Quasi-Metric Space I

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Abstract

Inspired by the work of Adhya and Ray, I provide my own proof of selected theorems and lemmas discussed in [1]. Original theorems should appear, in due course, in a future article.

Theorem 1. *Every singleton set in a μ-T_1 strong generalised topological space is μ-closed.*

Proof. All singletons in a singleton set are elements of the power set $\mathcal{P}(X)$. This means that there exist $B_1, B_2 \in \mu$, for each pair $x, y \in X$ (with $x \neq y$), such that $x \in \{x\}, y \notin \{x\}$ and $y \in \{y\}, x \notin \{y\}$. Because $x, y \in \mu$ and $X \in \mu$, due to (X, μ) being a strong generalised topological space, $\emptyset$ can only be in μ if X is both μ-closed and μ-open. If each singleton set, $\{x\} \in X$, is not μ-closed then $X = \bigcup_{x \in X} \{x\}$ is not μ-closed. This is a contradiction. Using the same logic, every singleton set in a μ-T_1 strong generalised topological space is also μ-open. $\square$

Theorem 2. *A metric space is Lebesgue if and only if every pseudo-Cauchy sequence having distinct terms clusters in it.*

Proof. ($\implies$) Let x_m and x_n both cluster to x. Let $d(x_m, x) < \delta_1$ when $m \in \mathbb{N} > k_1$ and let $d(x_n, x) < \delta_2$ when $n \in \mathbb{N} > k_2$. Assume that the function, f, is uniformly continuous - this means that $\forall \epsilon \, \exists \, \delta > 0$ such that if x_1 and $x_2 \in X$ with $d(x_1, x_2) < \delta$ then $d(f(x_1), f(x_2)) < \epsilon \; \forall x_1, x_2 \in X$. Let $k = \max\{k_1, k_2\}$ - this means that when $n, m > k$, $d(x_n, x_m) < d(x_m, x) + d(x_n, x) < \delta_1 + \delta_2 = \delta$. Therefore $d(f(x_n), f(x_m)) < f(\delta) = \epsilon$. Allowing $f(x) = y$ implies that both $f(x_m)$ and $f(x_n)$ cluster to y.
($\impliedby$) Let, $\forall \; \delta/2 > 0$, $d(x_n, x) < \delta/2 \; \forall n > k_x \in \mathbb{N}$. Therefore $d(x_n, x_m) < d(x_n, x) + d(x_m, x) < \delta/2 + \delta/2 = \delta \; \forall n, m \in \mathbb{N} > k_x$. Because f is continuous, this implies that (for $f : X \to Y$ such that $x \mapsto f(x) = y$), $d(f(x_n), f(x_m)) < \epsilon$, $\forall \; n, m \in \mathbb{N} > k_y$ (for all $x \in X$). This implies that $d(f(x_n), f(x)) < \epsilon/2$ $\forall n > k_y \in \mathbb{N}$. This means that f is also uniformly continuous. $\square$

Lemma 3. *Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of the same index r. A sequence (x_n, y_n) is G-Cauchy in $(X \times Y, d_{XY})$ if and only if (x_n) and (y_n) are G-Cauchy in (X, d_X) and (Y, d_Y) respectively.*

Proof. ($\Longrightarrow$) Let (x_n, y_n) by G-Cauchy in $(X \times Y, d_{XY})$. Choose $\epsilon > r$. Then $\exists k \in \mathbb{N}$ such that $d_{XY}((x_n, y_n), (x_{n+1}, y_{n+1})) < \epsilon$, $\forall n \geq k$. That is $d_X(x_n, x_{n+1}), d_Y(y_n, y_{n+1}) < \epsilon\ \forall n \geq k$. Then (x_n) and (y_n) are G-Cauchy in (X, d_X) and (Y, d_Y), respectively.
($\Longleftarrow$) Let (x_n) and (y_n) be G-Cauchy in (X, d_X) and (Y, d_Y), respectively. Choose $\epsilon > r$. Then $\exists k_1, k_2 \in \mathbb{N}$ such that $d_X(x_n, x_{n+1}) < \epsilon\ \forall\ n \geq k_1$ and $d_Y(y_n, y_{n+1}) < \epsilon\ \forall n \geq k_2$. Set $k =$max$\{k_1, k_2\}$. Then $d_X(x_n, x_{n+1})$, $d_Y(y_n, y_{n+1}) < \epsilon\ \forall n \geq k$. Hence (x_n, y_n) is G-Cauchy in $(X \times Y, d_{XY})$. □

Lemma 4. *Let (X, d_X) and (Y, d_Y) be g-quasi metric spaces of the same index r. A sequence (x_n, y_n) is pseudo-Cauchy in $(X \times Y, d_{XY})$ if and only if (x_n) and (y_n) are pseudo-Cauchy in (X, d_X) and (Y, d_Y) respectively.*

Proof. ($\Longrightarrow$) Let (x_n, y_n) by pseudo-Cauchy in $(X \times Y, d_{XY})$. Choose $\epsilon > r$. Then $\exists k \in \mathbb{N}$ such that $d_{XY}((x_n, y_n), (x_{n+1}, y_{n+1})) < \epsilon$, $\forall n \geq k$. That is $d_X(x_p, x_q), d_Y(y_p, y_q) < \epsilon\ \forall\ p, q \geq k$. Then (x_n) and (y_n) are pseudo-Cauchy in (X, d_X) and (Y, d_Y), respectively.
($\Longleftarrow$) Let (x_n) and (y_n) be pseudo-Cauchy in (X, d_X) and (Y, d_Y), respectively. Choose $\epsilon > r$. Then $\exists k_1, k_2 \in \mathbb{N}$ such that $d_X(x_p, x_q) < \epsilon\ \forall\ p, q \geq k_1$ and $d_Y(y_p, y_q) < \epsilon\ \forall\ p, q \geq k_2$. Set $k =$max$\{k_1, k_2\}$. Then $d_X(x_p, x_q), d_Y(y_p, y_q) < \epsilon$ $\forall\ p, q \geq k$. Hence (x_n, y_n) is pseudo-Cauchy in $(X \times Y, d_{XY})$. □

References

[1] Sugata Adhya and A. Deb Ray. On a generalization of quasi-metric space. *arXiv*, 2023.