Statistical Analysis of the Foundations of Quantum Mechanics I:

The fundamental laws of Quantum Dynamics

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Abstract

In this paper the laws of quantum dynamics are enunciated based upon statistical considerations of a generic quantum system. A new concept called quantropy is introduced and related to the Feynman path integral formalism as well as the orthodox Copenhagen Interpretation.

1. Introduction

There are many consistent interpretations of quantum mechanics [1]. All of them are equivocal [2]. However, the standard Copenhagen Interpretation (CI) provided by Bohr et. al. is the most popular and has stood the test of time [2]. The postulates of quantum mechanics are standard. Then, its just shut up and calculate. There are no proper laws of quantum dynamics as there are in Newtonian mechanics. Also, there systematic law describing the transition from classical to quantum or vice versa.

The Feynman Path Integral (FPI) formalism is a third way to understand much of the quantum [3]. The other two being Heisenberg's matrix method and Schrödinger's wave mechanics. The wave mechanics was given a statistical interpretation by Born [4] called the Born rule.

In this paper, we then do the following:

a. Enunciate systematically the laws of quantum dynamics by deriving them mathematically from a set of statistical postulates.
b. Relate the paths in the FPI formalism to the semi-classical theory by means of new notions.
c. Derive a relationship relating quantum dynamical randomness (not to be confused with entropy) and the Born probability density.

The present work takes a completely different route to build the theory of Quantum Mechanics for bosons as well as fermions taking thereby definitive steps in the directions of construction of a statistical theory of quantum dynamics.

Through the canonical theory, one arrives at the Legendre transform

$$L = \sum_i p_i \dot{q}_i - H \quad \text{(1-a)}$$

Or

$$\sum_i p_i \dot{q}_i - L = H \quad \text{(1-b)}$$

where, p_i is the momentum canonically conjugate to the generalized coordinate q_i; $\dot{q}_i$ are the corresponding generalized velocities. L is the Lagrangian functional and H is the Hamiltonian which in the quantum domain satisfies the Schrödinger's postulate

$$\hat{E}\psi = \hat{H}\psi \quad (2)$$

with $\hat{E}$, the energy operator, and ψ, the wave function. H, the Hamiltonian functional in Classical mechanics becomes a linear operator in quantum mechanics.

The Lagrangian, L, in (1-a,b) is a measure of how active a system is.

The principle of least action

$$\delta \int L \, dt = 0 \quad (3)$$

therefore, measures the laziness of the system.

2. Some Background

The first postulate of QM tells us that every quantum system is characterized by physical observables. For each such physical observable there exists a linear Hermitian operator which operates on an eigen vector and yields an eigenvalue which is one of the outcomes of the measurement.

Let the ket $|\psi\rangle$ be the eigen vector and a_i be the eigenvalue for $\hat{A}$, the corresponding linear Hermitian operator so that for one of the identical quantum measurements, the first postulates translates as

$$\hat{A}|\psi\rangle = a_i|\psi\rangle \quad (4)$$

The 1-form or bra, $\langle X|$, spans an abstract vector space dual to that spanned by $|\psi\rangle$ so that a physical inner product space whose fluents are such as $\psi(x)$ and are given by the inner product

$$\psi(x) = \langle X|\psi\rangle \quad (5)$$

The probability density that the quantum system in the state $\psi(x)$ is given by

$$P = \psi^*\psi \quad \text{(6-a)}$$

Or

$$P = |\psi|^2 \quad \text{(6-b)}$$

where, ψ^* is the wave function complex conjugate to ψ. The more precise statement of the Born rule may be found in [4].

3. The Main Thurst

Consider now the actional $S = \int L\, dt$. That the nature is lazy is described by eq(3), viz., $\delta S = 0$. That is, we vary the paths between two fixed points, find S for each possible path and choose the path that gives the least action. This path is the classical path as given by Newton's law. This action has the dimensions of $\hbar$. And, Planck's quantum condition tells us that "action is quantized".

Now, action is a function of time.

$$\frac{dS}{dt} = L \tag{7}$$

$$\delta S = \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \tag{8}$$

Clearly,

$$\frac{\partial S}{\partial q_k} = \frac{\partial L}{\partial \dot{q}_k} = p_k \tag{9}$$

So that, S is a function of the generalized coordinates q_k.

As such

$$\dot{S} = \sum_i \frac{\partial S}{\partial q_i} \dot{q}_i + \frac{\partial S}{\partial t} \tag{10}$$

Which upon using eqs (7) and (9), becomes

$$\dot{S} = L = \sum_i p_i \dot{q}_i + \frac{\partial S}{\partial t} \tag{11}$$

Comparing with eq(1-a), we have

$$\frac{\partial S}{\partial t} = -H \tag{12}$$

We thus see that, S is a function of time t, and

$$dS = \sum_i p_i \dot{q}_i - H\, dt = dA - H\, dt \tag{13}$$

Where A is action $\int p\, dq$. We thus write,

$$S = S(q_k; t) \tag{14}$$

From eq(13), it is clear that, dS is a perfect differential.

As already mentioned, the alternative quantization procedure for quantizing a classical system is due to Feynman and is based on the quantity S, the other one is the canonical quantization procedure to Dirac. In QM, we need an apparatus which is 'classical' to measure quantum effects. For such an apparatus, Dirac suggested that, the wavefunction should be,

$$\phi_{apparatus} \sim \exp\left(\frac{i}{\hbar} \int L\, dt\right) \tag{15}$$

Feynman boldly proposed that, even though the classical motion requires $\delta S = 0$, for calculation of quantum amplitude (i.e., the wavefunction), all paths will contribute and,

$$S = S_{Cl} + S_1 + S_2 + S_3 + \cdots \tag{16}$$

Here S_{Cl} is the classical Euler-Lagrange path, $S_1, S_2, \ldots, S_k, \ldots$ are the paths near and around classical paths where $\int L\, dt$ is evaluated. The Feynman amplitude is then,

$$\Psi = constant\ \times \exp\left(\frac{i}{\hbar} \sum_{all\ paths} S_k\right) \tag{17}$$

In this way, Feynman proposed an alternate formulation of QM. S has the dimensions of $\hbar$. Naturally, the phase will violently oscillate unless S_k is in the $\hbar$ neighborhood of the classical path. Thus the classical limit is restored as $\hbar \to 0$.

When an apparatus makes a quantum measurement, the system falls into one of the states which correspond to one of S_k. Firstly, there are the beables [5] and then there are the observables. We redefine beables here as the states that the quantum system is actually in and the measuring apparatus being classical, the apparatus knocks the system from a beable to an observable.

The fundamental ansatz that we start with is that the most important quantity, the quantum mechanical probability $P = |\psi|^2$ is defined by

$$\ln P = \sum_r f(n_r) \tag{18}$$

Now, we fix the following ansatz,

$$\sum_r n_r = N \tag{19-a}$$

and

$$\sum_r n_r \gamma_r = H \tag{19-b}$$

Here, γ_r corresponds to the path corresponding to the $\hbar - nbd$ action mentioned above. It corresponds to the kinetic energy of the quantum system. The subscript r corresponds to the $r - th$ copy of the quantum system following one of the Feynman paths. We now fix up Lagrange multipliers as,

$$\alpha + \beta\gamma_r = \frac{\partial f}{\partial n_r} \tag{20}$$

Then,

$$\delta \ln P = \sum_r \frac{\partial f}{\partial n_r} \delta\, n_r = \sum_r (\alpha + \beta\gamma_r) \delta\, n_r$$
$$= \alpha \sum_r \delta n_r + \beta \sum_r \gamma_r \delta n_r \tag{21}$$

Since n_r is fixed by ansatz (19-a),

$$\sum_r \delta n_r = 0 \tag{22}$$

From (19-b),

$$\sum_r n_r \delta\gamma_r + \sum_r \gamma_r \delta n_r = \delta H \tag{23}$$

The first term in eq(23) corresponds to the quantum system sampling the various paths. This leads to some kind of quantal work. Thus,

$$\sum_r n_r \delta\gamma_r = \sum_r \sum_i n_r \frac{\partial \gamma_r}{\partial \dot{q}_i} \delta \dot{q}_i \tag{24}$$

Which translates to

$$p_i = \sum_r n_r \frac{\partial \gamma_r}{\partial \dot{q}_i} \tag{25}$$

The second term in eq(23) is proportional to the amount of activity of the system. This then is the negative of the variational Lagrangian δL given by

$$\sum_r \gamma_r \delta n_r = -\delta L \tag{26}$$

Therefore, combining eqs (23) to (26), we have

$$\sum_i p_i \, \delta \dot{q}_i - \delta L = \delta H$$

which translates to eq(1-b). Thus in essence, the Legendre transform is the first law of quantum dynamics more precisely that

1.) the circulation is conserved in phase space. Even more fundamental is the fact that the phase space is quantized. The Schrödinger equation given by (2) is also an alternative to the first law that the Hamiltonian evolves in time according to eq (2).

On the other hand,

$$\delta \ln P = \beta \sum_r \gamma_r \delta n_r = -\beta \, \delta L \tag{27}$$

Therefore, $\beta\ \delta L$ is a total differential and β, the integrating factor of the Lagrangian. The statistical theory leads naturally to the second law of quantum dynamics which we enunciate as follows:

2.) δL has an integrating factor namely

$$\delta K = -\frac{1}{\tau}\delta L \tag{28}$$

where, K is a new quantity called the quantropy. It is like entropy in thermodynamics but not the same. It corresponds to least activity of the system as can interpreted from (28). This implies that the collapse of the wavefunction upon quantum measurement corresponds to least activity of the quantum system. Hence, the negative sign. The state of maximum randomness and quantropy are oppositely related. τ is the inverse Euclidean time. It shows that time is something that arises from the average property of the collection of the Feynman paths. For dimensional consistency, we insert

$$\beta = \frac{1}{\hbar\tau} \tag{29}$$

So,

$$\delta \ln P = -\frac{1}{\hbar\tau}\delta L$$

or

$$\delta K = -\hbar\,\delta \ln P$$

or therefore,

$$K = -\hbar \ln P \tag{30}$$

Or alternately,

$$K = -\hbar \ln|\psi|^2 \tag{30'}$$

Thus, the quantropy is related to the logarithm of the probability density of the quantum system, thus relating the Feynman formalism to the Born rule. From the above analysis, it is apparent that the normalization of the wavefunction tends to serve as a law of quantum dynamics. Say, the third law. We enunciate a minus first law of quantum dynamics as the law of conservation of information due to Susskind [6]. The Heisenberg Uncertainty Principle can be put up as the zeroth law.

3.) It is impossible for quantropy to become zero, since the quantum system is bound to be somewhere.

This is the formal statement of the third law of quantum dynamics. The evaluation of quantropy corresponds more or less to the **R-Process** (the Reduction Process) that Penrose speaks of in the context of the collapse of the wavefunction upon quantum measurement.

We pronounce the zeroth law thus,

In the case of absence of the system, the apparatus making the measurement detects uniform time. The Pauli theorem on time figures as the fourth law of quantum dynamics, stated thus,

4.) It is impossible to have a linear Hermitian Time operator, if the Hamiltonian is semi-bounded or discrete.

Just as temperature is a concept that holds macroscopically and breaks down at the individual molecular/ atomic level, time is also a concept that plays out a similar role in quantum mechanics.

4. Conclusions

The concept of quantropy is central in the afore discussion. It corresponds to both the **U-Process** (Unitary Process) and the **R-Process**. The conservation of information is inherent in the concept of quantropy. It is related to action and since action and entropy are related, it is related to entropy but is unlike entropy. The process of quantum measurement corresponds to a decrease in quantropy (note the negative sign). The laziness of the system is also encompassed in quantropy. It is interesting to see how quantropy relates to entanglement entropy and the Ryu-Takayanagi formula. It is also interesting to see the connection between quantropy and the most important principle of quantum mechanics: the principle of contextuality.

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6. References

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