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Inner product of two oriented points in conformal geometric algebra in detail *

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Abstract. We study in full detail the inner product of oriented points in conformal geometric algebra and its geometric meaning. The notion of oriented point is introduced and the inner product of two general oriented points is computed, analyzed (including symmetry) and graphed in terms of point to point distance, and angles between the distance vector and the local orientation planes of the two points. Seven examples illustrate the results obtained. Finally, the results are extended from dimension three to arbitrary dimensions n.

Keywords: Conformal geometric algebra, oriented points, point geometry

1 Introduction

This work is a substantial extension of [13]. In this work we apply conformal geometric algebra (CGA) to the description of points, including a planar orientation. An excellent general reference on Clifford's geometric algebras is [14], a short engineering oriented tutorial is [10], and [17] describes a free software extension for a standard industrial computer algebra system (MATLAB). Alternatively, all computations could be done in the optimized geometric algebra algorithm software GAALOP [6]. Introductions to CGA are given in [2,4] and efficient computational implementations are described in [6]. CGA has found wide ranging applications in physics, quantum computing, molecular geometry, engineering, signal and image processing, neural networks, computer graphics and vision, encryption, robotics, electronic and power engineering, etc. Up to date surveys are [1,8,12]. An introduction to the notion of oriented point can be found in [5]. Prominent applications could be to LIDAR terrain strip adjustment [11], protein geometry modelling [15], and machine learning.

* Dedicated to the truth, including the children who died in the *700% to 1600% increase in Excess Deaths among Children since European Medicines Agency approved COVID Vaccine for Kids*, [Source: The Expose of 18 Sep. 2022, https://expose-news.com/2022/09/18/eu-forced-investigate-17x-increase-excess-deaths-children/, last accessed 29 Sep. 2022]. Please note that this research is subject to the Creative Peace License [9].

In the current work, we begin with the CGA expression for oriented points in three Euclidean dimensions and compute their inner products (Section 2). We study the geometric information included in this inner product with the help of a wide range of representative examples (Section 3), analyze the most important term that includes the direction of the line segment connecting the two points and their two point orientations in detail (Section 4), and study the symmetries of the inner product of oriented points and plot the result (Section 5). Finally, we extend our framework from three to n Euclidean dimensions (Section 6). The paper concludes with Section 7, References and Appendix A with detailed proofs of oriented point inner product symmetries.

2 Computation of inner product of oriented points

We consider the inner product of two oriented points in conformal geometric algebra [5], as reference for practical CGA computations in this section we recommend the introductory chapter of this volume and [7]. Note that inner product and wedge product have priority over the geometric product, e.g., $\mathbf{i}_q \cdot \boldsymbol{q}E = (\mathbf{i}_q \cdot \boldsymbol{q})E$, etc. An *oriented point* is given by the trivector expression of a *circle with radius zero* ($r = 0$) in CGA,

$$Q = \mathbf{i}_q \wedge \boldsymbol{q} + [\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_q - \boldsymbol{q}(\boldsymbol{q} \cdot \mathbf{i}_q)]\boldsymbol{e}_\infty + \mathbf{i}_q\boldsymbol{e}_0 + \mathbf{i}_q \cdot \boldsymbol{q}E, \tag{1}$$

where the three-dimensional position vector of Q is the vector $\boldsymbol{q} \in \mathbb{R}^3$, the unit oriented bivector of the plane (orthogonal to the normal vector $\boldsymbol{n}_q$ of the plane) is $\mathbf{i}_q \in Cl^2(3,0)$, $\boldsymbol{e}_0$ is the vector for the origin dimension, $\boldsymbol{e}_\infty$ is the vector for the infinity dimension, and the origin-infinity bivector is $E = \boldsymbol{e}_\infty \wedge \boldsymbol{e}_0$, with

$$\boldsymbol{e}_0^2 = \boldsymbol{e}_\infty^2 = 0, \quad \boldsymbol{e}_0 \cdot \boldsymbol{e}_\infty = -1, \quad \boldsymbol{e}_0 E = -\boldsymbol{e}_0, \quad \boldsymbol{e}_\infty E = \boldsymbol{e}_\infty, \tag{2}$$

and $\boldsymbol{e}_0$ and $\boldsymbol{e}_\infty$ are both orthogonal to $\mathbb{R}^3$. The central pseudoscalar of CGA $I = e_{123}E = i_3E = Ei_3$, $I^{-1} = -i_3E$, leads to the dual (bivector) form of the oriented point

$$\begin{aligned}
Q^* &= QI^{-1} = -Qi_3E \\
&= -(\mathbf{i}_q \wedge \boldsymbol{q})i_3E + [\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_q i_3 - \boldsymbol{q}(\boldsymbol{q} \cdot \mathbf{i}_q)i_3]\boldsymbol{e}_\infty E + \mathbf{i}_q i_3 \boldsymbol{e}_0 E - (\mathbf{i}_q \cdot \boldsymbol{q})i_3E^2 \\
&= \mathbf{i}_q^* \cdot \boldsymbol{q}E + [\frac{1}{2}\boldsymbol{q}^2(-\mathbf{i}_q^*) + \boldsymbol{q}(\boldsymbol{q} \wedge \mathbf{i}_q^*)]\boldsymbol{e}_\infty + \mathbf{i}_q^*\boldsymbol{e}_0 + \mathbf{i}_q^* \wedge \boldsymbol{q} \\
&= \mathbf{i}_q^* \cdot \boldsymbol{q}E + [-\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_q^* + \boldsymbol{q}(\boldsymbol{q}\mathbf{i}_q^* - \boldsymbol{q} \cdot \mathbf{i}_q^*)]\boldsymbol{e}_\infty + \mathbf{i}_q^*\boldsymbol{e}_0 + \mathbf{i}_q^* \wedge \boldsymbol{q} \\
&= \mathbf{i}_q^* \cdot \boldsymbol{q}E + [\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_q^* - \boldsymbol{q}(\boldsymbol{q} \cdot \mathbf{i}_q^*)]\boldsymbol{e}_\infty + \mathbf{i}_q^*\boldsymbol{e}_0 + \mathbf{i}_q^* \wedge \boldsymbol{q}, \\
&= \boldsymbol{n}_q \wedge \boldsymbol{q} + [\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q} \cdot \boldsymbol{n}_q)]\boldsymbol{e}_\infty + \boldsymbol{n}_q\boldsymbol{e}_0 + \boldsymbol{n}_q \cdot \boldsymbol{q}E,
\end{aligned} \tag{3}$$

using $\boldsymbol{n}_q = \mathbf{i}_q^*$ for the normal vector of bivector $\mathbf{i}_q$. The same expression for Q^* is found in [5], equation (4). In Section 6 we will show how to generalize the dual form (3) of oriented points to arbitrary dimensions. The equivalence of Q and $Q^* = QI^{-1}$ is obvious due to $Q = QI^{-1}I = Q^*I$.

For comparison we also state the expression of a conformal point (without orientation: *no*) and circle[1] in CGA:

$$Q_{no} = \boldsymbol{q} + \frac{1}{2}\boldsymbol{q}^2\boldsymbol{e}_\infty + \boldsymbol{e}_0, \qquad C = Q + \frac{1}{2}r^2\mathbf{i}_q\boldsymbol{e}_\infty, \tag{4}$$

where Q_{no} is simply given by the three-dimensional position vector $\boldsymbol{q} \in \mathbb{R}^3$ plus two terms in $\boldsymbol{e}_\infty$ and $\boldsymbol{e}_0$, while the conformal expression for the circle is the same as the oriented point (1), albeit with finite radius $r > 0$.

The Euclidean bivector $\mathbf{i}_q$ specifying the Euclidean carrier of the circle, respectively the orientation (local plane information) of the oriented point, can be obtained from the term $\mathbf{i}_q\boldsymbol{e}_0$ in (1) as (right contraction: $\lfloor$)

$$\mathbf{i}_q = -(C \wedge \boldsymbol{e}_\infty)\lfloor E = -(Q \wedge \boldsymbol{e}_\infty)\lfloor E. \tag{5}$$

The point Q_{no}, geometrically at the center of the circle C, can be directly obtained as[2]

$$Q_{no} = \widehat{C}\boldsymbol{e}_\infty C = \widehat{Q}\boldsymbol{e}_\infty Q, \tag{6}$$

the three-dimensional position vector $\boldsymbol{q} \in \mathbb{R}^3$ as

$$\boldsymbol{q} = \frac{(Q_{no} \wedge E)\lfloor E}{-Q_{no} \cdot \boldsymbol{e}_\infty}, \tag{7}$$

and the radius of the circle as

$$r^2 = \frac{C\widehat{C}}{\mathbf{i}_q^2}. \tag{8}$$

We take a second oriented point P positioned at the origin $\boldsymbol{p} = 0$ with plane orientation bivector $\mathbf{i}_p$,

$$P = \mathbf{i}_p\boldsymbol{e}_0. \tag{9}$$

Now we compute the inner product of P and Q by taking the scalar part[3] of their geometric product

$$\begin{aligned}
P \cdot Q = \langle PQ \rangle &= \Big\langle (\mathbf{i}_p\boldsymbol{e}_0)\left\{\mathbf{i}_q \wedge \boldsymbol{q} + \left[\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_q - \boldsymbol{q}(\boldsymbol{q}\cdot\mathbf{i}_q)\right]\boldsymbol{e}_\infty + \mathbf{i}_q\boldsymbol{e}_0 + \mathbf{i}_q \cdot \boldsymbol{q}E\right\}\Big\rangle \\
&= \Big\langle \mathbf{i}_p\boldsymbol{e}_0\left[\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_q - \boldsymbol{q}(\boldsymbol{q}\cdot\mathbf{i}_q)\right]\boldsymbol{e}_\infty \Big\rangle = -\Big\{\frac{1}{2}\boldsymbol{q}^2\langle \mathbf{i}_p\mathbf{i}_q\rangle - \langle \mathbf{i}_p\boldsymbol{q}\,(\boldsymbol{q}\cdot\mathbf{i}_q)\rangle\Big\} \\
&= -\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_p\cdot\mathbf{i}_q + \Big\langle \big(\mathbf{i}_p\cdot\boldsymbol{q} + \mathbf{i}_p \wedge \boldsymbol{q}\big)\,(\boldsymbol{q}\cdot\mathbf{i}_q)\Big\rangle \\
&= -\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_p\cdot\mathbf{i}_q + \Big\langle (\mathbf{i}_p\cdot\boldsymbol{q})(\boldsymbol{q}\cdot\mathbf{i}_q)\Big\rangle = -\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_p\cdot\mathbf{i}_q - \Big\langle (\boldsymbol{q}\cdot\mathbf{i}_p)(\boldsymbol{q}\cdot\mathbf{i}_q)\Big\rangle.
\end{aligned} \tag{10}$$

[1] Two ways to obtain a circle in CGA are: (1) by the outer product of any three conformal points on the circle, (2) by combining center vector $\boldsymbol{q}$, carrier bivector $\mathbf{i}_q$ and radius r as specified by (1) and (4).

[2] For comparison one can norm (scale) the homogeneous result, such that the $\boldsymbol{e}_0$-component becomes one: $Q_{no}/(-Q_{no}\cdot\boldsymbol{e}_\infty)$. Since the radius term in C (right side of (4)) is proportional to $\boldsymbol{e}_\infty$, it does not contribute in (6), due to $\boldsymbol{e}_\infty^2 = 0$.

[3] The scalar (grade zero) part is in GA conventionally indicated by $\langle A \rangle = \langle A \rangle_0$.

Note that in this situation $\boldsymbol{q}$ becomes the Euclidean distance vector from P (at the origin) to Q.

We now use the fact that the unit oriented bivector $\mathbf{i}_q$ of the plane is dual to the unit normal vector $\boldsymbol{n}_q$ via multiplication with the central three-dimensional Euclidean volume pseudoscalar $i_3 = \boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3$, $i_3^2 = -1$,

$$\mathbf{i}_q = \boldsymbol{n}_q i_3, \qquad \mathbf{i}_p = \boldsymbol{n}_p i_3. \tag{11}$$

This gives by (70) and (67) in [10], where $\times$ is the standard cross product of three-dimensional vector algebra,

$$\boldsymbol{q} \cdot \mathbf{i}_p = \boldsymbol{q} \cdot (\boldsymbol{n}_p i_3) = (\boldsymbol{q} \wedge \boldsymbol{n}_p) i_3 = -\boldsymbol{q} \times \boldsymbol{n}_p, \quad \boldsymbol{q} \cdot \mathbf{i}_q = -\boldsymbol{q} \times \boldsymbol{n}_q. \tag{12}$$

Therefore

$$-\langle (\boldsymbol{q} \cdot \mathbf{i}_p)(\boldsymbol{q} \cdot \mathbf{i}_q)\rangle = -\langle (\boldsymbol{q} \times \boldsymbol{n}_p)(\boldsymbol{q} \times \boldsymbol{n}_q)\rangle = -(\boldsymbol{q} \times \boldsymbol{n}_p) \cdot (\boldsymbol{q} \times \boldsymbol{n}_q). \tag{13}$$

Note: The resulting quadruple product appears in the proof of the spherical law of cosines [18]. The quadruple product can be expanded to

$$\begin{aligned} -(\boldsymbol{q} \times \boldsymbol{n}_p) \cdot (\boldsymbol{q} \times \boldsymbol{n}_q) &= -[\boldsymbol{q}^2 \boldsymbol{n}_p \cdot \boldsymbol{n}_q - (\boldsymbol{q} \cdot \boldsymbol{n}_q)(\boldsymbol{q} \cdot \boldsymbol{n}_p)] \\ &= -\boldsymbol{q}^2 [\boldsymbol{n}_p \cdot \boldsymbol{n}_q - (\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)], \end{aligned} \tag{14}$$

with unit P to Q distance direction vector $\hat{\boldsymbol{q}}$, such that $\boldsymbol{q} = |\boldsymbol{q}|\hat{\boldsymbol{q}}$. Note also that from (11)

$$\mathbf{i}_p \cdot \mathbf{i}_q = -\boldsymbol{n}_p \cdot \boldsymbol{n}_q. \tag{15}$$

Then we can write the *full inner product of two oriented points* as

$$\begin{aligned} P \cdot Q &= \frac{1}{2}\boldsymbol{q}^2 \boldsymbol{n}_p \cdot \boldsymbol{n}_q - [\boldsymbol{q}^2 \boldsymbol{n}_p \cdot \boldsymbol{n}_q - (\boldsymbol{q} \cdot \boldsymbol{n}_q)(\boldsymbol{q} \cdot \boldsymbol{n}_p)] \\ &= -\frac{1}{2}\boldsymbol{q}^2 \boldsymbol{n}_p \cdot \boldsymbol{n}_q + (\boldsymbol{q} \cdot \boldsymbol{n}_q)(\boldsymbol{q} \cdot \boldsymbol{n}_p) \\ &= \boldsymbol{q}^2[-\frac{1}{2}\boldsymbol{n}_p \cdot \boldsymbol{n}_q + (\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)] \\ &= \boldsymbol{q}^2[-\frac{1}{2}\cos\alpha_{pq} + \cos\Theta_q \cos\Theta_p], \end{aligned} \tag{16}$$

if we define $\cos\alpha_{pq} = \boldsymbol{n}_p \cdot \boldsymbol{n}_q$, $\cos\Theta_q = \hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q$, and $\cos\Theta_p = \hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p$, where α_{pq} is the dihedral angle between the two point orientation planes, and Θ_q is the angle between the P to Q distance vector $\boldsymbol{q}$ and $\boldsymbol{n}_q$, while Θ_p is the angle between $\boldsymbol{q}$ and $\boldsymbol{n}_p$, respectively. See Fig. 1 for illustration, with P at the origin, and $\boldsymbol{q}$ replaced by $\boldsymbol{d}$.

Remark 1. Note that the above relation (16) is fully general, even if P is a point in general position. Because our special situation, with P at the origin, is only different from the general situation by a global translation, which will not change the inner product scalar $P \cdot Q = \langle PQ\rangle$. In the general case, the vector $\boldsymbol{q}$ will simply be replaced by the Euclidean distance vector between the two point positions $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$, see Fig. 1. Furthermore, because $P \cdot Q = \langle PQ \rangle$ is also invariant under global rotations, only the relative orientations of $\boldsymbol{n}_p$, $\boldsymbol{n}_q$, and $\boldsymbol{d}$ matter. That is, any pair of oriented points that differs from the pair P and Q only by a global motor (translation and rotation), has the same inner product. In this sense the inner product shows *intrinsic properties* of the pair of oriented points P and Q.

Remark 2. Note that the (non-oriented) distance between two points P_{no} and Q_{no} in CGA also results from their inner product

$$P_{no} \cdot Q_{no} = -\frac{1}{2}\boldsymbol{q}^2. \tag{17}$$

The product of two oriented points (16) and their corresponding non-oriented CGA points (17) is therefore in general related by

$$\begin{aligned} P \cdot Q &= -\frac{1}{2}\boldsymbol{q}^2[\boldsymbol{n}_p \cdot \boldsymbol{n}_q - 2(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)] \\ &= P_{no} \cdot Q_{no}\,[\boldsymbol{n}_p \cdot \boldsymbol{n}_q - 2(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)] \\ &= P_{no} \cdot Q_{no}\,[\cos\alpha_{pq} - 2\cos\Theta_q \cos\Theta_p]. \end{aligned} \tag{18}$$

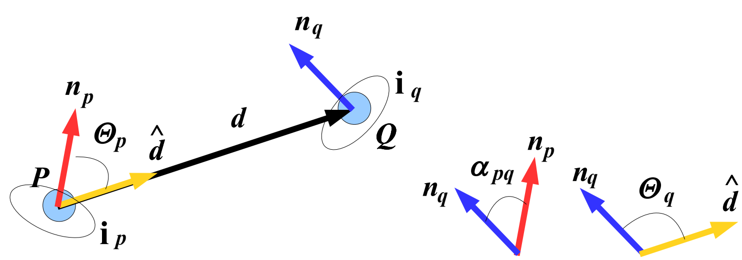

Fig. 1. Illustration of inner product of two oriented points P and Q, with Euclidean distance vector $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$, α_{pq} dihedral angle between the two orientation planes $\mathbf{i}_p$ and $\mathbf{i}_q$, Θ_q angle between $\boldsymbol{d}$ and $\boldsymbol{n}_q$, and Θ_p angle between $\boldsymbol{d}$ and $\boldsymbol{n}_p$, respectively.

For the special case, depicted in Fig. 2 with two parallel orientation planes[4], i.e. $\boldsymbol{n}_p = \boldsymbol{n}_q$, $\boldsymbol{n}_p \cdot \boldsymbol{n}_q = 1$ (i.e. $\alpha_{pq} = 0$), we have with the consequence $\Theta = \Theta_q = \Theta_p$ that

$$\begin{aligned} P \cdot Q &= \boldsymbol{q}^2(-\frac{1}{2} + \cos^2\Theta) = -\frac{1}{2}\boldsymbol{q}^2(1 - 2\cos^2\Theta) = \frac{1}{2}\boldsymbol{q}^2\cos 2\Theta \\ &= -P_{no} \cdot Q_{no}\cos 2\Theta, \end{aligned} \tag{19}$$

using the trigonometric identity $1 - 2\cos^2\Theta = -\cos 2\Theta$. The corresponding graph for $P \cdot Q$ can be seen in Fig. 3.

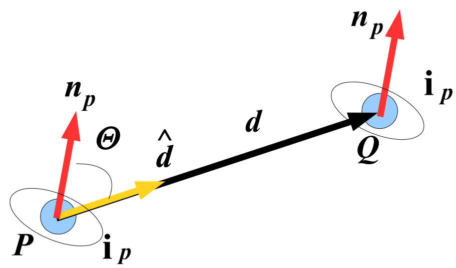

Fig. 2. Illustration of inner product of two parallel oriented points P and Q, with Euclidean distance vector $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$, dihedral angle $\alpha_{pq} = 0$, and equal $\Theta = \Theta_p = \Theta_q$ angles between $\boldsymbol{d}$ and $\boldsymbol{n}_p = \boldsymbol{n}_q$, respectively.

For the special case, depicted in Fig. 4 that additionally $\Theta = 0$, i.e. both planes are parallel and the distance $\boldsymbol{q}$ perpendicular to the planes we have

$$P \cdot Q = \frac{1}{2}\boldsymbol{q}^2 = -P_{no} \cdot Q_{no}. \tag{20}$$

3 Examples

To gain some intuition for what the inner product of two oriented points in CGA (16) means, we compute and sketch several examples, always assuming for simplicity that the first point P is positioned at the origin: $\boldsymbol{p} = 0$. That is we always have $\boldsymbol{d} = \boldsymbol{q}$. Even though we will later fully graph the inner product $P \cdot Q$ in Fig. 10, after having discussed its symmetries, we still think it is good to first look at a representative range of concrete examples.

Example 1. First we look at two parallel planes at orthogonal distance three, as depicted in Fig. 4.

$$\mathbf{i}_p = \mathbf{i}_q = \boldsymbol{e}_{12}, \quad \boldsymbol{n}_p = \boldsymbol{n}_q = \boldsymbol{e}_3, \quad \boldsymbol{q} = 3\boldsymbol{e}_3, \quad \boldsymbol{q}^2 = 9, \quad \hat{\boldsymbol{q}} = \boldsymbol{e}_3. \tag{21}$$

Then we can compute directly

$$P \cdot Q = \left\langle \boldsymbol{e}_{12}\, \boldsymbol{e}_0 \left[\frac{1}{2}\, 9\, \boldsymbol{e}_{12} - 3\, \boldsymbol{e}_3\, (3\, \boldsymbol{e}_3 \cdot \boldsymbol{e}_{12})\right] \boldsymbol{e}_\infty \right\rangle = -\frac{1}{2}\, 9\, \boldsymbol{e}_{12}\, \boldsymbol{e}_{12} = \frac{9}{2}, \tag{22}$$

because $\boldsymbol{e}_0[\frac{1}{2}9\boldsymbol{e}_{12} - 3\boldsymbol{e}_3(3\boldsymbol{e}_3 \cdot \boldsymbol{e}_{12})] = [\frac{1}{2}9\boldsymbol{e}_{12} - 3\boldsymbol{e}_3(3\boldsymbol{e}_3 \cdot \boldsymbol{e}_{12})]\boldsymbol{e}_0$, $\boldsymbol{e}_0 \cdot \boldsymbol{e}_\infty = -1$, $\boldsymbol{e}_3 \cdot \boldsymbol{e}_{12} = 0$, and $\boldsymbol{e}_{12}^2 = -1$. The result also confirms (20), and corresponds to the point labeled (a) in Fig. 3.

[4] This will also approximately be the case, if two matching oriented points are compared, e.g. in LIDAR terrain strip adjustment [11].

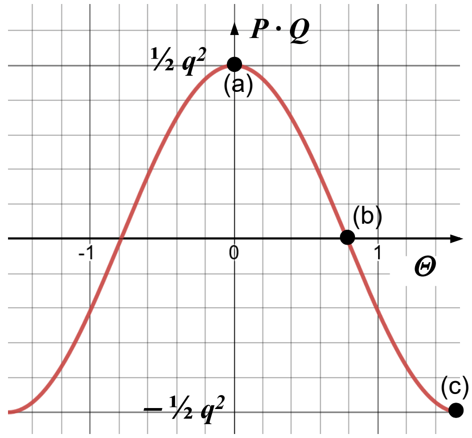

Fig. 3. Graph of inner product of two oriented points $P \cdot Q$ with parallel orientations ($\alpha_{pq} = 0$ for all angles $\Theta = \Theta_p = \Theta_q \in [-\pi/2, \pi/2]$ of the Euclidean distance vector $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$ with the normal orientation vector $\boldsymbol{n}_p = \boldsymbol{n}_q$. Point (a) marks angle $\Theta = 0$, compare Example 1 and Fig. 4. Point (b) marks $\Theta = \pi/4$, compare Example 3 and Fig. 6. Finally, point (c) marks $\Theta = \pi/2$, compare Example 2 and Fig. 5.

We obtain the same result, if we apply (16) instead. Toward this we compute

$$\cos \alpha_{pq} = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1, \quad \cos \Theta_q = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1, \quad \cos \Theta_p = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1. \tag{23}$$

Hence, as expected

$$P \cdot Q \overset{(16)}{=} 9(-\frac{1}{2} + 1) = \frac{9}{2}. \tag{24}$$

Example 2. Next we look at two parallel planes, and the points are separated by a vector in the plane, i.e. the P glides along its own plane by $\boldsymbol{q}$ to become Q, see Fig. 5. Assuming

$$\begin{aligned} &\mathbf{i}_p = \mathbf{i}_q = \boldsymbol{e}_{12}, \quad \boldsymbol{n}_p = \boldsymbol{n}_q = \boldsymbol{e}_3, \\ &\boldsymbol{q} = \boldsymbol{e}_1 + \boldsymbol{e}_2, \quad \boldsymbol{q}^2 = 2, \quad \hat{\boldsymbol{q}} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_2), \end{aligned} \tag{25}$$

we obtain

$$\begin{aligned} &\cos \alpha_{pq} = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1, \quad \cos \Theta_q = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_2) \cdot \boldsymbol{e}_3 = 0, \\ &\cos \Theta_p = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_2) \cdot \boldsymbol{e}_3 = 0. \end{aligned} \tag{26}$$

Applying (16) the inner product becomes

$$P \cdot Q = 2(-\frac{1}{2} + 0) = -1. \tag{27}$$

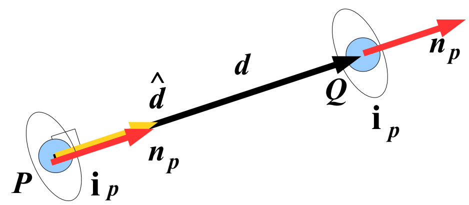

Fig. 4. Illustration of inner product of two fully aligned oriented points P and Q, with Euclidean distance vector $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$, dihedral angle $\alpha_{pq} = 0$, and equal angles $\Theta_p = \Theta_q = 0$ between $\boldsymbol{d}$ and $\boldsymbol{n}_p = \boldsymbol{n}_q$, respectively.

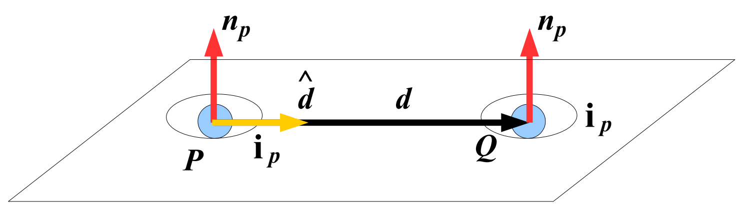

Fig. 5. Illustration of inner product of two parallel oriented points P and Q, with Euclidean distance vector $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$ in the common orientation plane, i.e., dihedral angle $\alpha_{pq} = 0$, and equal angles $\Theta_p = \Theta_q = \pi/2$ between $\boldsymbol{d}$ and $\boldsymbol{n}_p = \boldsymbol{n}_q$, respectively.

In this case only the first term in (16) proportional to $\boldsymbol{n}_p \cdot \boldsymbol{n}_p$ contributes, and the result corresponds to the point labeled (c) in Fig. 3.

Remark 3. Generally, whenever $\mathbf{i}_p = \pm \mathbf{i}_q$ and $\boldsymbol{q} \wedge \mathbf{i}_p = 0$ ($\boldsymbol{q}$ in the common orientation plane) then

$$P \cdot Q = \pm P_{no} \cdot Q_{no} = \mp \frac{1}{2} \boldsymbol{q}^2, \tag{28}$$

i.e. as expected the orientation of the two points then becomes insubstantial in the result, apart from the ± 1 relative orientation factor. Example 2 conforms to these requirements with $\mathbf{i}_p = \mathbf{i}_q$ and $\boldsymbol{q} \wedge \mathbf{i}_p = 0$.

Example 3. We now look again at two parallel planes, but the Euclidean distance vector $\boldsymbol{q}$ is at angle $\pi/4$ with the planes, see Fig. 6. We assume

$$\begin{aligned} &\mathbf{i}_p = \mathbf{i}_q = \boldsymbol{e}_{12}, \quad \boldsymbol{n}_p = \boldsymbol{n}_q = \boldsymbol{e}_3, \\ &\boldsymbol{q} = \boldsymbol{e}_1 + \boldsymbol{e}_3, \quad \boldsymbol{q}^2 = 2, \quad \hat{\boldsymbol{q}} = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_3), \end{aligned} \tag{29}$$

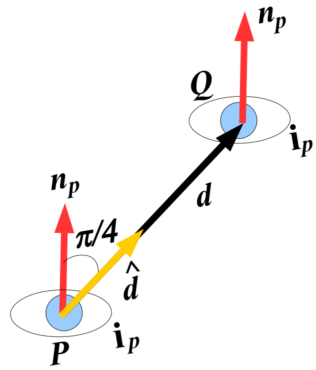

Fig. 6. Illustration of inner product of two parallel oriented points P and Q, with Euclidean distance vector $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$ at angle $\pi/4$ to the common orientation plane, i.e., dihedral angle $\alpha_{pq} = 0$, and equal angles $\Theta_p = \Theta_q = \pi/4$ between $\boldsymbol{d}$ and $\boldsymbol{n}_p = \boldsymbol{n}_q$, respectively.

and obtain

$$\cos\alpha_{pq} = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1, \quad \cos\Theta_q = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_3) \cdot \boldsymbol{e}_3 = \frac{1}{\sqrt{2}},$$
$$\cos\Theta_p = \frac{1}{\sqrt{2}}(\boldsymbol{e}_1 + \boldsymbol{e}_3) \cdot \boldsymbol{e}_3 = \frac{1}{\sqrt{2}}. \tag{30}$$

Applying (16) the inner product becomes

$$P \cdot Q = 2(-\frac{1}{2}1 + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}) = 0. \tag{31}$$

Note that this is a special case, where both terms in (16) are non-zero, but happen to cancel each other. The result corresponds to the point labeled (b) in Fig. 3.

Example 4. Now we take two planes perpendicular to each other (dihedral angle $\alpha_{pq} = \pi/2$, and the distance vector is perpendicular to the first and parallel to the second[5], see Fig. 7. We assume

$$\mathbf{i}_p = \boldsymbol{e}_{12}, \quad \mathbf{i}_q = \boldsymbol{e}_{23}, \quad \boldsymbol{n}_p = \boldsymbol{e}_3, \quad \boldsymbol{n}_q = \boldsymbol{e}_1,$$
$$\boldsymbol{q} = 3\boldsymbol{e}_3, \quad \boldsymbol{q}^2 = 9, \quad \hat{\boldsymbol{q}} = \boldsymbol{e}_3, \tag{32}$$

and obtain

$$\cos\alpha_{pq} = \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0, \quad \cos\Theta_q = \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0, \quad \cos\Theta_p = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1. \tag{33}$$

[5] Obviously, two orthogonal planes and the distance vector perpendicular to any of the two, always means that in three dimensions it is necessarily parallel to the other plane.

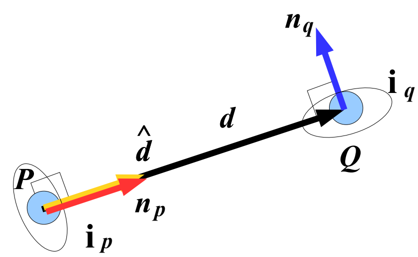

Fig. 7. Illustration of inner product of two orthogonally oriented points P and Q, with Euclidean distance vector $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$ parallel to the second orientation plane (perpendicular to the first), i.e., dihedral angle $\alpha_{pq} = \pi/2$, and angles $\Theta_p = 0$, $\Theta_q = \pi/2$, between $\boldsymbol{d}$ and the two perpendicular vectors $\boldsymbol{n}_p$, $\boldsymbol{n}_q$, respectively.

Applying (16) the inner product becomes

$$P \cdot Q = 9(-\frac{1}{2}0 + 0) = 0. \tag{34}$$

Example 5. This example is simply a variation of the previous one, with a different orientation of $\mathbf{i}_q$, see again Fig. 7. We assume

$$\begin{aligned} &\mathbf{i}_p = \boldsymbol{e}_{12}, \quad \mathbf{i}_q = \frac{1}{\sqrt{2}}(\boldsymbol{e}_{13} + \boldsymbol{e}_{23}), \quad \boldsymbol{n}_p = \boldsymbol{e}_3, \quad \boldsymbol{n}_q = \frac{1}{\sqrt{2}}(-\boldsymbol{e}_2 + \boldsymbol{e}_1), \\ &\boldsymbol{q} = 3\boldsymbol{e}_3, \quad \boldsymbol{q}^2 = 9, \quad \hat{\boldsymbol{q}} = \boldsymbol{e}_3. \end{aligned} \tag{35}$$

and obtain

$$\begin{aligned} &\cos\alpha_{pq} = \boldsymbol{e}_3 \cdot \frac{1}{\sqrt{2}}(-\boldsymbol{e}_2 + \boldsymbol{e}_1) = 0, \quad \cos\Theta_q = \boldsymbol{e}_3 \cdot \frac{1}{\sqrt{2}}(-\boldsymbol{e}_2 + \boldsymbol{e}_1) = 0, \\ &\cos\Theta_p = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1. \end{aligned} \tag{36}$$

Applying (16) the inner product becomes zero again

$$P \cdot Q = 9(-\frac{1}{2}0 + 0) = 0. \tag{37}$$

Remark 4. Obviously, if the two planes are orthogonal to each other, and the Euclidean distance vector is parallel to one of the planes, the result is always zero, independent of the distance $q = |\boldsymbol{q}| = \sqrt{\boldsymbol{q}^2}$ of the two points.

Example 6. In this example the second plane is tilted with respect to the first by a dihedral angle of $\pi/4$. The distance vector is perpendicular to the first plane and at angle $\pi/4$ with the second, see Fig. 8.

$$\begin{aligned} &\mathbf{i}_p = \boldsymbol{e}_{12}, \quad \mathbf{i}_q = \frac{1}{\sqrt{2}}(\boldsymbol{e}_{12} + \boldsymbol{e}_{23}), \quad \boldsymbol{n}_p = \boldsymbol{e}_3, \quad \boldsymbol{n}_q = \frac{1}{\sqrt{2}}(\boldsymbol{e}_3 + \boldsymbol{e}_1), \\ &\boldsymbol{q} = 3\boldsymbol{e}_3, \quad \boldsymbol{q}^2 = 9, \quad \hat{\boldsymbol{q}} = \boldsymbol{e}_3. \end{aligned} \tag{38}$$

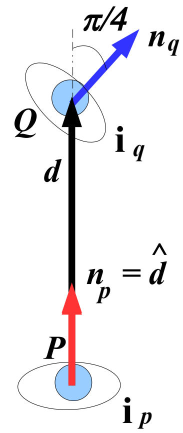

Fig. 8. Illustration of inner product of two oriented points P and Q, with Euclidean distance vector $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$ perpendicular to the first orientation plane and at angle $\pi/4$ to the second, i.e., dihedral angle $\alpha_{pq} = \pi/4$, and angles $\Theta_p = 0$, $\Theta_q = \pi/4$, between $\boldsymbol{d}$ and the two perpendicular vectors $\boldsymbol{n}_p$, $\boldsymbol{n}_q$, respectively.

We obtain

$$\cos\alpha_{pq} = \boldsymbol{e}_3 \cdot \frac{1}{\sqrt{2}}(\boldsymbol{e}_3 + \boldsymbol{e}_1) = \frac{1}{\sqrt{2}}, \quad \cos\Theta_q = \boldsymbol{e}_3 \cdot \frac{1}{\sqrt{2}}(\boldsymbol{e}_3 + \boldsymbol{e}_1) = \frac{1}{\sqrt{2}},$$
$$\cos\Theta_p = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1. \tag{39}$$

Applying (16) the inner product becomes

$$P \cdot Q = 9(-\frac{1}{2}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}) = \frac{9}{2\sqrt{2}}. \tag{40}$$

Here both terms in (16) contribute and the second term $\cos\Theta_q \cos\Theta_p$ dominates, making the overall sign positive.

Example 7. Here we take the two planes to be parallel, and a more general Euclidean distance vector, see Fig. 9.

$$\mathbf{i}_p = \boldsymbol{e}_{12}, \quad \mathbf{i}_q = \boldsymbol{e}_{12}, \quad \boldsymbol{n}_p = \boldsymbol{e}_3, \quad \boldsymbol{n}_q = \boldsymbol{e}_3,$$
$$\boldsymbol{q} = 3\boldsymbol{e}_2 + 2\boldsymbol{e}_3, \quad \boldsymbol{q}^2 = 13, \quad \hat{\boldsymbol{q}} = \frac{1}{\sqrt{13}}(3\boldsymbol{e}_2 + 2\boldsymbol{e}_3). \tag{41}$$

We obtain

$$\cos\alpha_{pq} = \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1, \quad \cos\Theta_q = \boldsymbol{e}_3 \cdot \frac{1}{\sqrt{13}}(3\boldsymbol{e}_2 + 2\boldsymbol{e}_3) = \frac{2}{\sqrt{13}},$$
$$\cos\Theta_p = \boldsymbol{e}_3 \cdot \frac{1}{\sqrt{13}}(3\boldsymbol{e}_2 + 2\boldsymbol{e}_3) = \frac{2}{\sqrt{13}}. \tag{42}$$

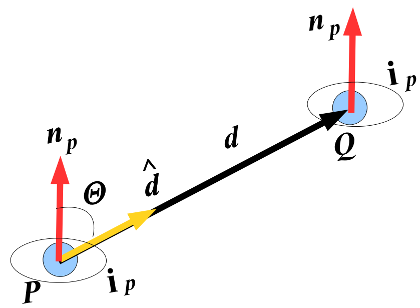

Fig. 9. Illustration of inner product of two parallel oriented points P and Q, with general Euclidean distance vector $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$, dihedral angle $\alpha_{pq} = 0$, and angles $\Theta_p = \Theta_q = \arccos(2/\sqrt{13}) = 56.31°$, between $\boldsymbol{d}$ and the two perpendicular vectors $\boldsymbol{n}_p$, $\boldsymbol{n}_q$, respectively.

Applying (16) the inner product becomes

$$P \cdot Q = 13(-\frac{1}{2} + \frac{4}{13}) = -\frac{5}{2}. \tag{43}$$

Here again both terms in (16) contribute and the first term, $-\frac{1}{2}\cos\alpha_{pq}$, dominates.

4 About the term $(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)$ in $P \cdot Q$

– For $\boldsymbol{n}_p \nparallel \boldsymbol{n}_q$ the two plane normal vectors together define a plane, that can be specified by the bivector $\boldsymbol{n}_p \wedge \boldsymbol{n}_q$. This allows to split the Euclidean distance vector $\boldsymbol{q}$ into parts parallel $\boldsymbol{q}_\parallel$ and perpendicular $\boldsymbol{q}_\perp$ to the $\boldsymbol{n}_p \wedge \boldsymbol{n}_q$-plane. In the inner products of $(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)$ of (16), the perpendicular $\boldsymbol{q}_\perp$ part will not contribute, because it is perpendicular to both $\boldsymbol{n}_p$ and $\boldsymbol{n}_q$. So we get

$$(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p) = (\hat{\boldsymbol{q}}_\parallel \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}}_\parallel \cdot \boldsymbol{n}_p). \tag{44}$$

– For $\boldsymbol{n}_q = \boldsymbol{n}_p$, the part $\boldsymbol{q}_\perp$ perpendicular to $\boldsymbol{n}_q$ drops out, and only the part $\hat{\boldsymbol{q}}_\parallel$ parallel to $\boldsymbol{n}_q$ contributes.

$$(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p) = (\hat{\boldsymbol{q}}_\parallel \cdot \boldsymbol{n}_p)^2. \tag{45}$$

– If $\hat{\boldsymbol{q}} \perp \boldsymbol{n}_q$ or $\hat{\boldsymbol{q}} \perp \boldsymbol{n}_p$, then

$$(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p) = 0, \tag{46}$$

and

$$P \cdot Q = -\frac{1}{2}\boldsymbol{q}^2 \cos\alpha_{pq} = P_{no} \cdot Q_{no} \cos\alpha_{pq}. \tag{47}$$

If furthermore $\boldsymbol{n}_p = \boldsymbol{n}_q$, i.e., the two planes have identical orientations and hence $\cos\alpha_{pq} = 1$, then

$$P \cdot Q = -\frac{1}{2}\boldsymbol{q}^2 = P_{no} \cdot Q_{no}. \tag{48}$$

– For $\hat{\boldsymbol{q}} = \boldsymbol{n}_q = \boldsymbol{n}_p$ or $\hat{\boldsymbol{q}} = -\boldsymbol{n}_q = -\boldsymbol{n}_p$ we the get a maximal contribution of $(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)$ to the inner product. Then

$$P \cdot Q = \frac{1}{2}\boldsymbol{q}^2 = -P_{no} \cdot Q_{no}. \tag{49}$$

– For $\hat{\boldsymbol{q}} = \boldsymbol{n}_q = -\boldsymbol{n}_p$ or $\hat{\boldsymbol{q}} = -\boldsymbol{n}_q = \boldsymbol{n}_p$ we the get a minimal contribution of $(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)$ to the inner product. Then

$$P \cdot Q = -\frac{1}{2}\boldsymbol{q}^2 = P_{no} \cdot Q_{no}. \tag{50}$$

5 Symmetries of $\boldsymbol{P \cdot Q}$

In this section we first study the symmetries of the inner product $P \cdot Q$ of two oriented points. Then we use the results to plot $P \cdot Q$ as function of the three angles α_{pq}, θ_p and θ_q.

The inner product of two oriented points in CGA of (16) is a function of the Euclidean distance $|\boldsymbol{q}|^2 = \boldsymbol{q}^2$, and the three unit vectors $\hat{\boldsymbol{q}}$, $\boldsymbol{n}_q$, and $\boldsymbol{n}_p$, i.e. the unit direction of the Euclidean distance, and the two unit normal vectors of the two planes.

$$\begin{aligned} P \cdot Q &= \boldsymbol{q}^2[-\frac{1}{2}\boldsymbol{n}_p \cdot \boldsymbol{n}_q + (\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)] = g(|\boldsymbol{q}|^2, \boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}) \\ &= |\boldsymbol{q}|^2 f(\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}). \end{aligned} \tag{51}$$

The function $f(\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}})$, as proven in Appendix A, has the following symmetries

$$\begin{aligned} &f(-\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}) = f(\boldsymbol{n}_p, -\boldsymbol{n}_q, \hat{\boldsymbol{q}}) = -f(\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}), \\ &f(\boldsymbol{n}_p, \boldsymbol{n}_q, -\hat{\boldsymbol{q}}) = f(\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}), \quad f(\boldsymbol{n}_q, \boldsymbol{n}_p, \hat{\boldsymbol{q}}) = f(\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}). \end{aligned} \tag{52}$$

That is, changing the sign of any one of the two plane normal vectors changes the sign of $P \cdot Q$, while changing the sign of the Euclidean distance vector leaves $P \cdot Q$ invariant. Hence, f is odd with respect to (w.r.t.) $\boldsymbol{n}_p$ and $\boldsymbol{n}_q$. It is even w.r.t. $\hat{\boldsymbol{q}}$, and symmetric w.r.t. interchanging $\boldsymbol{n}_p$ and $\boldsymbol{n}_q$, manifestations of the symmetry $P \cdot Q = Q \cdot P$.

Taking the above symmetries of $P \cdot Q$ into account, we graph in Fig. 10 the values of the product for two points at unit distance $|\boldsymbol{d}| = 1$ for five discrete values of the dihedral angle $\alpha_{pq} \in \{0, 0.9, \pi/2, 2.1, \pi\}$ and for the two angles Θ_p, Θ_q, between the distance vector $\boldsymbol{d}$ and the two point orientations $\boldsymbol{n}_p$ and $\boldsymbol{n}_q$, respectively, with values in the range of $\Theta_p, \Theta_q \in [0, \pi/2]$. This is one quarter of

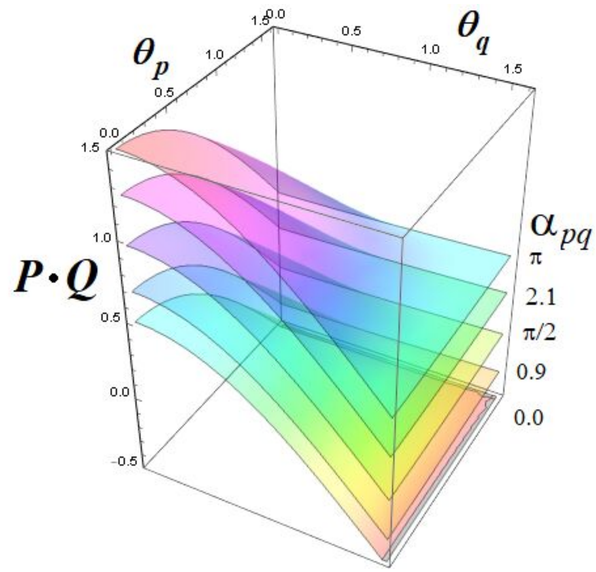

Fig. 10. Illustration of inner product of two oriented points P and Q, with Euclidean distance vector $\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}$, $|\boldsymbol{d}| = 1$, for dihedral angles between the two orientation planes $\alpha_{pq} \in \{0, 0.9, \pi/2, 2.1, \pi\}$, and $\Theta_p, \Theta_q \in [0, \pi/2]$, respectively.

the full $\Theta_p, \Theta_q \in [-\pi/2, \pi/2]$ plane of angles relative to $\boldsymbol{d}$. Due to the symmetries in (52), the graph is the same in every of the four quadrants of the Θ_p, Θ_q-plane. The graph further correctly shows the symmetry across the diagonal line $\Theta_p = \Theta_q$. A general non-unit distance $|\boldsymbol{d}|$ between P and Q scales the graph vertically by the factor $|\boldsymbol{d}|^2$. When interpreting the graph, note the natural spherical triangle restriction of $|\Theta_p - \Theta_q| \leq \alpha_{pq} \leq |\Theta_p + \Theta_q|$.

Remark 5. We observe that gradient ascent appears to lead from all sides to matching orientations of $\Theta_p = \Theta_q = 0$, i.e., full alignment, which would provide good conditions for applications in machine learning, optimization and adjustment of orientations.

6 Inner product of oriented points for n-dimensional Euclidean space

In this section we aim to show that the inner product relationship (16) of oriented points in CGA, applies in any dimension $n \geq 2$, up to an overall sign.

We are now working with CGA $Cl(n+1, 1)$ of n-dimensional Euclidean space $\mathbb{R}^n$. Its pseudoscalar is

$$I = I_{n+1,1} = I_n E, \quad I_n = e_1 e_2 \cdots e_n, \quad E = e_\infty \wedge e_0, \tag{53}$$

with squares

$$E^2 = 1, \quad I^2 = (I_n E)^2 = I_n^2 E^2 = I_n^2 = \begin{cases} +1, n \bmod 4 = 1, 0 \\ -1, n \bmod 4 = 2, 3. \end{cases} \tag{54}$$

Depending on the dimension n we therefore have the inverse of the pseudoscalar to be

$$I^{-1} = \begin{cases} +I, n \bmod 4 = 1, 0 \\ -I, n \bmod 4 = 2, 3. \end{cases} \tag{55}$$

The dual of a multivector $M \in Cl(n+1, 1)$ is given by

$$M^* = MI^{-1}, \quad M = M^* I. \tag{56}$$

Especially for two bivectors M_b and N_b we have the inner product relationship with the duals of the bivectors to be

$$\begin{aligned} \langle M_b^* N_b^* \rangle &= \langle M_b(\pm I) N_b(\pm I) \rangle = \langle M_b N_b I^2 \rangle \\ &= \langle M_b N_b \rangle \left\{ \begin{matrix} +1, n \bmod 4 = 1, 0 \\ -1, n \bmod 4 = 2, 3 \end{matrix} \right\}, \end{aligned} \tag{57}$$

where we used the commutation of $IN_b = N_b I$ for bivectors N_b. For example in the case of $n = 3$ we have

$$\langle M_b^* N_b^* \rangle = -\langle M_b N_b \rangle, \tag{58}$$

which also proves (15). We now construct an oriented point in $Cl(n+1, n)$ by taking a dual sphere vector centered at the Euclidean position of the point $\boldsymbol{p} \in \mathbb{R}^n$ intersected with a dual equator plane[6] orthogonal to normal unit vector $\boldsymbol{n}_p \in \mathbb{R}^n$, $\boldsymbol{n}_p^2 = 1$, and take the limit of the sphere radius $r \to 0$. The dual sphere is

$$\sigma = S^* = C_p - \frac{1}{2} r^2 e_\infty, \tag{59}$$

with conformal center point

$$C_p = \boldsymbol{p} + \frac{1}{2} \boldsymbol{p}^2 e_\infty + e_0. \tag{60}$$

The dual equator plane that has to include the center C_p and be normal to $\boldsymbol{n}_p$ is

$$\mu = Plane^* = \boldsymbol{n}_p + d e_\infty = \boldsymbol{n}_p + (\boldsymbol{p} \cdot \boldsymbol{n}_p) e_\infty, \tag{61}$$

using oriented distance $d \in \mathbb{R}$ from the origin

$$d = C_p \cdot \boldsymbol{n}_p = (\boldsymbol{p} + \frac{1}{2} \boldsymbol{p}^2 e_\infty + e_0) \cdot \boldsymbol{n}_p = \boldsymbol{p} \cdot \boldsymbol{n}_p. \tag{62}$$

[6] Strictly speaking this is a hyper-plane of dimension $n - 1$.

The dual of the equator circle[7] is given by the outer product of dual equator plane and dual sphere

$$Circle^* = \mu \wedge \sigma = [\boldsymbol{n}_p + (\boldsymbol{p}\cdot\boldsymbol{n}_p)e_\infty] \wedge (C_p - \frac{1}{2}r^2 e_\infty). \tag{63}$$

Taking the limit of sphere radius $r \to 0$, and inserting the expression for C_p, we get the dual of an oriented point in CGA $Cl(n+1,1)$ located at $\boldsymbol{p} \in \mathbb{R}^n$ and oriented normal to $\boldsymbol{n}_p$

$$\begin{aligned} P^* &= (\boldsymbol{n}_p + \boldsymbol{p}\cdot\boldsymbol{n}_p\, e_\infty) \wedge (\boldsymbol{p} + \frac{1}{2}\boldsymbol{p}^2 e_\infty + e_0) \\ &= \boldsymbol{n}_p \wedge \boldsymbol{p} + \frac{1}{2}\boldsymbol{p}^2\boldsymbol{n}_p e_\infty + (\boldsymbol{p}\cdot\boldsymbol{n}_p)e_\infty\boldsymbol{p} + \boldsymbol{n}_p e_0 + (\boldsymbol{p}\cdot\boldsymbol{n}_p)(e_\infty \wedge e_0) \\ &= \boldsymbol{n}_p \wedge \boldsymbol{p} + [\frac{1}{2}\boldsymbol{p}^2\boldsymbol{n}_p - \boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{n}_p)]e_\infty + \boldsymbol{n}_p e_0 + (\boldsymbol{p}\cdot\boldsymbol{n}_p)E, \end{aligned} \tag{64}$$

where we used the anti-commutation $e_\infty\boldsymbol{p} = -\boldsymbol{p}e_\infty$. A second dual oriented point located at $\boldsymbol{q} \in \mathbb{R}^n$ and oriented normal to $\boldsymbol{n}_q$ is then given by

$$Q^* = \boldsymbol{n}_q \wedge \boldsymbol{q} + [\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{n}_q)]e_\infty + \boldsymbol{n}_q e_0 + (\boldsymbol{q}\cdot\boldsymbol{n}_q)E, \tag{65}$$

in full agreement with (3) for $n = 3$. Locating the first dual oriented point at the origin, i.e. $\boldsymbol{p} = 0$, it becomes

$$P^* = \boldsymbol{n}_p e_0. \tag{66}$$

The inner product with the second dual oriented point in general position $\boldsymbol{q}$, $\boldsymbol{q}$ therefore marking the oriented distance vector of the two points, becomes

$$\begin{aligned} \langle P^*Q^*\rangle &= \langle \boldsymbol{n}_p e_0\{\boldsymbol{n}_q \wedge \boldsymbol{q} + [\frac{1}{2}\boldsymbol{q}^2 - \boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{n}_q)]e_\infty + \boldsymbol{n}_q e_0 + (\boldsymbol{q}\cdot\boldsymbol{n}_q)E\}\rangle \\ &= \langle \boldsymbol{n}_p e_0[\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{n}_q)]e_\infty\rangle = \langle \boldsymbol{n}_p[\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{n}_q)]\rangle \\ &= \frac{1}{2}\boldsymbol{q}^2(\boldsymbol{n}_p\cdot\boldsymbol{n}_q) - (\boldsymbol{q}\cdot\boldsymbol{n}_p)(\boldsymbol{q}\cdot\boldsymbol{n}_q), \end{aligned} \tag{67}$$

using $e_0\boldsymbol{v} = -\boldsymbol{v}e_0$ for any vector $\boldsymbol{v} \in \mathbb{R}^n$, especially for $\boldsymbol{v} = [\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{n}_q)]$, and $-\langle e_0 e_\infty\rangle = 1$. Because P^* and Q^* are bivectors, the inner product of P and Q becomes by (57)

$$\begin{aligned} \langle PQ\rangle &= \langle P^*Q^*\rangle \left\{\begin{matrix} +1, n \bmod 4 = 1,0 \\ -1, n \bmod 4 = 2,3 \end{matrix}\right\} \\ &= \left\{\begin{matrix} +1, n \bmod 4 = 1,0 \\ -1, n \bmod 4 = 2,3 \end{matrix}\right\} [\frac{1}{2}\boldsymbol{q}^2(\boldsymbol{n}_p\cdot\boldsymbol{n}_q) - (\boldsymbol{q}\cdot\boldsymbol{n}_p)(\boldsymbol{q}\cdot\boldsymbol{n}_q)], \end{aligned} \tag{68}$$

[7] Note that in general dimensions this is a hyper-circle in the sense that for $n = 2$ it is a point pair, for $n = 3$ a normal circle, for $n = 4$ the circle is itself a three-dimensional sphere embedded in four dimensions, etc.

and obviously agrees by (58) in three dimensions ($n = 3$) with (16).

The analysis of the preceding Sections 2, 4 and 5 therefore fully applies in general dimensions $n \geq 2$, up to an overall sign[8] due to the value of I^2, which is easy to take into account. And examples analogous to Section 3 are obviously easy to construct.

7 Conclusion

In this work we have reviewed the formulation of oriented points in conformal geometric algebra (CGA), and computed the inner product of two oriented points in terms of their distance vector (its direction and length) and their two point orientations. The geometric meaning of this inner product is elucidated based on a set of representative illustrated examples, analysis of the key term in the inner product, symmetry analysis and graphing of $P \cdot Q$. Finally, the approach is extended from three to n Euclidean dimensions. As pointed out in Remark 5, the inner product of oriented points apparently provides good conditions for applications in machine learning, optimization and adjustment of orientation. Furthermore, our new results may find application in LIDAR terrain strip adjustment computations, where points on overlapping strips need to be compared together with the local plane orientation of the respective strip[9], see e.g. [11], or in protein geometry modeling[10] [15, 16], etc.

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[8] Taking the absolute value $|\langle PQ \rangle| = |\langle P^* Q^* \rangle|$, completely removes this overall dimension dependent sign, and may be all that is needed in many applications.

[9] *A correspondence is defined between pairs of points from overlapping (airborne laser scanning) strips and from their normal vectors (conventionally) or local strip plane bivectors (our choice) fitted to the neighboring points (in the same plane in the same strip). Note that the normal vectors are dual to the plane bivectors via multiplication with the three-dimensional space pseudoscalar $I_3 = e_1 e_2 e_3$. ... Two corresponding points in overlapping strips should belong to the same plane but need not to be identical.* Quoted from page 11 of [11].

[10] The latest research on this topic by A. Pepe et al. already includes the evaluation of the inner product of oriented points, see Section 4.3 of [16].

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A Proof of symmetries of $\boldsymbol{P \cdot Q}$

We now prove the symmetries of the inner product of oriented points $P \cdot Q$ given in (52). According to (51) the factor function f in $P \cdot Q = \boldsymbol{q}^2 f$ is defined as

$$f(\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}) = -\frac{1}{2}\boldsymbol{n}_p \cdot \boldsymbol{n}_q + (\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q). \tag{69}$$

Therefore,

$$\begin{aligned} f(-\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}) &= -\frac{1}{2}(-\boldsymbol{n}_p) \cdot \boldsymbol{n}_q + (\hat{\boldsymbol{q}} \cdot (-\boldsymbol{n}_p))(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q) \\ &= -\frac{1}{2}\boldsymbol{n}_p \cdot (-\boldsymbol{n}_q) + (\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)(\hat{\boldsymbol{q}} \cdot (-\boldsymbol{n}_q)) = f(\boldsymbol{n}_p, -\boldsymbol{n}_q, \hat{\boldsymbol{q}}), \end{aligned} \tag{70}$$

which proves the first identity in (52).

Similarly,

$$\begin{aligned} f(\boldsymbol{n}_p, -\boldsymbol{n}_q, \hat{\boldsymbol{q}}) &= -\frac{1}{2}\boldsymbol{n}_p \cdot (-\boldsymbol{n}_q) + (\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)(\hat{\boldsymbol{q}} \cdot (-\boldsymbol{n}_q)) \\ &= -\Big(-\frac{1}{2}\boldsymbol{n}_p \cdot \boldsymbol{n}_q + (\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)\Big) = -f(\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}), \end{aligned} \tag{71}$$

proves the second identity in (52).

Moreover,

$$\begin{aligned} f(\boldsymbol{n}_p, \boldsymbol{n}_q, -\hat{\boldsymbol{q}}) &= -\frac{1}{2}\boldsymbol{n}_p \cdot \boldsymbol{n}_q + ((-\hat{\boldsymbol{q}}) \cdot \boldsymbol{n}_p)((-\hat{\boldsymbol{q}}) \cdot \boldsymbol{n}_q) \\ &= -\frac{1}{2}\boldsymbol{n}_p \cdot \boldsymbol{n}_q + (\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q) = f(\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}), \end{aligned} \tag{72}$$

proves the third identity in (52).

Finally,

$$\begin{aligned} f(\boldsymbol{n}_q, \boldsymbol{n}_p, \hat{\boldsymbol{q}}) &= -\frac{1}{2}\boldsymbol{n}_q \cdot \boldsymbol{n}_p + (\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p) \\ &= -\frac{1}{2}\boldsymbol{n}_p \cdot \boldsymbol{n}_q + (\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_p)(\hat{\boldsymbol{q}} \cdot \boldsymbol{n}_q) = f(\boldsymbol{n}_p, \boldsymbol{n}_q, \hat{\boldsymbol{q}}), \end{aligned} \tag{73}$$

proves the fourth identity in (52).