Fundamental Theorem of Algebra

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abstract

One of the immediate consequences of Cauchy's integral formula is Liouville's theorem, which states that an entire (that is, holomorphic in the whole complex plane $\mathbb{C}$) function cannot be bounded if it is not constant. This profound result leads to arguably the most natural proof of Fundamental Theorem of Algebra.

Key words: Liouville's theorem, roots of polynomials, number Pi

1. Introduction: Fundamental Theorem of Algebra

Definition: A function f is analytic on an open subset $R \subset \mathbb{C}$ if f is complex differentiable everywhere on R ; f is entire if it is analytic on all of $\mathbb{C}$.

Theorem 1.1 (Liouville). If $f(z)$ is analyitc and bounded in the complex plane, then $f(z)$ is constant.

Theorem 1.2 (Fundamental Theorem of Algebra). Let $p(z)$ be a polynomial with complex coefficients of degree n. Then $p(z)$ has n roots.

Proof. It is sufficient to show any $p(z)$ has one root, for by division we can then write

$$p(z) = (z - z_0)\, g(z) \tag{1}$$

with $g(z)$ of lower degree.

Note that if

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_0 \tag{2}$$

then as $|z| \longrightarrow \infty$, $|p(z)| \longrightarrow \infty$. This follows as

$$p(z) = z^n \left| a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n} \right| \tag{3}$$

Assume $p(z)$ is non-zero everywhere. Then $1/p(z)$ is bounded when $|z| \geq R$.

Also, $p(z) \neq 0$, so $1/p(z)$ is bounded for $|z| \leq R$ by continuity. Thus, $1/p(z)$ is a bounded, entire function, which must be constant. Thus, $p(z)$ is constant, a contradiction which implies $p(z)$ must have a zero (our assumption).

2. Some formulas for Pi

The number Pi is defined by

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n + 1} \tag{4}$$

Entry 1. Define $P(z)$ by

$$P(z) = z^{16} + 32 z^{15} + 16 z^{14} - 896 z^{13} + 92 z^{12} + 3488 z^{11} + 240 z^{10} - 768 z^9 + 326 z^8 - 1440 z^7 + 240 z^6 - 384 z^5 + 92 z^4 - 32 z^3 + 16 z^2 + 1 \tag{5}$$

Then

$$P(z) = 0 \Longrightarrow z = \{\{z_1 \to -30.5141\}, \{z_2 \to -5.89618\}, \{z_3 \to -1.8269\}, \{z_4 \to -1.13565\}, \{z_5 \to -0.118916 - 0.27328\, i\}, \{z_6 \to -0.118916 + 0.27328\, i\}, \{z_7 \to -0.0948143 - 0.430469\, i\}, \{z_8 \to -0.0948143 + 0.430469\, i\}, \{z_9 \to 0.0138181 - 0.484924\, i\}, \{z_{10} \to 0.0138181 + 0.484924\, i\}, \{z_{11} \to 0.0687008 - 0.460176\, i\}, \{z_{12} \to 0.0687008 + 0.460176\, i\}, \{z_{13} \to 0.368954\}, \{z_{14} \to 0.872912\}, \{z_{15} \to 2.32587\}, \{z_{16} \to 4.06756\}\} \tag{6}$$

Entry 2. The root $u = z_{13} = 0.3689 \ldots$ via radicals

$$u = \frac{T}{1 + \sqrt{1 + T^2}} \tag{7}$$

where

$$T = \frac{A + \sqrt{A^2 - B + Y}}{2} + \frac{1}{2} \sqrt{\left(A + \sqrt{A^2 - B + Y}\right)^2 - 2\left(Y - \sqrt{Y^2 - 4 B}\right)} \tag{8}$$

$$A = 2 - \sqrt{3} \;\; ; \;\; B = \frac{\sqrt{6} - \sqrt{2} - 1}{2} \tag{9}$$

$$Y = \frac{\sqrt{6} - \sqrt{2} - 1}{6} - \left(\frac{2}{3} \sqrt{\frac{11\,488 + 3871 \sqrt{2} - 6624 \sqrt{3} - 2241 \sqrt{6}}{3}} + \frac{1177 + 3587 \sqrt{2} - 588 \sqrt{3} - 2139 \sqrt{6}}{216}\right)^{1/3} + \left(\frac{2}{3} \sqrt{\frac{11\,488 + 3871 \sqrt{2} - 6624 \sqrt{3} - 2241 \sqrt{6}}{3}} - \frac{1177 + 3587 \sqrt{2} - 588 \sqrt{3} - 2139 \sqrt{6}}{216}\right)^{1/3} \tag{10}$$

Entry 3.

$$\pi = 48 \sum_{n=1}^{\infty} u^{4n-1} \sum_{k=0}^{n-1} \left(\frac{2^{4k+1}}{4k+1} \binom{2n+2k-1}{2n-2k-1} - \frac{2^{4k+3}}{4k+3} \binom{2n+2k+1}{2n-2k-1} u^2\right) \tag{11}$$

Entry 4. Define $Q(w)$ by

$$Q(w) = w^{16} + 16 w^{14} + 32 w^{13} + 92 w^{12} + 384 w^{11} + 240 w^{10} + 1440 w^9 + 326 w^8 + 768 w^7 + 240 w^6 - 3288 w^5 + 92 w^4 + 896 w^3 + 16 w^2 - 32 w + 1 \tag{12}$$

Then

$$Q(w) = 0 \Longrightarrow w = \{\{w_1 \to -2.71036\}, \{w_2 \to -1.14559\}, \{w_3 \to -0.429946\}, \{w_4 \to -0.317352 - 2.1257\, i\}, \{w_5 \to -0.317352 + 2.1257\, i\}, \{w_6 \to -0.245848\}, \{w_7 \to -0.0587148 - 2.06051\, i\}, \{w_8 \to -0.0587148 + 2.06051\, i\}, \{w_9 \to 0.0327717\}, \{w_{10} \to 0.169601\}, \{w_{11} \to 0.487995 - 2.21556\, i\}, \{w_{12} \to 0.487995 + 2.21556\, i\}, \{w_{13} \to 0.547376\}, \{w_{14} \to 0.880551\}, \{w_{15} \to 1.3388 - 3.07668\, i\}, \{w_{16} \to 1.3388 + 3.07668\, i\}\} \tag{13}$$

Entry 5. The root $v = w_9 = 0.03277 \ldots$ via radicals

$$v = \frac{S}{1 + \sqrt{1 + S^2}} \tag{14}$$

$$S = \frac{A + \sqrt{A^2 - B + Y}}{2} - \frac{1}{2} \sqrt{\left(A + \sqrt{A^2 - B + Y}\right)^2 - 2\left(Y - \sqrt{Y^2 - 4 B}\right)} \tag{15}$$

$$A = 2 - \sqrt{3}\ ;\ B = \frac{\sqrt{6} - \sqrt{2} - 1}{2} \tag{16}$$

$$Y = \frac{\sqrt{6} - \sqrt{2} - 1}{6} - \left(\frac{2}{3}\sqrt{\frac{11\,488 + 3871\sqrt{2} - 6624\sqrt{3} - 2241\sqrt{6}}{3}} + \frac{1177 + 3587\sqrt{2} - 588\sqrt{3} - 2139\sqrt{6}}{216}\right)^{1/3} + \left(\frac{2}{3}\sqrt{\frac{11\,488 + 3871\sqrt{2} - 6624\sqrt{3} - 2241\sqrt{6}}{3}} - \frac{1177 + 3587\sqrt{2} - 588\sqrt{3} - 2139\sqrt{6}}{216}\right)^{1/3} \tag{17}$$

Entry 6.

$$\pi = 48\sum_{n=0}^{\infty} v^{4n+1}\sum_{k=0}^{n}\left(\frac{2^{4k+1}}{4k+1}\binom{2n+2k}{2n-2k} - \frac{2^{4k+3}}{4k+3}\binom{2n+2k+2}{2n-2k}v^2\right) \tag{18}$$

Entry 7. Define r by

$$r = \frac{1}{3}\left(1 - 7\cdot 2^{2/3}\left(-4 + 3\sqrt{78}\right)^{-1/3} + \left(2\left(-4 + 3\sqrt{78}\right)\right)^{1/3}\right) \tag{19}$$

Then

$$r^3 - r^2 + 5r - 1 = 0 \tag{20}$$

$$\pi = 8\sum_{n=0}^{\infty} r^{4n+1}\sum_{k=0}^{n}\left(\frac{2^{4k+1}}{4k+1}\binom{2n+2k}{2n-2k} - \frac{2^{4k+3}}{4k+3}\binom{2n+2k+2}{2n-2k}r^2\right) \tag{21}$$

Entry 8. Define x_1 by

$$x_1 = 1 + \sqrt[3]{1 + 3\sqrt[3]{1 + 3\sqrt[3]{1 + \ldots}}} = 1 + 2\sin\left(\frac{7\pi}{18}\right) \tag{22}$$

Then

$$x_1^3 - 3x_1^2 + 1 = 0 \tag{23}$$

$$\pi = \frac{6}{\sqrt{x_1}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{x_1}\right)^n\sum_{k=0}^{[n/3]}\frac{2^{-2k}}{2n-6k+1}\binom{n-2k}{k}\binom{2n-4k}{n-2k}\binom{2n-6k}{n-3k}^{-1} \tag{24}$$

Entry 9. Define x_2 by

$$x_2 = \frac{1}{\sqrt{3 + \frac{1}{\sqrt{3+\ldots}}}} = \left(3 + \left(3 + (3 + \ldots)^{-1/2}\right)^{-1/2}\right)^{-1/2} = 2\sin\left(\frac{5\pi}{18}\right) - 1 \tag{25}$$

Then

$$x_2^3 + 3x_2^2 - 1 = 0 \tag{26}$$

$$\pi = \frac{2}{x_2}\sum_{n=0}^{\infty}(-1)^n (x_2)^n\sum_{k=0}^{[n/2]}\frac{(-1)^k 3^{-n+2k}}{(2k+1)(n-2k)!}\left(\frac{1}{2} - k\right)_{n-2k} \tag{27}$$

Remark: $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$, $(a)_0 = 1$.

Entry 10. Define x_3 by

$$x_3^3 = \left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3} + \ldots\right)^{3/2}\right)^{3/2}\right)^{3/2} \tag{28}$$

$$x_3 = 1 - 2\sin\left(\frac{\pi}{18}\right) \tag{29}$$

Then

$$x_3^3 - 3x_3^2 + 1 = 0 \tag{30}$$

$$\pi = \frac{2}{x_3}\sum_{n=0}^{\infty}(x_3)^n \sum_{k=0}^{[n/2]} \frac{(-1)^k\, 3^{-n+2k}}{(2k+1)\,(n-2k)!}\left(\frac{1}{2}-k\right)_{n-2k} \tag{31}$$

Remark: $(a)_n = a\,(a+1)\,(a+2)\,...\,(a+n-1)$, $(a)_0 = 1$.

Entry 11. For $k = 2, 3, 4, ...$, define y_k by

$$y_k = 2 - \left(2 - \left(2 - (2 - ...)^{-k}\right)^{-k}\right)^{-k} \tag{32}$$

Then

$$y_k^{k+1} - 2\,y_k^k + 1 = 0 \tag{33}$$

$$\pi = 3\cdot\sum_{n=0}^{\infty}2^{-3n}\left(\frac{1}{y_k}\right)^n \sum_{m=0}^{\left[\frac{n}{k+1}\right]} \frac{(-1)^m\, 2^{(3k-1)m}}{2n-2km+1}\binom{2n-2km}{n-km}\binom{n-km}{m} \tag{34}$$

Remarks:

$$\frac{1}{y_k} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + ...\right)^{k+1}\right)^{k+1} \quad , k = 2, 3, 4, ... \tag{35}$$

$$y_k = 2 - 2^{-k}\sum_{m=0}^{\infty}\frac{1}{(m+1)\,2^{(k+1)m}}\binom{k(m+1)+m-1}{m} \quad , \; k = 2, 3, 4, ... \tag{36}$$

Entry 12. Define s by

$$s = \sqrt[5]{\frac{75-21\sqrt{10}}{125}} + \sqrt[5]{\frac{75+21\sqrt{10}}{125}} - \sqrt[5]{\frac{72\sqrt{10}+225}{125}} + \sqrt[5]{\frac{72\sqrt{10}-225}{125}} \tag{37}$$

Then

$$s^5 + 15\,s - 12 = 0 \tag{38}$$

$$\pi = \frac{5\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n s^{n+1}\sum_{k=0}^{[n/5]}\frac{(-1)^k}{2n-10k+1}\left(\frac{5}{12}\right)^{n-5k}\left(\frac{1}{12}\right)^k\binom{n-4k}{k} \tag{39}$$

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