a cubic ellipsoid nucleus model: charge radius and excess neutrons

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Abstract

In this paper we analyze the charge radius of the nucleus in the light of the cubic ellipsoid geometric model. [5]
The goal here is to verify the model and its assumption that the excess neutrons are located in its envelope and expand its understanding and thus possibly also to gain new insights from it. The results match the experimental data quite well and strengthen so the model assumption. We also raise some new hypotheses as a conclusion of the calculations regarding the density of the nucleus, that might increase with the number of nucleons until it reaches a finite value in the vicinity of Argon.

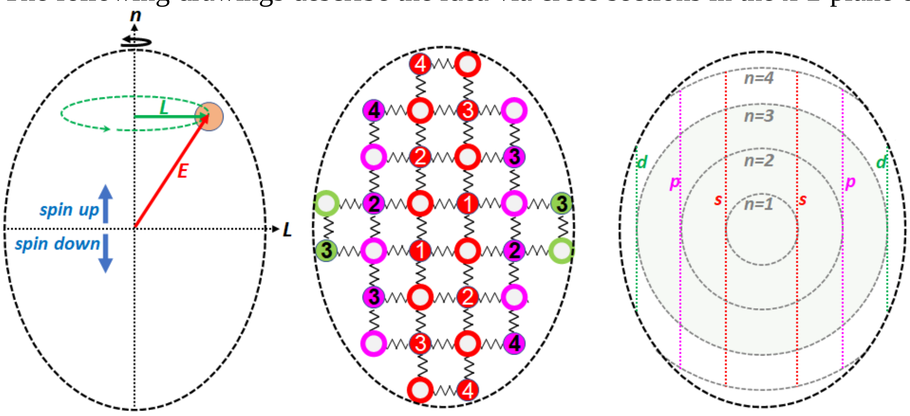
Content

The model at a glance

A brief description of the model [5]:

- The nucleus has an ellipsoid shape.
- The nucleons are connected in a cubic form.
- Protons are connected to neutrons (**p-n**).
- Neutrons are connected mainly to protons.
- The protons are populated and organized in shells in the nucleus in a full analogy to those of the electrons in the atom.
- The energy layers (principal quantum number **n**) grow along the **z**-axis of the nucleus in its both directions (more precisely **n** grows with its distance from the origin).
- The perpendicular distance from the **z**-axis in the **x-y**-plane reflects the angular momentum (**L**) and so the orbitals.
- The upper half of the ellipsoid is referred to as spin-up and the lower part as spin-down.
- The nucleus possibly rotates around its **z**-axis.

The following drawings describe the idea via cross sections in the **x-z**-plane of the nucleus.



***1**: a nucleon in the nucleus*   ***2**: the bonds between the nucleons*   ***3**: the energy levels of the nucleus*

1. One nucleon (**circle**) is observed inside the ellipsoid (dashed line) that encloses the nucleons and schematically defines the nucleus surface:
   - the distance from the origin represents its energy **E**.
   - the distance from the **z**-axis depicts it angular momentum **L**.
   - the nucleons in the upper half have spin up, and in the lower one spin down.
2. The bonds between the nucleons are shown for visibility as springs.
   - **Protons**: full circles of the **s**, **p** and **d** sub-orbitals. **Neutrons**: hollow circles.
3. The circles of equal energy states **n** in the ellipsoid.
   - the lines mark the development of the **s**, **p** and **d** sub-orbitals along the **z**-axis.
   - The **s** line crosses all **n** circles from 1 to 4 (**s1** to **s4**).
   - The **p** line begins by **n**=2 and reaches till **n**=4 (**p2** to **p4**).
   - The **d** line begins by **n**=3 and reaches the ellipsoid border, before it reaches the **n**=4 circle, and therefore there are no **d4** states at this stage (only **d3**).

Introduction

The subjects we deal with in this paper are:

- The charge radius of small nuclei:
  - hydrogen and helium show a large deviation of their size (charge radius) from the expected value; we try to explain this.
  - Lithium and Beryllium are already in a reasonable range of size, yet we want to gain through them some tangible feeling to the structure of the nuclei.
- An estimation for the charge radius of the noble gases was implemented and the results seem to support the model.
- According to the model the excess neutrons (the unpaired ones that their number exceeds the number of protons) are located on the nucleus envelope; we tested this through a comparison between $\frac{\mathbf{R_c}}{\sqrt[3]{\mathbf{A}}}$ and $\frac{\mathbf{R_c}}{\sqrt[3]{\mathbf{2 \cdot Z}}}$ and the results seem to support the model.

  The notation is:
  - A: the atomic mass (the number of nucleons).
  - Z: the atomic number (the number of protons).
  - R_c: the (measured) charge radius.

We use the following data for the calculations:

- $d_0 \approx (r_n + r_p)$: the distance between two neighboring nucleons in the nucleus.
- d_0 = 1.62 fm [5]
- r_n = 0.80 fm: the neutron radius. [3]
- r_p = 0.84 fm: the proton radius. [4]

We define *single layer*: a layer with sub-orbital that appears only once in a nucleus; for instance in Neon the second layer is a *single layer*, because of the P sub-orbital. If this occurs in two different subsequent layers, we call it a *double layer*; for instance in Argon the third layer is a *double layer*, because of the P sub-orbitals.
Additional examples:
Single layers:

- layer 4 by Krypton (a single D sub-orbital)
- layer 6 by Radon (a single F)

Double layers:

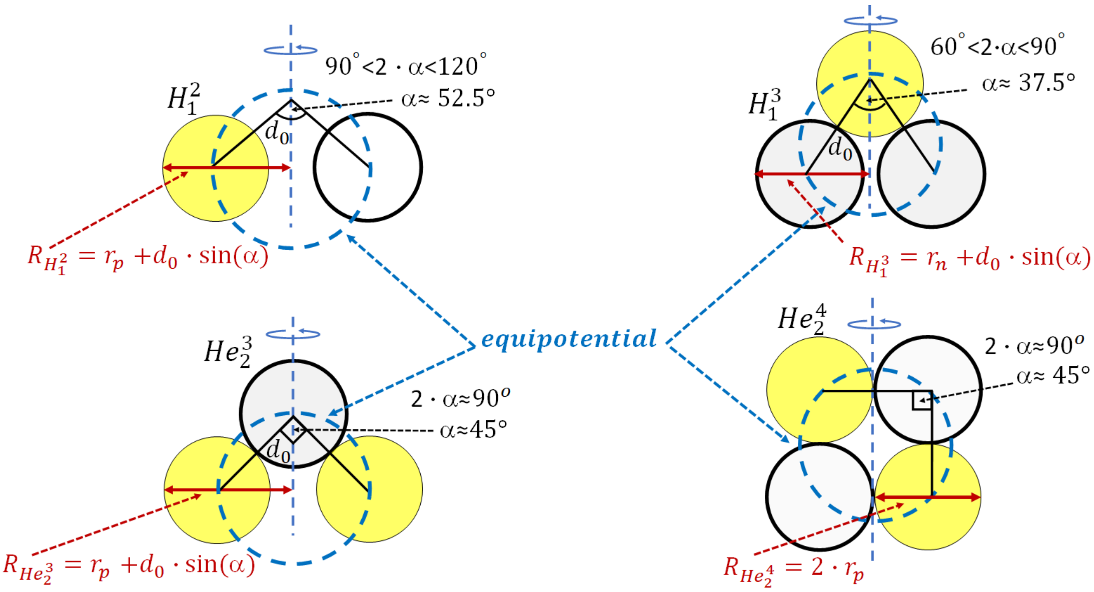
- layer 5 by Xenon (two D orbitals)
- layer 7 by Oganesson (two F orbitals)

Results

Hydrogen and Helium: charge radii

The charge radii of hydrogen and helium decrease as the number of their nucleons increases; we relate this to the total nuclear force that is increasing with the number of nucleons for these nuclei and so also their density.
Following drawings explain this idea:

- He_2^4: the nucleon bonds have an angle of about $2 \cdot \alpha = 90°$ and so the charge radius is: $R_{\mathrm{He}_2^4} \approx 2 \cdot r_p$.
- He_2^3: the 90° angle remains unchanged, but to keep symmetry with respect to the z-axis, the nucleus it turned in 45°, resulted in: $R_{He_2^3} \approx r_p + d_0 \cdot \sin(45°)$.
- H_1^3: the nucleus is very similar to He_2^3, but the protons and neutrons are swapped, so the two neutrons are a bit nearer, due to the lack of the electric repulsion. We estimate that the angle is between that of a right-angled and an isosceles triangle. This means an angle of $60° < 2 \cdot \alpha < 90°$ ($\alpha \approx \frac{30+45}{2} = 37.5$).
  We get: $R_{H_1^3} \approx r_n + d_0 \cdot \sin(37.5°)$.
- H_1^2: we treat the nucleus in a similar manner to the last two nuclei, although the top nucleon is missing; we estimate the angle therefore to be slightly larger than that of He_2^4, because of the smaller total attraction between the nucleons, so our estimation is $90° < 2 \cdot \alpha < 120°$ ($\alpha \approx \frac{45+60}{2} = 52.5$), and we get: $R_{H_1^2} \approx r_n + d_0 \cdot \sin(52.5°)$.
- **Remark**: we assume that the centers of the nucleons lie on an equipotential circle.



nucleus	R_{calc}	R_{meas}	rel. error	formula
H_1^2	2.13	2.14	0.8%	$r_p + d_0 \cdot \sin(52.5°)$
H_1^3	1.79	1.76	1.8%	$r_n + d_0 \cdot \sin(37.5°)$
He_2^3	1.99	1.97	1.0%	$r_p + d_0 \cdot \sin(45°)$
He_2^4	1.68	1.68	0.3%	$2 \cdot r_p$

Data of the charge radius: [2].
Remark: the results agree with the experimental data, but we note that this is only a rough estimation and not necessarily a proof of the model.

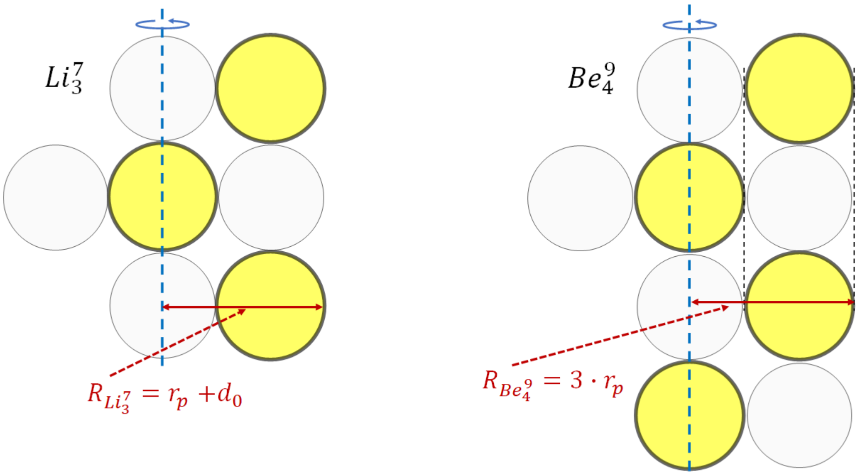
Lithium and Beryllium: charge radii

We observe now the charge radii of lithium and beryllium and try to fit the model to the experimental data.
We get good approximation using the formula:

- $R_{Li_3^7} = r_p + d_0$
- $R_{Be_4^9} = 3 \cdot r_p$

and explain it so:

- Li: the neutron and proton have the relationship as expected from the model.
- Be: maybe due to the larger repulsion from the additional proton, the protons determine the distances between the columns, but this could also be due to other reasons.



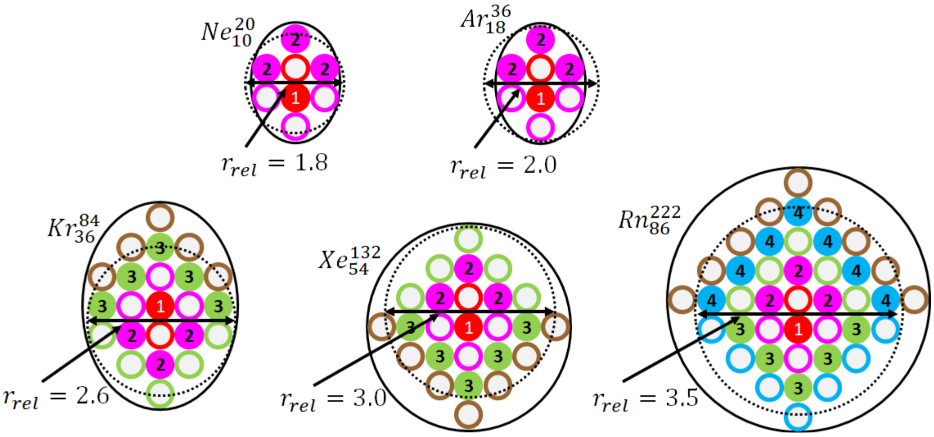
nucleus	R_{calc}	R_{meas}	rel. error	formula
Li_3^7	2.46	2.44	0.7%	$R_n + d_0$
Be_4^9	2.52	2.52	0.0%	$3 \cdot R_p$

Data of the charge radius [2].

Remark: we don't give here an explanation why this happens, but only show how this can be described geometrically or visually by the model.

Noble gases: charge radii

The noble gas nuclei of **Ne**, **Ar**, **Kr**, **Xe** and **Rn** possess according to the model complete shells, so the estimation of their charge radius is possibly easier. About Oganesson there is not enough experimental data.
The charge radius of the noble gasses is calculated in the x-y plane; a circle is drawn from the center of the nucleus till the "nearest" outside proton edge. This is a rough estimation, but it gives good results.
A double layer is slightly wider than a single layer; (the reason is not understood by the model; it could be that the excess neutron mix a bit with the neutron, but other reasons are possible as well; for instance due to rotation or precession of the nucleus;) therefore, Argon is wider than Neon; Xenon is wider than Krypton and in addition there are extra neutron in its envelope that increase the radius.



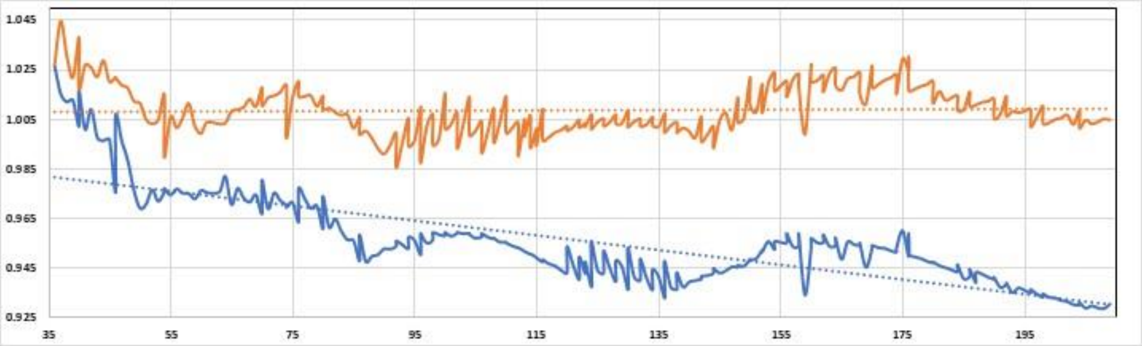
nucleus	R_{rel}	R_{calc}	R_{meas}	rel. error
Ne_{10}^{20}	1.8	2.92	3.01	3.0%
Ar_{18}^{36}	2.0	3.24	3.39	4.4%
Kr_{36}^{86}	2.6	4.21	4.19	0.6%
Xe_{54}^{132}	3.0	4.86	4.79	1.5%
Rn_{86}^{222}	3.5	5.67	5.69	0.4%

Data of the charge radius [2].

- R_{meas}: the measured radius.
- r_{rel}: relative radius, the number of nucleons contained in the radius as taken from the drawing.
- R_{calc}: the calculated radius: $R_{calc} \approx r_{rel} \cdot d_0$.
- d_0: the radius of proton + the radius of the neutron. $d_0 = r_p + r_n = 1.62\ fm$ [5].

The charge radius

According to the liquid drop model an approximately constant value would be expected for the ratio between the nucleus charge radius $\mathbf{R_c}$ and the third root of **A**, the atomic mass or the number of nucleons $\sqrt[3]{\mathbf{A}}$. We expect therefore to get $\frac{\mathbf{R_c}}{\sqrt[3]{\mathbf{A}}} \approx$ **constant**.
In reality this ratio decreases as the number of nucleons grows. This could mean a change of the nucleus density or nucleons distribution in the nucleus for larger nuclei or that the nucleus charge is concentrated toward its center by p-n pairs and the excess neutrons are located in its envelope, as the model assumes.
The number of nucleons **A** would therefore better be replaced by (twice) the number of protons **2·Z** (assuming that the number of neutrons is not smaller than that of the protons).
The following graph shows this by comparing between $\frac{\mathbf{R_c}}{\sqrt[3]{\mathbf{A}}}$ and $\frac{\mathbf{R_c}}{\sqrt[3]{\mathbf{2\cdot Z}}}$ for nuclei from Ar_{18} up to Cm_{96} vs. **Z** (for nuclei smaller than Ar_{18} the number of protons and neutrons is quite equal so there is no major different between the two).



Graph: a comparison between $\frac{R_{measured}}{A^{1/3}}$ *and* $\frac{R_{measured}}{(2\cdot Z)^{1/3}}$ (*raw data from* [2]***).***
Dotted lines: linear fit.

The following table summarizes the calculations of the above data.

	$\frac{R_{measured}}{A^{1/3}}$	$\frac{R_{measured}}{(2Z)^{1/3}}$
max	1.027	1.045
min	0.928	0.985
Δ = max-min	0.098	0.059
average	0.955	1.009
standard dev.	0.018	0.010

The result for $(2 \cdot Z)^{\frac{1}{3}}$ fits the data better than the one for $A^{\frac{1}{3}}$. The reason is that the protons and neutrons build a core with an equal number of both, and the excess neutrons are located in the envelope of the nucleus, and so don't influence its charge distribution.
Data of the charge radius [2].

Discussion of the results and conclusion

The results strengthen the model, yet no decisive proof was delivered. The conclusions are:

- The charge radius results for hydrogen and helium are well explained by the model.
- The charge radius of lithium and beryllium were briefly analyzed and strengthen the assumption of the model structure as well.
- The charge radius of the noble gas nuclei also strengthen the assumption of the model structure.
- The excess neutrons of the nuclei, beyond the number equal to that of the protons, seem to be located in its envelope, as the model predicts.

Sources and references

1. *Tables of Nuclear Data: Japan Atomic Energy Agency (JAEA)*
2. *Charge Radius: International Atomic Energy Agency (IAEA)*
3. *Neutron radius: Povh, B.; Rith, K.; Scholz, C.; Zetsche, F. (2002). Particles and Nuclei: An Introduction to the Physical Concepts. Berlin: Springer-Verlag. p. 73*
4. *Proton radius: Yong-Hui Lin, Hans-Werner Hammer and Ulf-G. Meißner: New insights into the nucleon's electromagnetic structure; Physical Review Letters, https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.128.052002*
5. *a cubic ellipsoid geometric model of the atomic nucleus and its mass formula - Ronen Yavor - (viXra)*