Covering and Connectedness

Frederick Temple *

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Abstract

The aim of this article is to provide proofs for theorems and propositions found in [1] and [2].

Definition 1. The function f is said to be $\mathcal{V}$-continuous if for any $x \in X$ there is a neighbourhood U of x and $V \in \mathcal{V}$ such that $f(U) \subseteq V$.

Definition 2. U refines V if for any $U \in \mathcal{U}$ there exists a $V \in \mathcal{V}$ such that $U \subseteq V$ (this is denoted by $U \prec V$).

Definition 3. A chain in $\mathcal{U}$ that joins x and y is a finite sequence $U_1, U_2, \ldots, U_n$ of elements in $\mathcal{U}$ such that $x \in U_1, y \in U_n$ and $U_i \cap U_{i+1} \neq \emptyset, \forall\, i \in \{1, 2, \ldots, n-1\}$.

Proposition 4. *If $f : X \to Y$ is $\mathcal{V}$-continuous, $g : Y \to Z$ is $\mathcal{W}$-continuous and $g(\mathcal{V}) \prec \mathcal{W}$, then $g \circ f : X \to Z$ is $\mathcal{W}$-continuous.*

Proof. Because f is $\mathcal{V}$-continuous, there exist a neighbourhood U of $x \in X$ such that, for a neighbourhood V of $y \in Y$, $f^{-1}(V) \subseteq U$. Similarly, because g is $\mathcal{W}$-continuous, there exists a neighbourhood V of $y \in Y$ such that, for a neighbourhood W of $z \in Z$, $g^{-1}(W) \subseteq V$ (because $g(\mathcal{V}) \prec \mathcal{W}$). Therefore $f(U) \subseteq V$, which leads to $g(f(U)) \subseteq g(V) \subseteq W$. Therefore $g \circ f : X \to Z$ must be $\mathcal{W}$-continuous. $\square$

Theorem 5. *The topological space X is connected if and only if it is chain connected in X.*

Proof. ($\implies$) BWOC, If X is not chain connected then for $x \in U_1, y \in U_n$ $\exists$ $i \in \{1, 2, \ldots, n-1\}$ such that $U_i \cap U_{i+1} = 0$. If this is true for all $x, y \in X$ then this implies a disconnected space (contradicting the connectedness of the topological space).
($\impliedby$) If X is chain connected if for every $x, y \in X$ there exists a chain connecting x with y. If the topological space is not connected then $\exists\, U, V \subset X$ such that $U \cap \bar{V} = \emptyset$, $\bar{U} \cap V = \emptyset$ and $U \cup V = X$. If $x \in U$ and $y \in V$, it is impossible to find a chain connecting x with y, so it must be the case that a chain connected set implies a connected topological space. $\square$

*pfrejituh@gmail.com

Corollary 6. *If topological space X is chain connected that it is also path connected.*

Proof. If x, y are chain connected in X there there's a chain of open sets connecting x to y, where $x \in U_1$ and $y \in U_n$ and, $\forall\ i \in \{1, 2, \ldots, n-1\}$, $U_i \cap U_{i+1} \neq 0$. This means that a path can be connected from x to a point in $U_1 \cap U_2$, and so on, until y is reached. ☐

Theorem 7. *If $f : X \to Y$ is a continuous function and $C \subseteq X$ is chain connected in X, then $f(C)$ is chain connected in Y.*

Proof. Because $C \subseteq X$ is connected, this implies $\forall\ x, y \in C, x \in U_1, y \in U_n$ and, $\forall\ i \in \{1, 2, \ldots, n-1\}$, $U_i \cap U_{i+1} \neq 0$. This means that, for a $\mathcal{V}$-continuous function f, $f(x) \subseteq V_1$ and $f(y) \subseteq V_n$. Thus $x \subseteq f^{-1}(V_1)$ and $y \subseteq f^{-1}(V_n)$. If Y is a disjointed topological space then $f(C) \subseteq Y = A \cup B$, $\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$; A and B are nonempty open sets in Y. Let $G = C \cap f^{-1}(A)$ and $H = C \cap f^{-1}(B)$. Because C is chain connected, it must be the case that $G \cap H \neq \emptyset$. But $f(G) = f(C) \cap A \subseteq A$ and $f(H) = f(C) \cap B \subseteq B$. Hence $G \cap H \subseteq f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) \subset f^{-1}(\bar{A} \cap B) = f^{-1}(\emptyset) = \emptyset$. Therefore $G \cap H = \emptyset$. This is a contradiction. ☐

Corollary 8. *If $f : X \to Y$ is a homeomorphism, then $C \subseteq X$ is chain connected in Xif and only if $f(C)$ is chain connected in Y.*

Proof. If f(C) is chain connected then this implies $\forall\ y_1, y_2 \in f(C), y_1 \in V_1, y_2 \in V_n$ and, $\forall\ i \in \{1, 2, \ldots, n-1\}$, $V_i \cap V_{i+1} \neq 0$. Therefore, if f is a homeomorphism (which means that X and Y are homeomorphic), this means that there's a one-to-one correspondence between elements of X and elements of Y. This means that if elements y_1 and y_2 are connected in Y then it must be the case that elements $f^{-1}(y_1) = x_1 \in U_1$ and $f^{-1}(y_2) = x_2 \in U_2$ [$\forall\ i \in \{1, 2, \ldots, n-1\}$, $f^{-1}(V_i) \cap f^{-1}(V_{i+1}) \neq 0$] are also chain connected (neighbourhoods are preserved). ☐

Theorem 9. *If C_i are chain connected sets in X_i, $\forall i \in I$, then $\prod_{i \in I} C_i$ is a chain connected set in $\prod_{j \in I} X_j$ equipped with the product topology.*

Proof. Let $p_i : \prod_{j \in I} X_j \to X_i$ be the i-th canonical projection. This is continuous for all X_j. Let $h : X_i \to Y_i$ be a continuous function and $C_i \subseteq X_i$ be chain continuous. If $h \circ p_i$ is not chain continuous then, it must be the case, that h is not chain continuous, which is a contradiction. ☐

Definition 10. The star of the set A with respect to the covering $\mathcal{U}$ of X in X is the set

$$st(A, \mathcal{U}) = \bigcup \{U \in \mathcal{U} \mid U \cap A \neq \emptyset\}.$$

The infinite star of the set A with respect to the covering $\mathcal{U}$ of X in X is the set

$$st^{\infty}(A, \mathcal{U}) = \cup_{n=1}^{\infty} st^n(A, \mathcal{U}).$$

If $A = \{x\}$, $st(\{x\}, \mathcal{U}) = st(x, \mathcal{U})$. The star degree of $n > 1$ of A and $\mathcal{U}$ in X is $st^n(A, \mathcal{U}) = st(st^{n-1}(A, \mathcal{U}))$.

Theorem 11. *The set C is chain connected in X if and only if for every $x \in C$ and every covering $\mathcal{U}$ of X, $C \subseteq st^{\infty}(x,\mathcal{U})$.*

Proof. ($\Longrightarrow$) If C is chain connected then there exist open sets connecting $x \in C \subseteq X$ to $y \in C \subseteq X$, where $x \in U_1$ and $y \in U_n$ and, $\forall\, i \in \{1, 2, \ldots, n-1\}$, $U_i \cap U_{i+1} \neq 0$. Therefore $\exists\, U \in \mathcal{U}$ such that $U \cap x \neq \emptyset$. Therefore $C \subseteq st(x,\mathcal{U}) \subseteq st^{\infty}(x,\mathcal{U})$.
($\Longleftarrow$) If $C \in st^{n}(x,\mathcal{U}) \subseteq st^{\infty}(x,\mathcal{U})$ then $\exists\, U_1 \in \mathcal{U}$ such that $U_1 \cap x \neq \emptyset$. By definition, $st^{n}(x,\mathcal{U}) = st(st^{n-1}(A,\mathcal{U}))$. When $n = 2$, $st^{2}(A,\mathcal{U}) = st(st^{1}(x,\mathcal{U}))$ - this means that that $\exists\, U_2 \in \mathcal{U}$ such that $U_2 \cap U_1 \neq \emptyset$. Continuing in this way will lead to a finite sequence $U_1, U_2, \ldots, U_n$ of elements in $\mathcal{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$, $\forall\, i \in \{1, 2, \ldots, n-1\}$. □

Definition 12. Let X be a topological space and $x \in Y \subseteq X$. The chain connected component of the point x of Y in X, denoted by $V_{YX}(x)$, is the biggest chain connected subset of Y in X that contains x.

Theorem 13. *Let $C_i, i \in I$, be a family of chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $\bar{C_{i_0}} \cap \bar{C_i} \neq \emptyset$, then the union $\cup_{i \in I} \bar{C_i}$ is chain connected in X.*

Proof. The proof of this can be found in [1]. □

Proposition 14. *The set of all chain connected subsets of Y in X consist of all of chain connected components of Y in X and their subsets.*

Proof. $A = A_{YX}(x)$ denotes the set of all points $y \in Y$ such that for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y. A is a partially ordered set because if there exists an $A_1 \in \cup C_i$ and $A_2 \in \cup C_i$ $(i \in I)$ such that, for every covering $\mathcal{U}$ of X, there exists a chain in $\mathcal{U}$ that connects x and y, then it's necessarily the case that either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.
I will use Zorn's lemma for this proof. $A_{YX}(x)$ is always nonempty because $\{x\} \in A_{YX}(x)$ (this is true because $x \underset{\mathcal{U}}{\sim} x$). It is known that for every two points $y, z \in A_{YX}(x)$, and for every covering $\mathcal{U}$ of X, there exist chains in $\mathcal{U}$ from y to x and from x to z. It follows that their union is a chain in $\mathcal{U}$ from y to z. Therefore take $V_{YX}(x)$ (the maximal element) to be the union of all $A_{YX}(x)$ in $\mathcal{U}$. □

Proposition 15. *For every $x \in Y$, one have $V_{YY}(x) \subseteq V_{YX}(x)$ $= \cup_{y \in V_{YX}(x)} V_{YY}(y)$.*

Proof. The proof of this can be found in [1]. □

Proposition 16. *For every $x \in Y$, $V_{YX}(x) = Y \cap V_{XX}(x)$. Each chain connected component of X in X contains at most one chain connected component of Y in X.*

Proof. The initial and terminal points of the chain $V_{XX}(x)$ must be the elements $x \in X$ and $x' \in X$, respectively. Therefore, the longest chain, denoted by $Y \cap V_{XX}(x)$, must have at least one point $y \in Y$ that isn't the initial or terminal point of the chain.
Suppose that we have a choice function $f : Y \cap V_{XX}(x) \to \cup A_{XX}(x)$ that picks the largest chain - with at least one point in Y that isn't the initial point, x, or terminal point of the chain, x'. This choice function cannot pick two chain connected components, because the longest chain will be a member of the union of all chains from x to x' (with at least one point $y \in Y$ in the chain that isn't the initial or terminal point in the chain) that have a non-zero overlap (based on Lemma 3.1 from [1]). It should be noted that $Y \cap V_{XX}(x) \subseteq A_{XX}(x)$. The function f is a choice function because for every $C \in Y \cap V_{XX}(x) \subseteq A_{XX}(x), f(C) \in Y \cap V_{XX}(x)$. It should be noted that, by construction, $Y \cap V_{XX}(x) = V_{YX} \subseteq \cup A_{XX}(x)$, but it must also be the case that $\cup A_{XX}(x) \subseteq V_{YX}(x)$, due to the maximality of $V_{YX}(x)$ in $\mathcal{U}$. Therefore $\cup A_{XX}(x) = V_{YX}(x)$. Therefore each chain connected component of X in X contains at most one chain connected component of Y in X.

□

Proposition 17. *The chain connected components of X are closed sets, i.e., for every $x \in X$, $V(x) = \bar{V}(x)$.*

Proof. If $\bar{V}(x)$ is connected then obviously $V(x) \subseteq \bar{V}(x)$ is also connected. But $V(x)$ is the largest connected chain, thus it must also be the case that $\bar{V}(x) \subseteq V(x)$. Therefore $V(x) = \bar{V}(x)$. □

Proposition 18. *Let $x \in X$ and $C(x)$ be a connected component of X. Then $C(x) \subseteq V(x)$.*

Proof. This is trivially true because $V(x)$ is the largest chain connected subset of Y in X that contains x. □

Theorem 19. *Quasicomponents and chain connected components in a topological space X coincide, i.e., for every $x \in X$, $Q_X(x) = V_{XX}(x)$.*

Proof. The proof of this can be found in [1]. □

Combining proposition 15, proposition 16 and theorem 19 leads to

Proposition 20. *For every $x \in Y$, we have $Q_Y(x) = V_{YY}(x) \subseteq \cup_{y \in V_{YX}} V_{YY}(y) = V_{YX}(x) \subseteq V_{XX}(x) = Q_X(x)$.*

Proof. We know that $Y \subseteq X$. From theorem 19, we know that $Q_X(x) = V_{XX}(x)$. For $x \in Y$, it is obviously the case that $Q_Y(x) = V_{YY}(x)$. By proposition 15, $V_{YY}(x)$ is a subset of the union of all elements $y \in V_{YY}(x)$ - this union is equal to $V_{YX}(x)$. Obviously (because $Y \subseteq X$) it must be the case that $V_{YX}(x) \subseteq V_{XX}(x) = Q_X(x)$. □

References

[1] A. Velkoska Z. Misajleski, N. Shekutkovski. Chain connected sets in a topological space. *Kragujevac Journal of Mathematics*, 43(4):575 – 586, 2019.

[2] Emin Durmishi. Continuity up to a covering and connectedness. *arXiv*, 2023.