac{1}{2!}(ht)^2 + \frac{1}{3!}(ht)^3 + \ldots$$

Then when ht is small we have

$$Z = \frac{f_o}{f_t} - 1 = e^{ht} - 1 \approx ht$$

On the other hand, by the Doppler Effect:

$$Z \approx \frac{v}{c} \qquad \text{or} \quad v = H\,(ct) = Hd$$

When v is relatively small compared to c; c is the speed of light; and d is the distance in kilometers.

Hubble's law states that the recessional velocity of a galaxy is proportional to its distance from the observer. Mathematically, this can be expressed as

$$v = Hd = H_0\, ct$$

Where v is the recessional velocity in km/s, and c is speed of light, d is the distance to the galaxy in Megaparsecs, H = 69.8 km/s/megaparsec [F1] is the Hubble constant, and $H_0 = \frac{H}{MPC}$.

Therefore,

$$Z \approx \frac{v}{c} = H_0 t \text{ or } h = H_0$$

Therefore

$$h = H_0 = \frac{H}{MPC}$$

Here we use

$$MPC = \text{Megaparsecs} = 30{,}856{,}775{,}814{,}913{,}673{,}000 \text{ km}$$

Then

$$H_0 = \frac{69.8}{MPC} = 2.2620674\times10^{-18} \left(\frac{1}{s}\right)$$

Where s denotes seconds and the frequency unit is in Hertz.

Throughout this paper, we denote H_o by

$$H_o = \frac{H}{MPC}$$

We also call H_o the Hubble Constant in terms of Hertz or just the Hubble Constant.

Example 1. How long does it take for blue light to change to red light?

A typical blue light wavelength is 450 nm while red light is 700 nm.

$$t = \frac{\ln(\frac{700}{450})}{H_o} = \frac{0.441833}{H_o} = 0.195323\times10^{18} \text{ (seconds)} = 6.19364\times10^{9} \text{ (years)}$$

So it takes about 6.19 billion years for light to change from blue to red, and also to traverse 6.19 billion light years.

Example 2. Using redshift to calculate distance

$$t = \frac{\ln(\frac{f_o}{f_t})}{H_o} = \frac{\ln(Z+1)}{H_o}$$

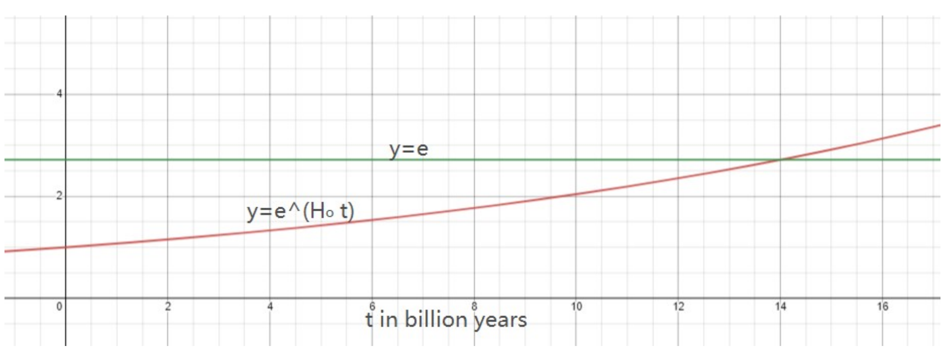
$$d = c\,t$$

If we denote time in billions of years, then we get d as billions of light years (BLY). 1BLY = 31536000000000000 Seconds or 3.1536E+16, H_o* 3.1536E+16 = 0.07133655753

We can calculate the following distances.

Galaxy or Cosmo Object	Z Redshift	LN(Z+1)	Distance(BLY)	Estimated Proper Distance
Cosmic Microwave	1098	7.0022	98.13	N/A
GN-Z11	11.1	2.4932	34.94	32
UDFY-38135539	8.6	2.2618	31.70	30
A1689-ZD1	7.6	2.1518	30.16	N/A
GRB090423	8.2	2.2192	31.10	30
ULAS J1342+0928	7.54	2.1448	30.06	29.36
TN J0924-2201	5.2	1.8245	25.57	N/A
SDSSJ1148+5251	6.42	2.0042	28.09	N/A

The following graph shows y = Z+1. The t^2 term shows why red shift grows with the acceleration $= \frac{1}{2}(H_o)^2$ and y = e is the time the Big Bang occurred. Here Z = $\frac{f_o}{f_t} - 1 = e^{H_o t} - 1 = Ht + \frac{1}{2!}(H_o t)^2 + \frac{1}{3!}(H_o t)^3 + \ldots$



2. Light frequency is lost constantly for each wave cycle

Let us check

$$\frac{\Delta f}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

Where f(t) is the frequency of the light at time t. By equation (1) and the Taylor series we have:

$$\frac{\Delta f}{\Delta t} = \frac{f_o e^{-H_o(t+\Delta t)} - f_o e^{-H_o t}}{\Delta t} \quad = \quad f_o e^{-H_o t}\, \frac{e^{-H_o \Delta t} - 1}{\Delta t}$$

$$\frac{\Delta f}{\Delta t} = f_o e^{-H_o t}\, \frac{-H_o \Delta t + \frac{1}{2} H_o^2 \Delta t^2 + O(H_o^3 \Delta t^3)}{\Delta t}$$

$$\frac{\Delta f}{\Delta t} = f_o e^{-H_o t} \left(-H_o + \frac{1}{2} H_o^2 \Delta t + O(H_o^3 \Delta t^2) \right)$$

Assume $\Delta t = \frac{1}{f(t)}$ is the time it takes for light to travel 1 wave cycle; f(t) is the frequency and

$$\Delta f = -H_o + \frac{1}{2} H_o^2 \Delta t + O(H_o^3 \Delta t^2)$$

Because $\frac{1}{2} H_o^2 \Delta t + O(H_o^3 \Delta t^2)$ is very small, we have the frequency loss, H_o, which is the Hubble Constant.

The Law of Frequency Loss of Light. For light traveling in a vacuum space, it will lose exactly equal to the Hubble Constant $H_o = \frac{69.8}{MPC} = 2.2620674 \times 10^{-18}\ (\frac{1}{s})$ in frequency (units of Hertz) for every wavelength it travels.

$$\Delta f = -H_o \tag{4}$$

Equation (1) can be considered equivalent to equation (4). Let us see that (4) implies (1). Assume light travels n cycles as follows:

f_o	f_o-H_o	f_o-2H_o	f_o-2H_o	……..	f_o-nH_o
$\frac{c}{f_o}$	$\frac{c}{f_o - H_o}$	$\frac{c}{f_o - 2H_o}$	$\frac{c}{f_o - 3H_o}$		$\frac{c}{f_o - nH_o}$

The distance (D)

$$D = t\cdot c = \frac{c}{f_o} + \frac{c}{f_o - H_o} + \frac{c}{f_o - 2H_o} + \frac{c}{f_o - 3H_o} + ... + \frac{c}{f_o - nH_o}$$

Then the time t

$$t = \frac{1}{f_o} + \frac{1}{f_o - H_o} + \frac{1}{f_o - 2H_o} + \frac{1}{f_o - 3H_o} + ... + \frac{1}{f_o - nH_o} \qquad (5)$$

let $N = \left[\frac{f_o}{H_o}\right]$ be the integer part of $\frac{f_o}{H_o}$. Let $a = \frac{f_o}{H_o} - N$, here $0 \le a < 1$. We can rewrite equation (5) as

$$t = \frac{1}{H_o}\left(\frac{1}{N + a} + \frac{1}{N\text{-}1 + a} + \frac{1}{N\text{-}2+a} + \frac{1}{N\text{-}3+a} + ... + \frac{1}{N\text{-}n+a}\right)$$

Based on Theorem 3 in appendix, we have

$$H_o t = \ln(N+a) + \frac{1}{2(N+a)} - O\left(\frac{1}{(N+a)^2}\right) - \ln(N\text{-}n\text{-}1+a) - \frac{1}{2(N\text{-}n\text{-}1+a)} + O\left(\frac{1}{(N\text{-}n\text{-}1+a)^2}\right)$$

$$e^{H_o t} = \frac{N+a}{N\text{-}n\text{-}1+a}\exp\left(\frac{1}{2(N+a)} - O\left(\frac{1}{(N+a)^2}\right) - \frac{1}{2(N\text{-}n\text{-}1+a)} + O\left(\frac{1}{(N\text{-}n\text{-}1+a)^2}\right)\right)$$

When N-n is big or we have

$$e^{H_o t} = \frac{N+a}{N\text{-}n\text{-}1+a}\left(1+O\left(\frac{1}{(N+a)}\right)\right)\left(1-O\left(\frac{1}{(N\text{-}n\text{-}1+a)}\right)\right)$$

$\frac{1}{N+a} = \frac{H_o}{f_o}$ and $\frac{1}{N\text{-}n\text{-}1+a} = \frac{H_o}{f_t}$ are very small for normal frequency of light, and $\frac{N+a}{N\text{-}n\text{-}1+a} = \frac{f_o}{f_o - (n+1)H_o}$ then we have (1) $f_t = f_o e^{-H_o t}$.

The equations of (1) and (4) have a different number of higher order H_o terms. But since light repeats itself every cycle, even if the frequency changes, it is most likely discrete. And (4) is the same for all frequencies of light and thus each cycle. In this way, equation (4) is more likely to be true than equation (1) as it is more applicable over shorter periods of time, but the two work equally over longer periods. Equation (4) thus also reveals more about the nature of light.

Example 3. Life of light. We assume light is dead if its frequency is smaller than H_o. Based on the law of light frequency loss, we can extend equation (5) until we find that the life of light (t) is

$$t = \frac{1}{f_o} + \frac{1}{f_o\text{-}H_o} + \frac{1}{f_o\text{-}2H_o} + \frac{1}{f_o\text{-}3H_o} + \ldots + \frac{1}{f_o\text{-}(N\text{-}1)H_o}$$

Based on theorem (3) in the appendix, we have

$$t = \frac{1}{H_o}\left(\ln(N+a) + \frac{1}{2(N+a)} + \gamma - D(a) - \frac{1}{12(N+a)^2} + O\left(\frac{1}{(N+a)^4}\right) \right)$$

Then

$$t = \frac{1}{H_o}(\ln(N+a) + \gamma - D(a))$$

For visible violet light with a wavelength of 420 nm:

$f_o = 713 \times 10^{12}$ Hertz

$\ln(f_o) = 34.20$, $\ln(H_o) = - 40.6$

we assume $a = 0$, $D(0) = 0$

$\gamma = 0.57721$

then $t = \frac{75.408}{H_o} = 33.33 \times 10^{18}$ (seconds) = 1057.07×10^9 (years)

So the life of visible violet light is about 1.05 trillion years. The last wave of the light would only have a very small frequency, H_o, or 14 billion years if it were to travel 1 cycle.

The following table demonstrates the life of light:

Light	Frequency	Power of T	$\ln(f)-\ln(H_o) + \gamma$	Seconds	BL Years
Gamma Rays	300	18	88.35777826	3.90607E+19	1238.606
X-rays	30	18	86.05519317	3.80428E+19	1206.329
Violet Light	713	12	75.40796665	3.33359E+19	1057.075
Red Light	1	12	68.83848523	3.04317E+19	964.9834
Microwave	160	9	67.00590377	2.96216E+19	939.2942
Radio wave	3	6	56.12158696	2.48099E+19	786.7169
ELF Radio	3	1	44.6086615	1.97203E+19	625.3278

So for a visible red light with 1 TH to decay to a microwave of 160 GH, this would take about 964.98 – 939.29 = 25.69 billion years.

3. On the expansion of the universe

The law of frequency loss contradicts the idea that the universe is expanding. We see light from far away losing frequency, and one of the theorized causes is the universe's expansion. We have assumed the expanding theory for over 100 years. But it could be that light naturally loses frequency or energy while it travels through space. The law of frequency says for any light it loses the same amount of energy for every cycle of the wave. It means it travels through the space at speed of light, just like rowing a boat on a lake, is not for free, it needs energy.

Let us carefully examine the theory of expansion to see why it contradicts this law.

Let f be the frequency, λ the wavelength $\lambda = \frac{c}{f}$, where c is speed of the light. For the next cycle, frequency becomes $f_1 = f\text{-}H_o$ or the wavelength $\lambda_1 = \frac{c}{f\text{-}H_o}$, and time $t_1 = \frac{1}{f\text{-}H_o}$

$$\lambda_1 = \frac{c}{f\text{-}H_o} = \frac{c}{f}\left(1 + \frac{H_o}{f\text{-}H_o}\right) = \lambda\left(1 + H_o t_1\right)$$

$$\lambda_1 = \lambda\left(1 + H_o t_1\right) \qquad (6)$$

If the law of frequency loss, equation (6), is caused by the universe expanding, then (6) is true for all lengths λ and in all directions, and (6) says universe dilates in all direction at same rate H_o. Let us look at the next cycle:

$$\lambda_2 = \frac{c}{f\text{-}2H_o},\ t_2 = \frac{1}{f\text{-}H_o} + \frac{1}{f\text{-}2H_o}$$

and

$$\lambda\left(1 + H_o t_2\right) = \frac{c}{f}\left(1 + \frac{H_o}{f\text{-}H_o} + \frac{H_o}{f\text{-}2H_o}\right) = \frac{c}{f}\left(\frac{f}{f\text{-}H_o} + \frac{H_o}{f\text{-}2H_o}\right) = \frac{c}{f}\left(\frac{f(f\text{-}H_o) - H_o^2}{(f\text{-}H_o)(f\text{-}2H_o)}\right)$$

Then

$$\lambda_2 = \frac{c}{f\text{-}2H_o} = \lambda\left(1 + H_o t_2\right) + \lambda\left(\frac{H_o^2}{(f\text{-}H_o)(f\text{-}2H_o)}\right) \qquad (7)$$

and

$$\lambda_3 = \frac{c}{f\text{-}3H_o} = \lambda\left(1 + H_o t_3\right) + \lambda\left(\frac{H_o^2}{(f\text{-}H_o)(f\text{-}2H_o)} + \frac{2H_o^2}{(f\text{-}2H_o)(f\text{-}3H_o)}\right)$$

The 2nd and 3rd cycles are slightly larger than the expansion theory predicts. If one thinks this extra term in (7) is too small to be convincing, then let us look at further for λ_2, λ_3 ,…, λ_n for n cycles, as the wavelength $\lambda = \frac{c}{f}$ becomes $\lambda_n = \frac{c}{f\text{-}nH_o}$ after time

$$t_n = \frac{1}{f_o\text{-}H_o} + \frac{1}{f_o\text{-}2H_o} + \frac{1}{f_o\text{-}3H_o} + \ldots + \frac{1}{f_o\text{-}nH_o} = \sum_{k=1}^{n} \frac{1}{f\text{-}mH_o}$$

If this change is due to the expansion of our universe, then by (6) it should be $\lambda_n = \lambda\left(1 + H_o t_n\right)$.

But, first for any $m \geq 0$ we have

$$\frac{f}{f\text{-}mH_o} + \frac{H_o}{f\text{-}(m+1)H_o} = \frac{f}{f\text{-}(m+1)H_o} - \frac{mH_o^2}{(f\text{-}mH_o)(f\text{-}(m+1)H_o)}$$

Or

$$\frac{f}{f\text{-}(m+1)H_o} - \frac{f}{f\text{-}mH_o} = \frac{H_o}{f\text{-}(m+1)H_o} + \frac{mH_o^2}{(f\text{-}mH_o)(f\text{-}(m+1)H_o)}$$

Summing them through m = 0 to m = n-1 then we have

$$\sum_{k=0}^{n-1} \left(\frac{f}{f\text{-}(m+1)H_o} - \frac{f}{f\text{-}mH_o}\right) = \sum_{k=0}^{n-1} \left(\frac{H_o}{f\text{-}(m+1)H_o} + \frac{mH_o^2}{(f\text{-}mH_o)(f\text{-}(m+1)H_o)}\right)$$

or

$$\frac{f}{f\text{-}nH_o} - 1 = \sum_{k=1}^{n} \frac{H_o}{f\text{-}mH_o} + \sum_{k=1}^{n-1} \frac{mH_o^2}{(f\text{-}mH_o)(f\text{-}(m+1)H_o)}$$

or

$$\frac{f}{f\text{-}nH_o} = 1 + \sum_{k=1}^{n} \frac{H_o}{f\text{-}mH_o} + \sum_{k=1}^{n-1} \frac{mH_o^2}{(f\text{-}mH_o)(f\text{-}(m+1)H_o)}$$

for $\lambda_n = \frac{c}{f\text{-}nH_o}$, $\lambda = \frac{c}{f}$ and multiplying the above equation by λ we have

$$\lambda_n = \frac{c}{f\text{-}nH_o} = \lambda\left(1 + H_o t_n\right) + \lambda \sum_{k=1}^{n-1} \frac{mH_o^2}{(f\text{-}mH_o)(f\text{-}(m+1)H_o)}$$

Because $\frac{1}{(f\text{-}mH_o)(f\text{-}(m+1)H_o)} \geq \frac{1}{f^2}$ then

$$\lambda_n \geq \lambda\left(1 + H_o t_n\right) + \lambda\left(\frac{H_o^2}{f^2}\frac{n(n\text{-}1)}{2}\right) \tag{8}$$

Now let us look at a gamma ray with a wavelength of 10^{-11} meters, with $n = 10^{11}$, $f_o= nc$; where c is speed of light and

$$t_n = \frac{1}{f_o-H_o} + \frac{1}{f_o-2H_o} + \frac{1}{f_o-3H_o} + ... + \frac{1}{f_o-nH_o} = \sum_{k=1}^{n} \frac{1}{f_o-mH_o}$$

Then

$$1/(c- H_o) \geq t_n \geq 1/c$$

That is the time taken for the gamma ray to travel n cycles. Let us choose a radio wave with wavelength of $\lambda= t_nc/(1+ H_ot_n)$ meters, then the frequency $f = (1+ H_ot_n)/t_n =1/t_n + H_o$.

$t_1 = \frac{1}{f-H_o} = t_n$ then

$$\lambda_1 = \lambda\left(1 + H_ot_n\right)$$

Let us assume this length change is due to the Universe expanding. Now let us look at $N = \lambda 10^{11}$ gamma rays each with the same wavelength of 10^{-11} meters, lining them up and filled then in a space of length λ where these spaces after time t_n and by equation (8) is notated as:

For each small gamma ray

$$\lambda_n \geq 10^{-11} \left(1 + H_ot_n\right) + 10^{-11}\left(\frac{H_o^2}{f_o^2} \frac{n(n-1)}{2}\right)$$

and

$$N\lambda_n \geq \lambda \left(1 + H_ot_n\right) + \lambda\left(\frac{H_o^2}{f_o^2} \frac{n(n-1)}{2}\right) = \lambda \left(1 + H_ot_n\right) + \lambda\left(\frac{H_o^2}{2c^2}\right)$$

where $N\lambda_n$ should be the same as λ_1 since both are caused by the Universe expanding the same length λ for the same length of time t_n. Then $\left(\frac{H_o^2}{c^2}\right) \leq 0$ is impossible, since although it is indeed very small, it is still a positive number.

In this way, equation (8) contradicts the theory that the Universe is expanding and dilating in all directions at same rate H_o, i.e equation (6). This proves the contradiction. If the universe is not expanding, then the Big Bang theory is also put to question. The cosmic microwave background would be light that has taken 25.69 billion years to decay into microwave. Furthermore, our telescopes can only see light around 14 billion years old because the light of the star and galaxy change to infrared ray. After 14 billion years, $t^* H_0 = 14^* 0.07133655753 = 0.998$ close to 1 the frequency change is $f = f_0 /e$, this would change most of the galaxy's light to infrared light. That is why it is difficult for our telescopes to observe. So, should we develop more infrared telescopes that can see further than 14 billion light years away, we may observe that our universe is more than 14 billion years old.

4. A project to test this theory and derive the Hubble Constant

In this section, we are introducing a way to obtain the Hubble Constant H_0.

Based on the law of frequency loss of light, we have $\Delta f / f = -tH_0$ where t is time and f is the initial frequency of the light. On the other hand $c = \lambda f$, then we have $\Delta f / f + \Delta \lambda / \lambda = 0$ by the Product Rule. Then

$$\Delta \lambda / \lambda = t H_0$$

This is spectroscopic resolution. We can obtain H_0 through

$$H_0 = \Delta \lambda / (\lambda t)$$

Hubble Constant H_o is about $\frac{69.8}{\text{MPC}} = 2.2620674\times10^{-18} \left(\frac{1}{s}\right)$

We need a spectroscopic resolution accuracy of 10^{19} if t = 1 second. We can capture the light through a laser component of two highly reflective mirrors and a stimulate emission to keep the light live for 1 year = 31536000 seconds, or $3.15*10^7$ seconds. Let us say the light is 200 nm in wavelength. To find H_o we need to find $\Delta \lambda / \lambda$ with an accuracy of 10^{-12} (i.e spectroscopic resolutions 10^{12}). 200nm $*10^{12}$ = 200 km. To calculate the different of wavelength we can use Michelson interferometer. So it needs 200km big Michelson interferometer to see the fringe change when we switch the light to measure the change of wavelength

$$10^{12} \lambda_1 + \text{diff} = 10^{12} \lambda_2$$

The diff < 200nm

Therefore diff $* 10^{-12} = \Delta \lambda$ will get we need the accuracy. To set the 200km Michelson interferometer, we could use the fiber to set it up due to reflection the light is not change the frequency. But rotating Earth should change the reflection frequency like Sagnac effect. We need overcome the Earth rotating effecting on the frequency change.

The other is to use X ray laser. The wavelength could be the wavelength of the 14.4 keV Mössbauer photon it is about 0.86nm, 0.86nm $*10^{12}$ = 86m. The equipments have to be 86m apart and displacement accuracy at 0.001nm and also keep the X ray laser for 1 year. These all should be a challenge for today's tech.

5. Appendix: some mathematics knowledge

Theorem 1: If a(t) is a non-zero decreasing function for all real numbers $t \geq 0$ and for any positive real number t_1, t_2

$$a(t_1) \cdot a(t_1) = a(t_1 + t_2) \qquad (a1)$$

Then there is positive number H, such that

$$a(t) = e^{-Ht}$$

Here the number e is Euler's number or the base of the natural logarithm.

Proof: Let $t_1 = t_2 = 0$ in the equation (1), we have

$$a(0)^2 = a(0)$$

Because a( ) is non zero decreasing function and then $a(0) \neq 0$. Hence $a(0) = 1$ and $a(1) < 1$. We can write $a(1) = e^{-H}$ where $H = \ln(a(1))$ and ln( ) is natural logarithmic function.

For any positive integer n, repeatedly using equation (1), we have

$$a(n) = a(1)a(n-1) = a(1)^2a(n-2) = ... = a(1)^n$$

Then $a(n) = a(1)^n = e^{-Hn}$

Also, for any positive integer m, repeatedly using the equation again, we have

$$a(1) = a(\frac{1}{m})a(\frac{m-1}{m}) = a(\frac{1}{m})^2a(\frac{m-2}{m}) = ... = a(\frac{1}{m})^m$$

Or

$$a(\frac{1}{m}) = a(1)^{\frac{1}{m}}$$

And for any fraction $\frac{n}{m}$, we have

$$a(\frac{n}{m}) = a(\frac{1}{m})a(\frac{n-1}{m}) = a(\frac{1}{m})^2a(\frac{n-2}{m}) = ... = a(\frac{1}{m})^n$$

So $a(\frac{n}{m}) = a(\frac{1}{m})^n = a(1)^{\frac{n}{m}} = e^{-H\frac{n}{m}}$.

Then for any fractional number t we have $a(t) = e^{-Ht}$. Because for any real number we can find that the decreasing fractional number sequence $\{t_k\}$ has lim $t_k = t$. because a() is decreasing function then we have

$$a(t_1) \geq a(t_2) \geq a(t_3) \geq \cdots \geq a(t)$$

then $a(t) \leq \lim_{k\to\infty} a(t_k) = \lim_{k\to\infty} e^{-Ht_k} = e^{-Ht}$.

On the other hand, we can find an increasing fractional number sequence which has limit t, we have $a(t) \geq e^{-Ht}$. Then $a(t) = e^{-Ht}$ QED

Harmonic number h_n is defined by

$$h_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

Theorem 2. For harmonic number h_n we have

$$h_n = \ln(n) + \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}}$$

Where γ is Euler's constant defined as

$$\gamma = \int_1^{\infty} \frac{x - [x]}{x[x]} dx = 0.5772156649...$$

Where [x] is defined as integer part of x.

And B_{2k} is Bernoulli number, with

$$B_2 = \frac{1}{6} \quad B_4 = -\frac{1}{30} \quad B_6 = \frac{1}{42} \quad B_8 = -\frac{1}{42} \quad ....$$

Defined as the following Taylor series:

$$\frac{x}{e^{x}-1} = \sum_{k=0}^{\infty} \frac{B_n x^n}{n!}$$

For a proof of theorem 2, please refer to reference [A1]

Also, we have:

$$h_n = \ln(n) + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + O(\frac{1}{n^4}) \qquad \text{(a2)}$$

For a general harmonic number:

$$h_n(a) = \frac{1}{1+a} + \frac{1}{2+a} + \frac{1}{3+a} + \ldots + \frac{1}{n+a} = \sum_{k=1}^{n} \frac{1}{k+a}$$

Where $0 \le a \le 1$ is a real number.

Consider the Digamma function $\psi(x)$ defined as

$$\psi(x) = \frac{d}{dx}\ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}$$

Where $\Gamma(x)$ gamma function is defined as

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$$

We have

$$\Gamma(x+1) = x\Gamma(x), \quad \frac{d\Gamma(x+1)}{dx} = \Gamma(x) + x\Gamma'(x), \quad \frac{\Gamma'(x+1)}{\Gamma(x+1)} = \frac{\Gamma'(x)}{\Gamma(x)} + \frac{1}{x}$$

$$\psi(x+1) - \psi(x) = \frac{1}{x}$$

Then

$$h_n(a) = \frac{1}{1+a}+\frac{1}{2+a}+\frac{1}{3+a}+...+\frac{1}{n+a} = \psi(n+a+1) - \psi(1+a)$$

For the Digamma function $\psi(x)$ [A2] we also have

$$\psi(1+x) = \ln(x) + \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kx^{2k}} \quad \text{and}$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+a}\right) = \psi(1+a) + \gamma$$

We have the following

Theorem 3. For general harmonic numbers we have

$$h_n(a) = \ln(n+a) + \frac{1}{2(n+a)} - \psi(1+a) - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(n+a)^{2k}} \tag{a3}$$

Where B_{2k} is Bernoulli number and

$$\text{Lim } (h_n - h_n(a)) = D(a) \equiv \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+a}\right) = \psi(1+a) + \gamma \tag{a4}$$

The proof theorem can be seen in reference [A2].

Combining (3) and (4):

$$h_n(a) = \ln(n+a) + \frac{1}{2(n+a)} + \gamma - D(a) - \frac{1}{12(n+a)^2} + O(\frac{1}{(n+a)^4}) \tag{a5}$$

Where $0 \le a \le 1$ is a real number and $O(x) < \frac{x}{120}$. **QED**

Let us compute D(a) . Because

$$\frac{1}{k+a} = \frac{1}{k} + \frac{1}{k}\sum_{j=1}^{\infty}\left(\frac{-a}{k}\right)^{j} = \frac{1}{k} + \sum_{j=1}^{\infty}\frac{(-a)^{j}}{k^{j+1}}$$

We have related to Riemann-zeta function $\zeta(x) = \sum_{k=1}^{\infty}\frac{1}{x^{k}}$

$$D(a) = -\sum_{j=1}^{\infty}(-a)^{j}\,\zeta(j+1) = (1-a)\sum_{k=1}^{\infty}a^{2k-1}\zeta(2k) - \sum_{k=1}^{\infty}a^{2k}(\zeta(2k+1)-\zeta(2k))$$

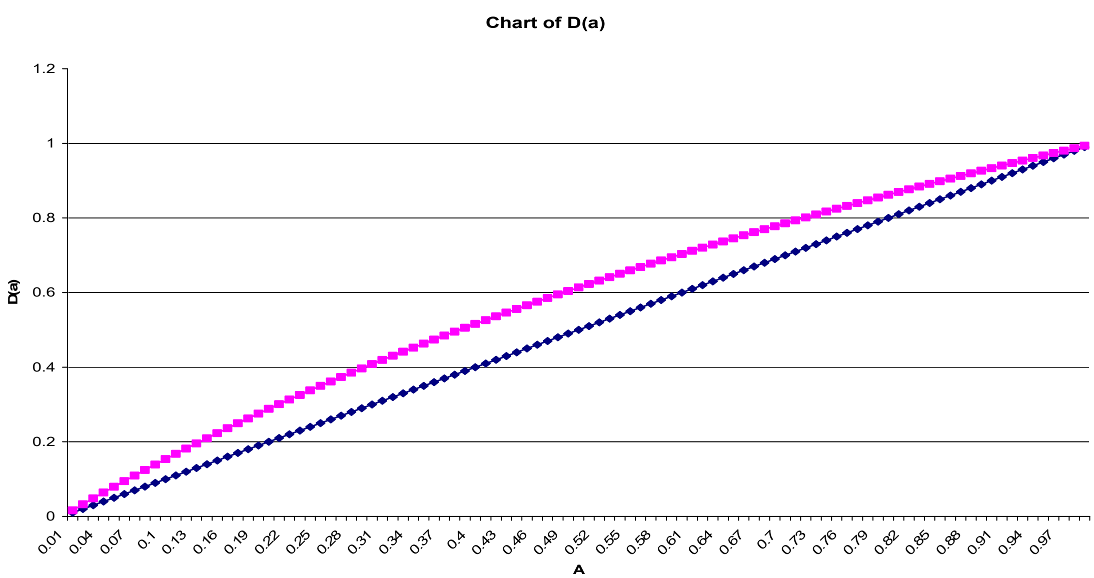
$$D(a) = \frac{a}{1+a} + (1-a)a\sum_{k=0}^{\infty}a^{2k}(\zeta(2k+2)-1) + \sum_{k=1}^{\infty}a^{2k}(\zeta(2k)-\zeta(2k+1)) \qquad \text{(a6)}$$

Because

$$|\zeta(2k)-1| = \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \frac{1}{4^{2k}} + \ldots < \frac{1}{2^{2k}} + \frac{1}{(2k-1)2^{2k-1}} < \frac{1}{2^{2k-1}} \quad \text{and}$$

$$|\zeta(2k)-\zeta(2k+1)| = \frac{1}{2^{2k+1}} + \frac{2}{3^{2k+1}} + \frac{3}{4^{2k+1}} + \ldots < \frac{1}{2^{2k-1}}$$

The last 2 series have been dominated by $(\frac{a}{2})^{2k} \leq (\frac{1}{2})^{2k}$. We can use (a6) to estimate the D(a).



The above is the function chat looks like the red is D(a) and blue is D = a.

Reference:

[A1] Arfken, George (1970), Mathematical methods for physicists (2nd ed.), Academic Press, Inc., ISBN 978-0120598519.

[A2] Abramowitz, M.; Stegun, I. A., eds. (1972). "6.3 psi (Digamma) Function.". Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (10th ed.). New York: Dover. pp. 258–259.

[F1] Freedman, Wendy L. (2021-09-01). "Measurements of the Hubble Constant: Tensions in Perspective*". The Astrophysical Journal. 919 (1): 16.

[F1]F*riedman, A. (1922). "Über die Krümmung des Raumes". Zeitschrift für Physik.* ***10*** *(1): 377–386. Bibcode:1922ZPhy...10..377F.*

[G1] Geoffrey Burbidge Noncosmological Redshifts The Astronomical Society of the Pacific. All rights reserved. Printed in U.S.A. Publications of the Astronomical Society of the Pacific, Volume 113, Number 786 **Citation** Geoffrey Burbidge 2001 *PASP* **113** 899

[G2] Goldhaber, G.; et al. (2001). "Timescale Stretch Parameterization of Type Ia Supernova B-Band Lightcurves". *Astrophysical Journal*. **558** (1): 359–386

[G3] *Goldhaber, Gerson (2009). "The Acceleration of the Expansion of the Universe: A Brief Early History of the Supernova Cosmology Project (SCP)". AIP Conference Proceedings.* ***1166****: 53.*

[H1] Hubble, Edwin (1929). "A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae". *Proceedings of the National Academy of Sciences of the United States of America*. **15** (3): 168–173.

[H2] Hubble, Edwin (1937). "Red-shifts and the distribution of nebulæ". *Monthly Notices of the Royal Astronomical Society*. **97** (7): 506.

[H3] *Hubble, Edwin (1936). "Effects of Red Shifts on the Distribution of Nebulae". The Astrophysical Journal.* ***84*** *(11): 621–627.*

[R1] Riess, Adam G. (January 1995). ""Using Type IA supernova light curve shapes to measure the Hubble constant"". The Astrophysical Journal. 438: L17

[S1] Shangyu Si et "14.4 keV X-Ray Michelson Interferometer Design and Core Component Characterization" Acta Optica Sinica Vol. 43, Issue 4, 0434002 (2023)

[S2] S*lipher, V. M. (1913). "The Radial Velocity of the Andromeda Nebula". Lowell Observatory Bulletin.* ***1*** *(8): 56–57. Bibcode:1913LowOB...2...56S.*